A FINITE-ELEMENT METHOD FOR TRANSVERSE VIBRATIONS OF BEAMS AND PLATES

by

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Hudson Matlock

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Development of Methods for Computer Simulation of Beam-Columns and Grid-Beam and Slab Systems
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The opinions, findings, and conclusions expressed in this publication are those of the authors and not necessarily those of the Bureau of Public Roads.
This report presents the results of an analytical study which was undertaken to develop an implicit numerical method for determining the transient and steady-state vibrations of elastic beams and plates. The study consists of (1) a theoretical analysis of the stability of difference equations used, (2) the formulation of the difference equations for the general solution of the beam and plate, and (3) a demonstration of the method by computer solutions of example problems. A supplemental report will describe the use of the associated computer programs for the beam and plate and will further illustrate the application of these programs to highway engineering problems.

Report 56-1 in the List of Reports provides an explanation of the basic procedures which are used in these programs. Although the programs are written in FORTRAN-63 for the CDC 1604 computer, minor changes would make these programs compatible with an IBM 7090 system. Copies of the programs and data cards for the example problems in this report may be obtained from the Center for Highway Research at The University of Texas.

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H. Salani
H. Matlock
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LIST OF REPORTS

Report No. 56-1, "A Finite-Element Method of Solution for Linearly Elastic Beam-Columns" by Hudson Matlock and T. Allan Haliburton, presents a finite-element solution for beam-columns that is a basic tool in subsequent reports.

Report No. 56-2, "A Computer Program to Analyze Bending of Bent Caps" by Hudson Matlock and Wayne B. Ingram, describes the application of the beam-column solution to the particular problem of bent caps.

Report No. 56-3, "A Finite-Element Method of Solution for Structural Frames" by Hudson Matlock and Berry Ray Grubbs, describes a solution for frames with no sway.

Report No. 56-4, "A Computer Program to Analyze Beam-Columns under Movable Loads" by Hudson Matlock and Thomas P. Taylor, describes the application of the beam-column solution to problems with any configuration of movable non-dynamic loads.


Report No. 56-6, "Discontinuous Orthotropic Plates and Pavement Slabs" by W. Ronald Hudson and Hudson Matlock, describes an alternating-direction iteration method for solving complex two-dimensional plate and slab problems with emphasis on pavement slabs.

Report No. 56-7, "A Finite-Element Analysis of Structural Frames" by T. Allan Haliburton and Hudson Matlock, describes a method of analysis for rectangular plane frames with three degrees of freedom at each joint.

Report No. 56-8, "A Finite-Element Method for Transverse Vibrations of Beams and Plates" by Harold Salani and Hudson Matlock, describes an implicit procedure for determining the transient and steady-state vibrations of beams and plates, including pavement slabs.

Report No. 56-9, "A Direct Computer Solution for Plates and Pavement Slabs" by C. Fred Stelzer, Jr., and W. Ronald Hudson, describes a direct method for solving complex two-dimensional plate and slab problems.

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ABSTRACT

A finite-element method is developed to determine the transverse linear deflections of a vibrating beam or plate. The method can be used to obtain numerical solutions to varied beam and plate vibration problems which cannot be readily solved by other known methods. The solutions for the beam and plate are separate formulations which have been programmed for a digital computer. Both solutions permit arbitrary variations in bending stiffness, mass density and dynamic loading. The static equations have been included in the development so that the initial deflections can be conveniently established. In the beam, the difference equations are solved by a recursive procedure. For the plate, the same procedure is combined with an alternating-direction technique to obtain an iterated solution. The numerical results demonstrate that the method is applicable to a wide range of vibration problems which are relevant to a beam or plate.
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CHAPTER 1. INTRODUCTION

Advances in science and technology have brought about an increasing need for solutions to structural problems in which dynamic behavior is an important factor. Classical solutions are available for a limited class of problems in this category. The development of the high-speed digital computer has made it feasible to obtain approximate numerical solutions for a vast number of heretofore unsolved problems.

The primary purpose of this investigation is to develop a finite-element method for determining the transverse time-dependent linear deflections of a beam or plate. The method is based on an implicit formula which was introduced by Crank and Nicolson (Ref 5)* to solve the second order heat flow problem.

Essentially, the beam or plate is replaced by an arbitrary number of finite elements and the time dimension is divided into discrete intervals. This representation readily permits the flexural stiffness, elastic restraints and the loading to be discontinuous. The governing partial linear differential equation is approximated by a difference equation and a numerical solution is obtained at specified intervals of time. The difference equation for the unknown deflection may be formulated explicitly or implicitly. In an explicit formula, there is only one unknown deflection in each difference equation, whereas, in an implicit formula, there are several unknown deflections in each equation. Thus the resulting set of difference equations must be solved simultaneously to obtain the unknown deflections.

Finite difference solutions for initial value problems are subject to

*See References on p 51.
instability. This can be illustrated by considering the following equation for an undamped transversely vibrating beam:

\[
EI \frac{\partial^4 w}{\partial x^4} + \rho \frac{\partial^2 w}{\partial t^2} = 0
\] (1.1)

In the foregoing, \(E\) is the modulus of elasticity, \(I\) is the moment of inertia, \(\rho\) is the mass density per unit length, \(w\) is the deflection, \(x\) is the distance along the beam and \(t\) is the time. For suitable boundary conditions and a given initial displacement, the beam will vibrate periodically. If the deflections are calculated from a solution of the partial differential equation, the contribution from the higher characteristic frequencies is usually negligible. However, in a finite difference solution, it is possible for the higher frequencies to cause the calculated deflections to become unbounded as time approaches infinity. In his book on difference methods, Richtmyer (Ref 11) discusses the equivalence of stability and convergence. For properly defined problems, stability insures convergence. Crandall (Ref 4) and other investigators have discussed the stability of finite difference approximations for Eq 1.1.

The stability criteria and pictorial representations of the explicit and implicit formulas for a beam and plate will be presented in the subsequent discussion. Both formulations have been programmed for a digital computer. However, the development of the equations and the numerical results will pertain to the implicit solution. As a convenience in establishing the initially deflected shape of a beam or plate, the equations of statics have been included in this development. All difference equations are based on the assumptions of linear elasticity and elementary beam and thin plate theories. The symbols adopted for use in this paper are defined where they first appear and are listed in the Nomenclature.
CHAPTER 2. STABILITY OF THE BEAM EQUATION

From a theoretical standpoint, the use of difference equations for the solution of a linear transient problem is complicated by stability requirements. In this discussion, a finite difference solution is stable if the solution is bounded as time approaches infinity. To facilitate a difference representation of the terms in the vibrating beam equation, it is convenient to establish a rectangular grid in an \( x,t \) plane. The coordinate axes for the grid are the beam and the time axes, and the lines in the grid intersect at mesh points. Any mesh point may be located by station numbers which are identified by the indices \( j \) and \( k \) with respect to the beam and time axes. The distances between the grid lines in the coordinate directions are fixed by the lengths of the beam increment \( h_x \) and the time increment \( h_t \). This grid is illustrated in Fig 1.

**Explicit Formula**

An examination of the explicit formula for a uniform beam will demonstrate the stability criterion which was first established by Collatz (Ref 1). The explicit difference approximation for Eq 1.1 is

\[
g^2 \left[ w_{j-2,k} - 4w_{j-1,k} + 6w_{j,k} - 4w_{j+1,k} + w_{j+2,k} \right] + w_{j,k-1} - 2w_{j,k} + w_{j,k+1} = 0 \tag{2.1}
\]

wherein

\[
g^2 = \frac{EI}{\rho} \frac{h_t^2}{h_x^2}
\]
Fig 1. Explicit operator for the transverse deflections of a uniform beam.
At $k = 0$, the initial deflections and velocities are specified. Therefore, the value of $w_{j,k+1}$ is the only unknown in the equation. In Fig 1, the operator associated with Eq 2.1 is superimposed on the rectangular grid. To solve for each unknown deflection at $k = 1$, the operator is applied successively at $j = 1, 2, \ldots, M-1$. The boundary conditions are introduced to establish the deflections at the ends of the beam. In a similar manner, the unknown deflections are calculated for $k = 2, 3, \ldots, \infty$.

For a beam with hinged ends and $M$ segments or increments, a solution to Eq 2.1 is assumed to be

$$w_{j,k} = A \sin \left( j\beta_n \right) e^{k\phi} \quad (2.2)$$

in which $A$ is a constant, $j = 0, 1, 2, \ldots, M$, and $k = 2, 3, 4, \ldots, \infty$.

Equation 2.2 is substituted into Eq 2.1 to establish

$$g^2 e^{k\phi} \left[ \sin \left( j-2 \right) \beta_n - 4 \sin \left( j-1 \right) \beta_n + 6 \sin \left( j\beta_n \right) \right. \left. - 4 \sin \left( j+1 \right) \beta_n + \sin \left( j+2 \right) \beta_n \right]$$

$$+ \sin j\beta_n \left[ e^{\left( k-1 \right)\phi} - 2 e^{k\phi} + e^{\left( k+1 \right)\phi} \right] = 0 \quad (2.3)$$

The following trigonometric identities are used to simplify Eq 2.3:

$$\sin \left( \theta \pm \gamma \right) = \sin \theta \cos \gamma \pm \cos \theta \sin \gamma$$

and

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

Hence, Eq 2.3 becomes

$$\frac{1}{e^{k\phi}} \left[ e^{\left( k-1 \right)\phi} - 2 e^{k\phi} + e^{\left( k+1 \right)\phi} \right] = -4g^2 \left( 1 - \cos \beta_n \right)^2 \quad (2.4)$$

This may be reduced to
\[ e^{2\phi} + e^{\phi} \left[ 4g^2 (1 - \cos \beta_n)^2 - 2 \right] + 1 = 0 \] (2.5)

On the boundaries, independent of \( k \), the deflections and moments are zero. Thus,

\[ w_{0,k} = w_{M,k} = \frac{\partial^2 w_{0,k}}{\partial x^2} = \frac{\partial^2 w_{M,k}}{\partial x^2} = 0 \] (2.6)

From Eq 2.2,

\[ \sin (M \beta_n) = 0 \]

Hence,

\[ M \beta_n = \pi, 2\pi, \ldots, n\pi \]

or

\[ \beta_n = \frac{n\pi}{M} \] (2.7)

where

\[ n = 1, 2, 3, \ldots, M-1 \]

Therefore, Eq 2.2 becomes

\[ w_{j,k} = \sum_{n=1}^{M-1} A_n \sin \left( j \frac{n\pi}{M} \right) e^{k\phi} \] (2.8)

The roots of the quadratic Eq 2.5 are substituted into Eq 2.8 so that

\[ w_{j,k} = \sum_{n=1}^{M-1} A_n \sin \left( j \frac{n\pi}{M} \right) \left[ C_1 (e^{\phi_1})^k + C_2 (e^{\phi_2})^k \right] \] (2.9)

where \( C_1 \) and \( C_2 \) are constants. In Eq 2.9, for \( w_{j,k} \) to be bounded for all values of \( k \), the roots of the quadratic, \( e^{\phi_1} \) and \( e^{\phi_2} \), must satisfy the condition that

\[ |e^{\phi_1}|, |e^{\phi_2}| \leq 1 \] (2.10)

This condition may be satisfied by defining \( g^2 \) in Eq 2.5. Thus, the limiting value of \( g^2 \) occurs when the discriminant
(16 \ g'^2 - 2)^2 - 4 \leq 0 \quad (2.11)

for \ \beta_n = \pi.

Expanding Eq 2.11 discloses that

\[ 4 \ g'^2 - 1 \leq 0 \]

and

\[ g'^2 \leq \frac{1}{4} \quad (2.12) \]

The preceding analysis is based on a uniform beam with hinged supports. For a stable solution, the maximum value of \( g'^2 \), or \( \frac{EI}{\rho} \frac{h'^2}{h_x} \), is prescribed by Eq 2.12. Because of this limitation, the explicit formula will not be used in the subsequent development of the dynamic beam equation.

**Implicit Formula**

In Fig 2, an implicit operator of the Crank-Nicolson (Ref 5) form is shown for Eq 1.1. All deflections at Station \( k+1 \) are unknown. The fourth derivative term that was previously at the \( k^{th} \) station has been divided equally between the stations at \( k-1 \) and \( k+1 \). For any Station \( j \), this implies that the deflection at Station \( k \) is an average of the sum of the deflections at Stations \( k-1 \) and \( k+1 \). At \( k=0 \), the initial deflections and velocities are specified. To solve for the unknown deflections at \( k=1 \), the operator is applied systemically at \( j = 1, 2, \ldots, M-1 \). This procedure establishes a set of simultaneous equations wherein each equation includes five unknown deflections. These equations may be solved by any convenient method. In a similar fashion, the unknown deflections are determined for \( k = 2, 3, 4, \ldots, \infty \).

The admissibility of the implicit formula can be established by a procedure suggested by Young (Ref 16). Let \( L(w) \) be the differential equation and \( G(w) \) be a Taylor series expansion of the terms in the implicit formula.
Fig 2. Implicit operator for the transverse deflections of a uniform beam.
about the point \( j,k \). When \( G(w) \) is subtracted from \( L(w) \), the remainder, or truncation error, is of the order \((h_x)^2\) and \((h_t)^2\). Furthermore, \( h_t \) is a given function of \( h_x \).

Thus the

\[
\lim_{h_x \to 0} \left[ L(w) - G(w) \right] = 0 \quad (2.13)
\]

and the admissibility of the implicit formula is established.

The implicit difference approximation to Eq 1.1 is

\[
\frac{\phi^2}{2} \left[ w_{j-2,k+1} - 4w_{j-1,k+1} + 6w_{j,k+1} - 4w_{j+1,k+1} + w_{j+2,k+1} + w_{j-2,k-1} - 4w_{j-1,k-1} + 6w_{j,k-1} - 4w_{j+1,k-1} + w_{j+2,k-1} \right] + w_{j,k-1} - 2w_{j,k} + w_{j,k+1} = 0 \quad (2.14)
\]

To establish the stability criterion, Eq 2.2 is substituted into Eq 2.14 to yield

\[
\frac{\phi^2}{2} \left\{ e^{(k+1)\phi} \left[ \sin (j-2) \beta_n - 4 \sin (j-1) \beta_n + 6 \sin (j+1) \beta_n - 4 \sin (j+2) \beta_n \right] + e^{(k-1)\phi} \left[ \sin (j-2) \beta_n - 4 \sin (j+1) \beta_n + 6 \sin (j+2) \beta_n \right] \right\} + \sin (j\beta_n) \left[ e^{(k-1)\phi} - 2e^{k\phi} + e^{(k+1)\phi} \right] = 0 \quad (2.15)
\]

The above equation reduces to

\[
\frac{1}{e^{(k+1)\phi} + e^{(k-1)\phi}} e^{(k-1)\phi} - 2e^{k\phi} + e^{(k+1)\phi} = 2g^2 (1 - \cos \beta_n)^2 \quad (2.16)
\]
and the quadratic equation becomes

\[ e^{2\phi} - e^\phi \left[ \frac{2}{1 + 2g^2 (1 - \cos \beta_n)^2} \right] + 1 = 0 \]  

(2.17)

The value of \( \beta_n \) is given in Eq 2.7. The roots of the quadratic satisfy Eq 2.10 for all \( g^2 > 0 \). Therefore, the implicit formula is stable for all positive values of \( EI, \rho, h_t, \) and \( h_x \).

The preceding discussion of stability has been based on free vibration of a uniform beam and well defined boundary conditions. Analytical proofs for more complicated cases are not feasible. For example, if the same beam has uniform rotational restraints \( r \), foundation springs \( s \), and an axial tension \( P \), the quadratic form becomes

\[ e^{2\phi} - e^\phi \left[ \rho + h_t^2 \left( \frac{2EI}{h_x^2} (1 - \cos \beta_n)^2 + \frac{s}{2} + \frac{r+P}{h_x^2} (1 - \cos \beta_n) \right) \right] + 1 = 0 \]  

(2.18)

An evaluation of stability from Eq 2.18 is not practicable. However, stable numerical solutions have been obtained for complex problems.

Crandall (Ref 4) has shown that the optimum implicit formula for a uniform beam has a truncation error of the order \( (h_t)^3 \). In a recent paper, Tucker (Ref 15) used an implicit formula which has a truncation error of the order \( (h_t) \). In this study, the general development of the beam and plate equations will be based on the Crank-Nicolson (Ref 5) implicit form which has a truncation error of the order \( (h_t)^2 \).
CHAPTER 3. DEVELOPMENT OF THE BEAM EQUATIONS

The finite-element beam solution consists of the static equation, the dynamic equation related to the initial velocities and the dynamic equation. The static equation is due to Matlock (Ref 9) and is discussed briefly herein. Central differences (Ref 3) are used in all derivations except where otherwise noted. The coordinate system which was described in the preceding chapter is applicable in the following development.

Static Equation

The beam segment in Fig 3 illustrates the static loads and elastic restraints which may be imposed on the beam to establish its initially deflected shape. A finite-element model of this segment has been developed by Matlock (Ref 9). Equation 3.1 is obtained by summing moments and forces on the beam segment in Fig 3.

\[
\frac{d^2 M_b}{dx^2} = q - sw + \frac{d}{dx} \left[ tc + (r + P) \frac{dw}{dx} \right] \quad (3.1)
\]

In the foregoing, \( M_b \) is the bending moment, \( q \) is the transverse load per unit length, \( s \) is the elastic stiffness of the foundation per unit length, \( t_c \) is an applied couple per unit length, \( r \) is a rotational restraint per unit length and \( P \) is an axial load. Combining Eq 3.1 with the differential equation for a beam

\[
\frac{d^2}{dx^2} \left[ EI \frac{d^2 w}{dx^2} \right] = \frac{d^2 M_b}{dx^2} \quad (3.2)
\]

establishes Eq 3.3

\[
\frac{d^2}{dx^2} \left[ EI \frac{d^2 w}{dx^2} \right] = q - sw + \frac{d}{dx} \left[ tc + (r + P) \frac{dw}{dx} \right] \quad (3.3)
\]
Fig 3. Beam segment with static loads and elastic restraints.

Fig 4. Beam segment with transient loads.
In a difference equation, the distributed quantities \( q, r, t, s \) are lumped as corresponding concentrated quantities \( Q, R, T, S \) at each incremental point along the beam. Equation 3.3 involves the derivative of a product of two variables. In transforming this differential equation to a difference equation, the left side of the equation is expanded from the outside to the inside in the following manner:

\[
\frac{d^2}{dx^2} \left( F \frac{d^2 w}{dx^2} \right) = \frac{1}{h^2_x} \left\{ \left( F \frac{d^2 w}{dx^2} \right)_{j-1} - 2 \left( F \frac{d^2 w}{dx^2} \right)_j + \left( F \frac{d^2 w}{dx^2} \right)_{j+1} \right\}
\]

\[
= \frac{1}{h^2_x} \left\{ F_{j-1} (w_{j-2} - 2w_{j-1} + w_j) - 2F_j (w_{j-1} - 2w_j + w_{j+1}) + F_{j+1} (w_j - 2w_{j+1} + w_{j+2}) \right\}
\]

(3.4)

In Eq 3.4, \( F \) represents the bending stiffness and \( h_x \) is the length of a beam increment. Similarly, Eq 3.3 is converted to the difference equation

\[
\left[ F_{j-1} - 0.25h_x (R_{j-1} + h_x P_{j-1}) \right] w_{j-2} - \left[ 2 \left( F_{j-1} + F_j \right) \right] w_{j-1}
\]

\[
+ \left[ F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j + 0.25h_x (R_{j-1} + h_x P_{j-1}) \right] w_j - \left[ 2 \left( F_j + F_{j+1} \right) \right] w_{j+1}
\]

\[
+ \left[ F_{j+1} - 0.25h_x (R_{j+1} + h_x P_{j+1}) \right] w_{j+2}
\]

\[
= h_x^3 Q_j - 0.5h_x^2 (T_{c,j-1} - T_{c,j+1})
\]

(3.5)

The application of this equation at each incremental point results in a set of simultaneous equations which is solved by a recursive procedure. This procedure and the boundary conditions will be discussed subsequently.
Dynamic Equation

The partial differential equation for the transverse vibrations of a beam can be derived from d'Alembert's principle. The concept of reversed effective forces, or inertial forces, in d'Alembert's principle is quite easily visualized. Imagine that the inertial and viscous drag forces and an externally applied force \( q(x,t) \) are superimposed on the beam segment which is shown in Fig 4. Thus the differential equation for a vibrating beam is

\[
\frac{\partial^2 w}{\partial x^2} \left[ F \frac{\partial^2 w}{\partial x^2} \right] = - \rho \frac{\partial^2 w}{\partial t^2} - d \frac{\partial w}{\partial t} + q(x,t) \quad (3.6)
\]

where \( d \) is the coefficient of viscous damping and the other symbols have the same meaning as before. The quantities \( r, s \) and \( P \), which affect the stiffness of a beam at any instant of time, are added to Eq 3.6 and this yields

\[
\frac{\partial^2 w}{\partial x^2} \left[ F \frac{\partial^2 w}{\partial x^2} \right] + sw - \frac{\partial}{\partial x} \left[ (r + P) \frac{\partial w}{\partial x} \right] + \rho \frac{\partial^2 w}{\partial t^2} + d \frac{\partial w}{\partial t} = q(x,t). \quad (3.7)
\]

The implicit representation of Eq 3.7* is

\[
Y_a w_{j+1,k+1} + Y_b w_{j-1,k+1} + \left[ Y_c + \frac{h_x^4}{h_t^4} \rho_j + \frac{h_x^4}{h_t^4} d_j \right] w_{j,k+1}
\]

\[
+ Y_d w_{j+1,k+1} + Y_e w_{j+2,k+1} = h_x^3 Q_{j,k} + \left[ 2 \frac{h_x^4}{h_t^4} \rho_j \right] w_{j,k}
\]

\[
- \left[ \frac{h_x^4}{h_t^4} \rho_j \right] w_{j,k-1} + \left[ \frac{h_x^4}{h_t^4} d_j \right] w_{j,k} - Y_a w_{j-2,k-1}
\]

\[
- Y_b w_{j-1,k-1} - Y_c w_{j-1,k-1} - Y_d w_{j+1,k-1} - Y_e w_{j+2,k-1}
\]

in which

\[
Y_a = \frac{1}{2} \left[ F_{j-1} - 0.25 h_x (R_{j-1} + h_x P_{j-1}) \right]
\]

* A derivation of the implicit formula for Eq 3.7 is given in Appendix 1.
In the foregoing, \( h_t \) is the length of time increment. The remaining symbols have been previously defined. In Eq 3.8, the unknown deflections at \( k+1 \) appear on the left side of the equation, and the known deflections at \( k \) and \( k-1 \) appear on the right side of the equation.

At the outset, the deflections and velocities at \( k=0 \) are given. With these initial conditions, the unknown deflections at \( k=1 \) are then calculated to begin the transient solution. This is accomplished by rewriting Eq 3.8 so that the generic indices \( k+1, k \) and \( k-1 \) become \( 1, \frac{1}{2} \) and \( 0 \) respectively. Furthermore, the initial deflections and velocities are introduced in the computational procedure in accordance with the following equations:

\[
\begin{align*}
\frac{\partial w}{\partial t} \bigg|_{j,0} &= \frac{-w_{j,0} + w_{j,\frac{1}{2}}}{h_t/2} \\
\rho \frac{\partial^2 w}{\partial t^2} \bigg|_{j,\frac{1}{2}} &= \rho_j \frac{w_{j,0} - 2w_{j,\frac{1}{2}} + w_{j,1}}{(h_t/2)^2}
\end{align*}
\]

(3.10)

(3.11)

The unknown deflections at \( k=\frac{1}{2} \) are eliminated by combining Eqs 3.10 and 3.11.
Consequently, the deflections at \( k=1 \) are calculated. Commencing at \( k=2 \) and thereafter, the solution progresses with time in accordance with Eq 3.8. This is demonstrated in Fig 5.

The effects of rotatory inertia and shear deformation have been omitted in the derivation of the dynamic equation. A discussion of these effects is given in Ref 12.

**Method of Solution for the Difference Equations**

There are several systematic procedures available to solve simultaneous equations. For an efficient machine procedure, it is convenient to use a method of elimination described by Matlock (Ref 9).

The difference equation, whether static or dynamic, may be written in the form

\[
\overline{a}_j w_{j-2,k} + \overline{b}_j w_{j-1,k} + \overline{c}_j w_{j,k} + \overline{d}_j w_{j+1,k} + \overline{e}_j w_{j+2,k} = \overline{f}_{j,k} \tag{3.12}
\]

\( k = 0, 1, 2, 3, \ldots, \infty \).

The terms \( \overline{a}_j \), \( \overline{b}_j \), \( \overline{c}_j \), \( \overline{d}_j \), \( \overline{e}_j \) and \( \overline{f}_{j,k} \) may be recognized by comparing the foregoing equation with either the static Eq 3.5 or the dynamic Eq 3.8. For instance, in Eq 3.8,

\[
\overline{a}_j = Y_a \\
\overline{b}_j = Y_b \\
\overline{c}_j = Y_c + \frac{h}{h_t} \rho_j \left( \frac{x}{h_t} \right) \frac{4}{4} d_j \\
\overline{d}_j = Y_d \\
\overline{e}_j = Y_e
\]

and

\[
\overline{f}_{j,k} = h^3 Q_{j,k-1} + \left[ 2 \frac{h}{h_t} \rho_j \right] w_{j,k-1} - \left[ \frac{h}{h_t} \rho_j \right] w_{j,k-2}
\]

(equation continued)
BEAM: 0, 1, 2, ..., M
PRESCRIBED BOUNDARIES AT 0, M (Illustrated above as a hinge)
TIME: 0, 1, 2, ..., ∞
DEFLS ARE KNOWN AT k-2, k-1, k
DEFLS ARE UNKNOWN AT k+1

Fig 5. Propagation of solution for unknown beam deflections.
The solution to Eq 3.12 is assumed to be

\[ w_{j,k} = A_j + B_j w_{j+1,k} + C_j w_{j+2,k} \]  \hspace{1cm} (3.13)

in which

\[ A_j = D_j (E_j A_{j-1} + \bar{a}_j A_{j-2} - \bar{r}_j, k) \]  \hspace{1cm} (3.14)

\[ B_j = D_j (E_j C_{j-1} + \bar{a}_j) \]  \hspace{1cm} (3.15)

\[ C_j = D_j (\bar{e}_j) \]  \hspace{1cm} (3.16)

\[ D_j = 1 / (E_j B_{j-1} + \bar{a}_j C_{j-2} + \bar{c}_j) \]  \hspace{1cm} (3.17)

\[ E_j = \bar{a}_j B_{j-2} + \bar{b}_j \]  \hspace{1cm} (3.18)

Proceeding from either end of the beam in what is called a forward direction, Equations 3.14 through 3.18 are applied at every station, including one fictitious station beyond each end of the beam. On the reverse pass, the unknown deflections are calculated from Eq 3.13.

**Boundaries and Specified Conditions**

Although the equations have not been established in a matrix array, it is convenient to consider the coefficients \( \bar{a}_j, \ldots, \bar{e}_j \) as terms in a quintuple-diagonal coefficient matrix and the unknown deflections and known loads as column matrices. The first and last equations represent the moment at the free edge of a beam, and the second and next-to-the-last equations represent the shear one-half increment inside the free edge. For a uniform beam with an unloaded free boundary, the first and second equations are
\[ w_{-1,k} - 2w_{0,k} + w_{1,k} = 0 \]  \hspace{1cm} (3.19)

and

\[ -w_{-1,k} + 3w_{0,k} - 3w_{1,k} + w_{2,k} = 0 \]  \hspace{1cm} (3.20)

Thus an approximation of the natural boundary conditions for zero moment and shear are automatically created by zero stiffness values beyond the ends of the beam.

Specified deflections are established by equating \( A_j \) to the desired deflection and setting \( B_j \) and \( C_j \) equal to zero in Eq 3.13. To specify a slope at the jth station, the coefficients \( A_j \), \( B_j \) and \( C_j \) at Stations \( j-1 \) and \( j+1 \) are recalculated on the basis of the reaction couple that must be developed about the jth station (Ref 9).
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CHAPTER 4. NUMERICAL RESULTS - BEAM

The static and dynamic equations that were developed in the preceding chapter have been programmed in FORTRAN for the Control Data Corporation 1604 computer. A listing of this program, DCBl, a guide for data input, and a summary flow diagram are in Appendix 4.

Verification of the Method

Table 1 illustrates the problems which have been selected to verify the method. The theoretical angular frequency of vibration for each problem is given in Timoshenko (Ref 12). The period of vibration corresponding to the lowest angular frequency was divided into an arbitrary number of time increments. For all problems, the number of beam increments is 10, the increment length is 12 in., the stiffness is $1.08 \times 10^9$ lb-in$^2$, and the mass density is $9.04 \times 10^{-3}$ lb-sec$^2$/in$^2$. Each beam has hinged support.

In Problems 1, 2 and 3, the time increments are $2.653 \times 10^{-4}$ sec, $5.306 \times 10^{-4}$ sec and $2.565 \times 10^{-3}$ sec. The initially deflected shape of each beam is established as one-half cycle of a sine wave. This is the fundamental mode of vibration of the beam. At $k=0$, the beam is released and the deflections are noted during the ensuing vibrations. The deflected shape of the beam at the conclusion of the first period is similar to its initial shape. This is illustrated in Table 1 by the recorded values of the initial deflections and the subsequent deflections at the end of the first period. These three problems demonstrate that a small time increment is desirable.

Problems 4 and 5 are similar to Problem 1 with the following alterations. In Problem 4, the axial load is $-3.70 \times 10^5$ lb and the time increment is
TABLE 1. A SUMMARY OF THE NUMERICAL RESULTS

<table>
<thead>
<tr>
<th>BEAM AT INITIAL CONDITIONS</th>
<th>NUMBER OF TIME INCREMENTS PER FUNDAMENTAL PERIOD OF VIBRATION BASED ON A THEORETICAL SOLUTION</th>
<th>INITIAL DEFLECTION (Inches)</th>
<th>SUBSEQUENT DEFLECTION (Inches)</th>
<th>TIME STATION</th>
<th>w</th>
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</thead>
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<td>99</td>
<td>-1.987</td>
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<td>101</td>
<td>7.937 x 10^{-3}</td>
</tr>
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</table>

* DEFLECTION IS ZERO IN THEORETICAL SOLUTION GIVEN BY TIMOSHENKO (REF.12)
3.752 \times 10^{-4} \text{ sec. In Problem 5, the uniform foundation spring is } 12.0 \times 10^3 \text{ lb/in/sta and the time increment is } 1.540 \times 10^{-6} \text{ sec.}

The beam in Problem 6 has zero initial deflections and a uniform initial velocity of 30 in/sec everywhere except at the supports. The time increment is $2.653 \times 10^{-4}$ sec. Theoretically, the deflections at the end of the first period are zero.

In Problem 7, a concentrated load of $1.0 \times 10^5 \text{ lb}$ is applied suddenly at the middle of the span and is removed at the end of the first period. The time increment is $2.653 \times 10^{-4}$ sec. At the conclusion of the first period and thereafter, the deflections are zero.

Excluding Problem 3, the maximum error in the numerical results based on the theoretical solutions is about 4%. Furthermore, these results confirm that the finite-element method described herein can be used to solve vibrating beam problems.

Example Problems

Two example problems have been selected to illustrate the versatility of the finite-element method. The partially embedded beam, which is described in Fig 6, is subjected to an axial load and a transient pulse. In addition to the hinged supports, there is a rotational restraint at the upper boundary. The soil modulus has been converted at each station to an equivalent elastic spring. A damping factor of $10.0 \text{ lb-sec/in}^2$ has been assumed arbitrarily.

Figure 6b shows the deflected shape of the beam at the conclusion of the pulse, or $k = 18$, and at a subsequent time. Figure 6c illustrates the response of a typical station on the beam.

The second example, which is sketched in Fig 7, is a three-span beam with a constant load moving along the beam at a uniform velocity of one beam increment per time increment. Figure 7b illustrates the response of the beam
Fig 6. Partially embedded beam subjected to a load pulse.
Fig 7. Moving load on a three-span beam.
at Station 20. Figure 7c is a plot of the beam deflections at the two indicated times.

For the Control Data Corporation 1604 computer, the execution time required for each solution is approximately 45 seconds.
CHAPTER 5. STABILITY OF THE PLATE EQUATION

A difference solution for the vibrating plate equation must meet the requirements of stability. The restrictions that have been established for the beam equation are not applicable to a plate, but the same procedures are involved. Therefore, the following development will parallel the previous work.

The equation for the transverse deflections of a vibrating plate is

\[ D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) + \rho \frac{\partial^2 w}{\partial t^2} = 0 \]  

(5.1)

where \( w \) is the deflection, \( D \) is the uniform flexural stiffness, \( x \) and \( y \) are the rectangular coordinate axes, \( t \) is time and \( \rho \) is the mass per unit area of the plate. The independent variables in Eq 5.1 are \( x \), \( y \) and \( t \). Therefore, a difference representation of the terms in the above equation requires a three-dimensional coordinate system in which \( x \), \( y \) and \( t \) are the three coordinate axes. A rectangular grid, whose lines are parallel to the \( x \) and \( y \) axes, is established at each interval of time. The intersections of these grid lines are known as mesh points. Any mesh point may be located by station numbers which are defined by the indices \( i \), \( j \) and \( k \) with respect to the coordinate axes. In the \( x \) or \( y \)-direction, the distance between adjacent grid lines is fixed by the length of the plate increment \( h_p \).

Explicit Formula

Explicitly, the finite difference formula for Eq 5.1 is

\[ u^2 \left\{ w_{i-2,j,k} + w_{i+2,j,k} + w_{i,j-2,k} + w_{i,j+2,k} + 20w_{i,j,k} ight\} 
- 8 \left[ w_{i-1,j,k} + w_{i+1,j,k} + w_{i,j-1,k} + w_{i,j+1,k} \right] 
\]

(equation continued)
Two initial conditions and eight boundary conditions are prescribed. At $k = 1$ and thereafter, the only unknown is $w_{i,j,k+l}$.

The operator corresponding to Eq 5.2 is shown in Fig 8. To solve explicitly for each unknown deflection at any time station, the operator is used successively at every mesh point in the $x,y$ plane. The boundary conditions are introduced to establish the deflections along the edges of the plate. In this manner, the solution marches forward with time.

For a rectangular plate with $M$ by $N$ increments and hinged supports along the edges, a solution is assumed to be of the form

$$w_{i,j,k} = Ae^{k\phi} \sin (i\alpha_m) \sin (j\beta_n)$$

where

$$i = 0, 1, 2, \ldots, M$$

$$j = 0, 1, 2, \ldots, N$$

and

$$k = 2, 3, \ldots, \infty.$$  

A substitution of Eq 5.3 into Eq 5.2 establishes that

$$u^2 e^{k\phi} \left[ \sin (j\beta_n) \left\{ \sin (i-2) \alpha_m - 8 \sin (i-1) \alpha_m + 20 \sin (i\alpha_m) 
- 8 \sin (i+1) \alpha_m + \sin (i+2) \alpha_m \right\} \right]$$

(equation continued)
where:

\[
\begin{align*}
W_{i,j,k+1} &= -u^2 \left[ W_{i-2,j,k} + W_{i+2,j,k} + W_{i,j-2,k} + W_{i,j+2,k} \right] \\
&\quad -2u^2 \left[ W_{i-1,j-1,k} + W_{i+1,j-1,k} + W_{i-1,j+1,k} + W_{i+1,j+1,k} \right] \\
&\quad +8u^2 \left[ W_{i-1,j,k} + W_{i,j-1,k} + W_{i+1,j,k} + W_{i,j+1,k} \right] \\
&\quad +(-20u^2 + 2) W_{i,j,k} - W_{i,j,k-1}
\end{align*}
\]

where \( u^2 = \frac{D}{\rho} \frac{h_f^2}{h_p^4} \)

Fig 8. Explicit operator for the transverse deflections of a uniform plate.
\[ + \sin (i\alpha_m) \left[ \sin (j-2) \beta_n - 8 \sin (j-1) \beta_n - 8 \sin (j+1) \beta_n \right] \\
+ \sin (j+2) \beta_n \right] + 2 \left[ \sin (i-1) \alpha_m \sin (j-1) \beta_n \\
+ \sin (i+1) \alpha_m \sin (j+1) \beta_n \right] + \sin (i\alpha_m) \sin (j\beta_n) \left[ e^{(k-1)\phi} \\
- 2e^{k\phi} + e^{(k+1)\phi} \right] = 0 \] 

(5.4)

A simplification of Eq 5.4 yields

\[ e^{-\phi} - 2 + e^{\phi} = -4u^2 \left\{ \left[ \cos \alpha_m - 2 \right]^2 + \left[ \cos \beta_n - 2 \right]^2 - 4 \\
+ 2 \cos \alpha_m \cos \beta_n \right\} \] 

(5.5)

Equation 5.5 reduces to

\[ e^{2\phi} + e^{\phi} \left\{ 4u^2 \left[ \cos \alpha_m - 2 \right]^2 + \cos \beta_n - 2 \right\} - 2 \right\} + 1 = 0 \] 

(5.6)

On the boundaries, independent of \( k \), the deflections must satisfy the following equations:

\[ W_{i,0,k} = W_{i,N,k} = 0 \] 

(5.7)

\[ W_{0,j,k} = W_{M,j,k} = 0 \] 

(5.8)

\[ W_{i,-1,k} = W_{i,1,k} \] 

(5.9)

\[ W_{i,N-1,k} = W_{i,N+1,k} \] 

(5.10)

\[ W_{-1,j,k} = W_{1,j,k} \] 

(5.11)

and

\[ W_{M-1,j,k} = W_{M+1,j,k} \] 

(5.12)

The boundary conditions are satisfied for
\[ \alpha_m = \frac{m}{M} \pi \quad m = 1, 2, \ldots, M-1 \]  
\[ \beta_n = \frac{n}{N} \pi \quad n = 1, 2, \ldots, N-1 \]  

Thus, Eq 5.3 becomes

\[ w_{i,j,k} = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} A_m A_n \sin \left( \frac{i m \pi}{M} \right) \sin \left( \frac{j n \pi}{N} \right) \left[ C_1 (e^{\phi_1})^k \right. \right. \\
\left. + C_2 (e^{\phi_2})^k \right] \]  

in which \( C_1 \) and \( C_2 \) are constants.

For stability,

\[ |e^{\phi_1}|, |e^{\phi_2}| \leq 1 \]  

An examination of Eq 5.6 shows that Eq 5.16 is satisfied if the discriminant

\[ \left\{ 4u^2 \left[ (\cos \alpha_m - 2)^2 + (\cos \beta_n - 2)^2 - 4 + 2 \cos \alpha_m \cos \beta_n \right] - 2 \right\}^2 \\
- 4 \leq 0 \]  

For \( \alpha_m = \beta_n = \pi \), Eq 5.17 reveals that

\[ u^2 \leq \frac{1}{16} \]  

For a stable explicit solution, the maximum value of \( u^2 = \frac{Dh^2}{\rho h_p^4} \) is predicted by Eq 5.18. For this reason, the explicit formula will not be used in the development of the dynamic plate equation.

To verify this stability criterion and to gain some insight of the behavior of an unstable solution, a numerical experiment was performed with an explicit plate program. The experiment consisted of five problems in which a square plate with hinged supports about the edges was divided into a 4 x 4 grid. For each problem, \( D \), \( \rho \) and \( h_p \) were constants and the time
increment \( h_t \) was calculated on the basis of a prescribed value for the ratio \( \frac{Dh}{\rho} \). The values for this ratio were 0.04, 0.05, 0.06, 0.08 and 0.1. On the basis of Eq 5.18, instability could be predicted for a ratio of 0.0625. At \( k=0 \), the initial deflections were specified. An examination of the computed deflections revealed a divergent oscillatory solution for the largest ratio. The deflections became increasingly larger at each successive time interval. At ratios of 0.05, 0.06 and 0.08, irregularities were noted in the computed deflections.

**Implicit Formula**

Figure 9 illustrates the implicit formula and operator for Eq 5.1. The fourth derivative terms that were previously at the \( k \)th station have been divided equally between the stations at \( k-1 \) and \( k+1 \). This assumes that the deflections at the \( k \)th station are an average of the sum of the corresponding deflections at Stations \( k-1 \) and \( k+1 \). All deflections at \( k+1 \) are unknown, whereas those at \( k \) and \( k-1 \) are known from previous solutions. Thus, for an implicit solution, a set of simultaneous equations must be solved.

The stability criterion for the implicit plate formula may be established by the same procedure that was employed for the explicit formula. Accordingly, Eq 5.3 is substituted into the equation that is shown in Fig 9. A separation of variables yields

\[
e^a \Phi - e^{\Phi} \left\{ \frac{2}{1+2u^2} \left[ (\cos \alpha_m - 2)^2 + (\cos \beta_n - 2)^2 + 2 \cos \alpha_m \cos \beta_n - 4 \right] \right\} + 1 = 0
\]

(5.19)

The roots of the preceding quadratic equation satisfy Eq 5.16 for all

\[
u^2 > 0
\]

(5.20)
where

\[ u^2/2 \left[ w_{i-2, j, k+1} + w_{i, j-2, k+1} + w_{i+2, j, k+1} + w_{i, j+2, k+1} \right] + u^2 \left[ w_{i-1, j-1, k+1} + w_{i-1, j+1, k+1} \right. \\
+ w_{i+1, j-1, k+1} + w_{i+1, j+1, k+1} \right] - 4u^2 \left[ w_{i-1, j, k+1} + w_{i, j-1, k+1} + w_{i, j+1, k+1} \right. \\
+ w_{i+1, j, k+1} \left. \right] + \left( 10u^2 + 1 \right) w_{i, j, k+1} = 2w_{i, j, k} \\
\]

\[ -u^2/2 \left[ w_{i-2, j, k-1} + w_{i, j-2, k-1} + w_{i+2, j, k-1} + w_{i, j+2, k-1} \right] - u^2 \left[ w_{i-1, j-1, k-1} + w_{i-1, j+1, k-1} + w_{i+1, j-1, k-1} + w_{i+1, j+1, k-1} \right. \\
+ w_{i+1, j+1, k-1} \left. \right] + 4u^2 \left[ w_{i-1, j, k-1} + w_{i, j-1, k-1} + w_{i+1, j, k-1} + w_{i, j+1, k-1} \right. \\
+ \left( -10u^2 - 1 \right) w_{i, j, k-1} \]

where

\[ \frac{u}{\rho} = \frac{D}{h_p^4} \]

Fig 9. Implicit operator for the transverse deflections of a uniform plate.
Hence, the implicit formula is stable for any choice of positive values for $D$, $\rho$, $h_p$ and $h_t$. In a subsequent chapter, the implicit formula will be employed to solve for the deflections of a nonuniform plate. Analytical proofs for other boundary conditions are not readily attainable. Nonetheless, stability is indicated by the fact that numerical solutions have been obtained for problems with other well defined boundaries.
CHAPTER 6. DEVELOPMENT OF THE PLATE EQUATIONS

The finite-element plate solution includes the static equation, the dynamic equation related to the initial velocities and the dynamic equation. Shear deformations, linear damping and the effects of rotatory inertia have been omitted.

Static Equation

Consideration of static equilibrium and the moment-curvature relationship (Ref 13) yields

$$\frac{\partial^2 w}{\partial x^2} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x \partial y} \right) \right] + \frac{\partial^2 w}{\partial y^2} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x \partial y} \right) \right]$$

$$+ 2 \frac{\partial^2 w}{\partial x \partial y} \left[ D \left( 1-\nu \right) \frac{\partial^2 w}{\partial x \partial y} \right] = q - sw \quad (6.1)$$

where

$$D = \frac{Eh^3}{12 \left( 1-\nu^2 \right)}$$

In the foregoing, h is the plate thickness, \( \nu \) is Poisson's ratio, s is the foundation modulus and q is the transverse static load. The coordinate system which was described in the preceding chapter is applicable in the following development.

In the finite-element solution, it is assumed that the increment length \( h_x \) in the x-direction does not necessarily equal the increment length \( h_y \) in the y-direction. Furthermore, the stiffness D and the lumped quantities S and Q may vary from one mesh point to another. The variation in D accounts for a changing plate thickness, but the plate properties are isotropic. The partial derivatives in Eq 6.1 are expanded in the same manner that was used for the beam equation. This establishes the difference equation.
In the above equation, the coefficients $X_1, ..., X_{11}, Y_1, ..., Y_{11}$ and $Z_1, ..., Z_9$ are defined in Appendix 2. A finite-element model of the plate has been developed by Hudson (Ref 6).

**Dynamic Equation**

The partial differential equation of motion for forced lateral vibration of a plate is

$$\frac{\partial^2}{\partial x^2} \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] + \frac{\partial^2}{\partial y^2} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right] + 2 \frac{\partial^2}{\partial x \partial y} \left[ D \left( 1-\nu \right) \frac{\partial^2 w}{\partial x \partial y} \right] + \rho \frac{\partial^2 w}{\partial t^2} = q(x,y,t) \quad (6.3)$$

where $q(x,y,t)$ is the imposed lateral force. The implied difference equation for Eq 6.3* is

$$\frac{1}{2} \left( X_1 \right) \left[ \frac{w_{k+1}}{w_{k-1}} \right]_{i-2,j} + \frac{1}{2} \left( X_2 + Y_7 + 2Z_2 \right) \left[ \frac{w_{k+1}}{w_{k-1}} \right]_{i-1,j} + \frac{1}{2} \left( X_3 + Y_3 + 2Z_5 \right) \frac{S_{i-1,j}}{h_x h_y} + \frac{\rho i_{i-1,j}}{h_t} \left[ \frac{w_{k+1}}{w_{k-1}} \right]_{i,j}$$

(equation continued)

* A derivation of the implicit formula for Eq 6.3 is given in Appendix 3.
The compact notation in brackets in Eq 6.4 implies a multiplication of the coefficients by the deflections at \( k+1 \) and \( k-1 \). In Eq 6.4, the solution for the unknown deflections at \( k+1 \) is dependent on the known deflections at \( k \) and \( k-1 \).

To begin the transient solution at \( k = 0 \), Eq 6.4 is modified so that the generic indices \( k+1 \), \( k \) and \( k-1 \) become \( i \), \( i \) and \( 0 \), respectively. In addition, the initial velocities and deflections are introduced in the computational procedure in accordance with the following equations:

\[
\frac{\partial w}{\partial t} |_{i,j,0} = \frac{-w_{i,j,0} + w_{i,j,\frac{1}{2}}}{h_0 / 2} \tag{6.5}
\]

and
The unknown deflections at \( k = \frac{1}{2} \) are eliminated by combining Eqs 6.5 and 6.6. Thus the deflections at \( k = 1 \) are calculated. Beginning at \( k = 2 \), the plate deflections are determined from Eq 6.4 for each time interval as the solution marches forward.

Method of Solution for the Difference Equations

To obtain a solution for the unknown deflections, either static or dynamic, the appropriate equation is applied at each mesh point within the interior of the plate, along all boundaries, and at one mesh point outside of these boundaries. For a square plate which has been divided into \( M \) intervals in both directions, this procedure will introduce \((M+3)^2 - 4\) unknowns in \((M+3)^2 - 4\) equations. In matrix form this becomes

\[
\begin{bmatrix}
\mathbf{B} \\
\mathbf{w}
\end{bmatrix} = \begin{bmatrix}
\mathbf{C}
\end{bmatrix}
\quad (6.7)
\]

\([\mathbf{B}]\) is a square matrix with a predominant number of zero terms, but the non-zero terms are not banded about the main diagonal. These equations may be solved conveniently by an iterative procedure which is known as an alternating-direction-implicit, or ADI, method. In a comparison with other iterative methods, Young (Ref 17) has shown for second order difference equations that the ADI method has the most rapid rate of convergence. Conte and Dames (Ref 2) were among the first to utilize the ADI method to solve for the static deflections of a plate. Tucker (Ref 14) used this method successfully to solve the static grid-beam problem.

The ADI method is comparable to line relaxation in the \( x \) and \( y \)-directions. Basically, for an ADI solution, Eq 6.4 is solved for the deflections
in an x system and the deflections in a y system at alternate iterations. Equation 6.8 shows the iterative procedure employed to solve Eq 6.4 for the x system at iteration \( n + \frac{1}{2} \).

\[
\frac{1}{2} (X_1) \left[ \overline{w_x}_{i-2,j,k+1} \right]^{n+\frac{1}{2}} + \frac{1}{2} (X_2 + Z_2) \left[ \overline{w_x}_{i-1,j,k+1} \right]^{n+\frac{1}{2}} + \left\{ \frac{1}{2} (X_3 + Z_5 + S_{i,j}) + \frac{\rho_{i,j}}{h^2} \lambda \left[ \overline{w_y}_{i,j,k+1} \right]^{n+\frac{1}{2}} \right. \\
+ \frac{1}{2} (X_4 + Z_8) \left[ \overline{w_x}_{i+1,j,k+1} \right]^{n+\frac{1}{2}} + \frac{1}{2} (X_5) \left[ \overline{w_x}_{i+2,j,k+1} \right]^{n+\frac{1}{2}} \\
= \frac{Q_{i,j,k}}{h_x h_y} + \lambda_m \left[ \overline{w_y}_{i,j,k+1} \right]^n - \sum \left[ X, Y, Z, \frac{\rho}{h^2} \right] \left[ w \right] (6.8)
\]

In the foregoing, \( \left[ \overline{w_x} \right]^{n+\frac{1}{2}} \) are the unknown deflections for the x system at iteration \( n + \frac{1}{2} \), and \( \left[ \overline{w_x} \right]^n \) and \( \left[ \overline{w_y} \right]^n \) are the known deflections from the \( n \)th iteration for the x and y systems, respectively. The summation term on the right hand side of Eq 6.8 implies a multiplication of the remaining X, Y, Z and \( \frac{\rho}{h^2} \) terms in Eq 6.4 with their respective deflections at iteration \( n + \frac{1}{2} \) or \( n \), or at a previous time interval. The closure parameter \( \lambda_m \) will be discussed subsequently. Equation 6.8 involves \( M+3 \) unknowns in \( M+3 \) equations along a single line of mesh points in the x-direction. An equation similar to Eq 6.8 can be written for the y system. One iteration consists of solving \( 2M+2 \) lines in the x and y-directions. The total number of equations solved in each iteration is \( (2M+2)(M+3) \).

Each equation has five non-zero terms banded about the main diagonal in the coefficient matrix. This quintuple-diagonal system of equations is solved by the same method which was described previously for the beam equations. The solution is reached when \( \left| \overline{w_x} - \overline{w_y} \right| \) is less than a specified closure tolerance.
Boundaries and Specified Conditions

For an unloaded free edge at \( x = a \), the following difference approximations for moment and shear are automatically satisfied in the plate solution by zero stiffness beyond the edge of the plate.

\[
\begin{align*}
\mathcal{w}_{a-1,j} - 2(1 + \nu) \mathcal{w}_{a,j} + \mathcal{w}_{a+1,j} + \nu (\mathcal{w}_{a,j-1} + \mathcal{w}_{a,j+1}) &= 0 \\
\mathcal{w}_{a-1,j} - (2 - \nu) \mathcal{w}_{a-1,j-1} + (2 - \nu) \mathcal{w}_{a,j-1} - \mathcal{w}_{a-2,j} + (3 + 2(2 - \nu)) \mathcal{w}_{a-1,j} &- (3 + 2(2 - \nu)) \mathcal{w}_{a,j} + \mathcal{w}_{a+1,j} - (2 - \nu) \mathcal{w}_{a-1,j+1} + (2 - \nu) \mathcal{w}_{a,j+1} &= 0
\end{align*}
\]

(6.9) and (6.10)

Equations 6.9 and 6.10 are equivalent to the Kirchhoff boundary conditions (Ref 13) which are

\[
\begin{align*}
\left( \frac{\partial^2 \mathcal{w}}{\partial x^2} + \nu \frac{\partial^2 \mathcal{w}}{\partial y^2} \right) &= 0 \\
\left( \frac{\partial^2 \mathcal{w}}{\partial x^2} + (2 - \nu) \frac{\partial^2 \mathcal{w}}{\partial x \partial y} \right) &= 0
\end{align*}
\]

(6.11) and (6.12)

In the numerical solution, a zero deflection is conveniently established by inserting very stiff elastic foundation springs at the desired mesh points. No provision has been made to prescribe the slope at any boundary. However, this could be accomplished by the same procedure that was used for a beam.

Closure Parameters

The scalars \( \lambda_1, \lambda_2, \ldots, \lambda_m \) in Eq 6.8 are closure parameters that accelerate the convergence of the iterative procedure. In fact, these parameters are the key to an efficient solution. For a symmetric problem (Ref 6), these parameters have been related to the eigenvalues of the difference equations along any line in either the \( x \) or \( y \) system.
The parameters for the static equation as it is formulated in this development may be determined from

\[
\frac{D}{h_x} \sum [w_{i-2,j} - 6w_{i-1,j} + 10w_{i,j} - 6w_{i+1,j} + w_{i+2,j}] = \lambda_m w_{i,j}
\]  

(6.13)

In the above equation, the plate stiffness \( D \) is a constant and the increment lengths \( h_x \) and \( h_y \) are equal. For hinged boundaries and \( M \) intervals, a solution is assumed to be

\[
w_{i,j} = \sin (i\alpha_m)
\]

(6.14)

where

\[
\alpha_m = \frac{\pi m}{M}
\]

This yields

\[
\lambda_m = \frac{D}{h_x^4} \left[ 4 \left( 1 - \cos \frac{\pi m}{M} \right) \left( 2 - \cos \frac{\pi m}{M} \right) \right]
\]

(6.15)

\[
m = 1, 2, \ldots, M-1
\]

There are \( M-1 \) parameters which are used in cyclic order in the static and dynamic equations. If the problem has mixed boundary conditions and non-uniform stiffness, the closure parameters may be estimated from Eq 6.15.

The closure parameters for each system are inversely proportional to \( h_x^4 \) and \( h_y^4 \). For an efficient solution, the iterative procedure must account for this variation in closure parameters. Ingram (Ref 7) has demonstrated that optimum closure is obtained if the calculated parameters for the \( x \) system are used in the solution of the \( y \) system and vice versa. This scheme has been included in the plate solution.
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The development of the plate equations in Chapter 6 has been assembled in a FORTRAN program for the Control Data Corporation 1604 computer. A listing of this program, DPII, a guide for data input, and a summary flow diagram are in Appendix 5. Four problems are used to interpret the computed results of the plate program. Problems 1, 2, and 3 are intended to illustrate the effect of variations in number of plate increments and length of time increment on the accuracy of the solution and on the amount of computation time required to propagate the solution through a given number of time increments. If the plate in initially deflected in the shape of its fundamental mode of vibration and then released, theoretically this deflected shape will be repeated at the end of each integer multiple of the fundamental period of vibration. The program was modified to permit specification of initial deflections, but since this is of little practical use it was not made a permanent part of the final version.

Problem 1: 4 X 4 Grid

A plate with hinged supports along the edges is divided into a 4 X 4 grid. The increment lengths \( h_x \) and \( h_y \) are 12 in., the uniform stiffness is \( 2.5 \times 10^6 \) lb-in., Poisson's ratio is 0.25, the mass density is \( 7.5 \times 10^{-4} \) lb-sec\(^2/\)in\(^3\), the increment of time \( h_t \) is \( 4.233 \times 10^{-4} \) sec and the closure parameters are \( 1.83 \times 10^2 \), \( 9.62 \times 10^2 \) and \( 2.24 \times 10^3 \) lb/in\(^3\). The theoretical period of vibration for the lowest angular frequency (Ref 12) is \( 30 h_t \). At \( k=0 \), the initial deflections of the plate are

\[
w_{i,j,0} = \sin \left( \frac{i\pi h_x}{L} \right) \sin \left( \frac{j\pi h_y}{L} \right)
\]

(7.1)

in which \( L \) is 48.0 in. This shape corresponds to a normal mode of vibration. The plate is then released. At the conclusion of the first period, or \( 30 h_t \),
the shape of the plate is similar to its initial shape. In Table 2, this similarity is shown for selected mesh points. The maximum variation between the initial deflections and the deflections at the conclusion of the first period is about 9 percent. For a closure tolerance of $1.0 \times 10^{-6}$ in., four iterations are required to solve for the unknown deflections for each time increment. The computer execution time is 1.2 minutes for 30 increments of time.

Problem 2: 8 x 8 Grid

For this problem, the plate is divided into an 8 x 8 grid. Thus, the increment lengths $h_x$ and $h_y$ are 6 in. and the summation in Eq 7.1 is changed accordingly. The remaining dimensions are the same as those in the preceding problem. The closure parameters are $2.93 \times 10^{3}$, $4.0 \times 10^{3}$, $5.0 \times 10^{3}$, $7.0 \times 10^{3}$, $1.0 \times 10^{4}$, $1.54 \times 10^{4}$, and $3.58 \times 10^{5}$ lb/in$^3$. Seven iterations are required for each time increment. The similarity between the initial deflections and the deflections at the end of the first period is illustrated in Table 2. The variation in the deflections is about 2 percent. The computer execution time is 6.3 minutes for 30 time increments.

Problem 3: 4 x 4 Grid and Reduced Time Increment

This problem is identical to Problem 1 with the exception that the time increment $h_t$ is $2.117 \times 10^{-4}$ sec, which is one-half of the value used in Problem 1. The deflections are shown in Table 2. Three iterations are required for each increment of time and the computer execution time is 1.7 minutes for 60 increments of time. The variation in the deflections for this problem is about 7 percent.

Problem 4: Moving Load on a Rectangular Plate

Three different solutions have been obtained for the uniform plate which
TABLE 2. A COMPARISON OF THE NUMERICAL RESULTS

\[
\begin{array}{|c|c|c|}
\hline
\text{MESH POINT} & \text{PROBLEM 1} & \text{PROBLEM 2} \\
\hline
\text{DEFL} & \text{TIME} & \text{DEFL} & \text{TIME} \\
\hline
1 & 0.5000 \text{ in.} & 0.5000 \text{ in.} & 0.4910 \text{ in.} \\
& 0.4580 \text{ in.} & 0.4910 \text{ in.} & \\
\hline
2 & 0.7071 \text{ in.} & 0.7071 \text{ in.} & 0.6944 \text{ in.} \\
& 0.6478 \text{ in.} & 0.6944 \text{ in.} & \\
\hline
3 & 1.0000 \text{ in.} & 1.0000 \text{ in.} & 0.9821 \text{ in.} \\
& 0.9161 \text{ in.} & 0.9821 \text{ in.} & \\
\hline
\text{PROBLEM 3} & & & \\
\text{DEFL} & \text{TIME} & & \text{NOT INCLUDED IN OUTPUT} \\
\hline
1 & 0.5000 \text{ in.} & 0.5000 \text{ in.} & 0.4910 \text{ in.} \\
& 0.4697 \text{ in.} & 0.4910 \text{ in.} & \\
\hline
2 & 0.7071 \text{ in.} & 0.7071 \text{ in.} & 0.6944 \text{ in.} \\
& \star \text{ in.} & \star \text{ in.} & \\
\hline
3 & 1.0000 \text{ in.} & 1.0000 \text{ in.} & 0.9821 \text{ in.} \\
& 0.9393 \text{ in.} & 0.9821 \text{ in.} & \\
\hline
\end{array}
\]

\( \tau = \text{FUNDAMENTAL PERIOD OF VIBRATION OF THEORETICAL PLATE} \)
is described in Fig 10. First, the static load is applied at \( i = 7 \) and the resulting static deflections are noted. For the two dynamic solutions, the initial velocities and deflections are zero and the moving load is applied successively at \( i = 0, 1, 2, \ldots, 15 \). In one solution, the velocity of the moving load is \( 9.45 \times 10^2 \) in/sec. For the other solution, the velocity of the moving load is \( 3.78 \times 10^3 \) in/sec. The deflections are noted when the load is at \( i = 7 \). Figure 10b illustrates the deflected shape of the plate for the three solutions. Figure 11 shows the contours of the deflections for the same solutions. The closure tolerance is \( 1.0 \times 10^{-6} \) in. and the closure parameters are \( 0.7, 1.0, 4.0, 6.0, \) and \( 11.0 \) lb/in\(^3\). The static solution requires 50 iterations. The dynamic solutions require 16 iterations for each time increment when \( h_t \) is \( 5.08 \times 10^{-2} \) sec and 5 iterations when \( h_t \) is \( 1.25 \times 10^{-2} \) sec.

This problem was selected to demonstrate the effect that the velocity of a moving load has on the response of a plate. For \( v = 9.45 \times 10^2 \) in/sec, the dynamic deflection at \( i = 7 \) is greater than the static deflection. However, for \( v = 3.78 \times 10^3 \) in/sec, the dynamic deflection at \( i = 7 \) is less than the static deflection and the traveling wave lags behind the moving load. This phenomenon was discussed by Reismann (Ref 10) in his theoretical solution for a long rectangular plate.
Hinged supports along all edges
Corners are held down

15 x 4 GRID

D = 2.5 x 10^6 lb - in

h_x = h_y = 4.8 in.

L = 192 in.

ν = 0.25

Q = 1.0 x 10^3 lb/sta

FOR ν = 9.45 x 10^2 in/sec

h_t = 5.08 x 10^2 sec

FOR ν = 3.78 x 10^3 in/sec

h_t = 1.27 x 10^2 sec

Fig 10. Moving load on a rectangular plate.
Fig 11. Contours of transverse deflections for a rectangular plate.
A finite-element method has been presented to determine the response of a vibrating beam or plate. The method is based on an implicit difference formula of the Crank-Nicolson form. An examination of the difference equations for a uniform beam and plate disclosed that the implicit formula is not subject to instability. Therefore, this formula has been used in the development of the beam and plate equations. Although several investigators have used difference equations to solve the equation of motion for a uniform beam, the general development described herein is applicable to nonuniform beams and plates.

For the beam equation, the development includes externally applied dynamic loading, rotational restraints, elastic foundation supports, axial loads and viscous damping. For the plate equation, the development is arbitrarily restricted to externally applied dynamic loading and elastic foundation supports. Separate computer programs have been written in FORTRAN-63 for the solutions of the beam and plate equations. Both programs permit the flexural stiffnesses, elastic restraints, mass densities and loads to be discontinuous. Numerical examples demonstrate that the programs will be useful in solving many diverse problems whose solutions are not easily attainable by other known methods.

The present beam program effectively uses about 60 percent of the core storage of the Control Data Corporation 1604 computer. In contrast, the plate program utilizes the entire core storage of the computer and is restricted to problems whose maximum grid dimensions are $15 \times 15$. This limitation can be alleviated by storing a portion of the program on auxiliary tape.

A future extension of the preceding development will incorporate nonlinear flexural stiffness, foundation supports and damping. In addition, coupling
between response of the beam, or slab, and response of a moving mass must be considered. The fundamental ideas and procedures described herein may have a potential application in shell dynamics and in other initial-value problems in engineering.
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APPENDIX 1

DYNAMIC BEAM EQUATION
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APPENDIX 1. DYNAMIC BEAM EQUATION

The partial differential equation for the vibrating beam has been shown to be

\[
\frac{\partial^2}{\partial x^2} \left[ F \frac{\partial^2 w}{\partial x^2} \right] + \frac{\partial}{\partial x} \left[ (r + P) \frac{\partial w}{\partial x} \right] + p \frac{\partial^2 w}{\partial t^2} + d \frac{\partial w}{\partial t} = q(x,t) \quad (Al.1)
\]

A finite difference form of the above equation is derived in the following manner. All symbols have been previously defined. Expansion of Eq Al.1 establishes

\[
\frac{1}{h_x} \left[ \left( F \frac{\partial^2 w}{\partial x^2} \right)_{j-1,k} - 2 \left( F \frac{\partial^2 w}{\partial x^2} \right)_{j,k} + \left( F \frac{\partial^2 w}{\partial x^2} \right)_{j+1,k} \right]
\]

\[+ (sw)_{j,k} - \frac{1}{2h_x} \left[ - (r + P) \frac{\partial w}{\partial x} \right]_{j-1,k} \]

\[+ \left( (r + P) \frac{\partial w}{\partial x} \right)_{j+1,k} \] \[+ \frac{\rho j}{h_t^2} \left[ w_{j,k-1} - 2w_{j,k} + w_{j,k+1} \right] \]

\[+ \frac{d j}{h_t} \left[ -w_{j,k} + w_{j,k+1} \right] = q_{j,k} \quad (Al.2)
\]

and

\[
\frac{1}{h_x^4} \left[ F_{j-1} \left( w_{j-2,k} - 2w_{j-1,k} + w_{j,k} \right) \right.
\]

\[ - 2 \left(F_{j} \left( w_{j-1,k} - 2w_{j,k} + w_{j+1,k} \right) \right) \]

\[+ \left(F_{j+1} \left( w_{j,k} - 2w_{j+1,k} + w_{j+2,k} \right) \right) + s_{j} w_{j,k} \]

(equation continued)
\[- \frac{1}{4h_x} \left[ - \left( r + P \right)_{j-1} \left( - \varphi_{j-2,k} + \varphi_{j,k} \right) \right. \]

\[+ \left. \left( r + P \right)_{j+1} \left( - \varphi_{j,k} + \varphi_{j+2,k} \right) \right] \]

\[+ \frac{\rho_j}{h_x} \left[ \varphi_{j,k-1} - 2 \varphi_{j,k} + \varphi_{j,k+1} \right] \]

\[+ \frac{d_j}{h_x} \left[ - \varphi_{j,k} + \varphi_{j,k+1} \right] = \varphi_{j,k} \quad \text{(A1.3)} \]

Furthermore, let

\[Q_{j,k} = h_x \varphi_{j,k} \quad \text{(A1.4)} \]

\[S_j = h_x \varphi_j \quad \text{(A1.5)} \]

and

\[R_j = h_x \varphi_j \quad \text{(A1.6)} \]

Equations A1.3, A1.4, A1.5 and A1.6 are combined to yield

\[\left[ F_{j-1} - 0.25 h_x \left( R_{j-1} + h_x P_{j-1} \right) \right] \varphi_{j-2,k} \]

\[- 2 \left[ F_{j-1} + F_j \right] \varphi_{j-1,k} + \left[ F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j \right. \]

\[+ 0.25 h_x \left( R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1} \right) \left] \varphi_{j,k} \right. \]

(equation continued)
\[-2 \left[ F_j + F_{j+1} \right] w_{j+1,k} + \left[ F_{j+1} - 0.25 \ h_x (R_{j+1} + h_x P_{j+1}) \right] w_{j+2,k} + \frac{h_x^4}{h_t^2} \rho_j \left[ w_{j,k-1} - 2 w_{j,k} + w_{j,k+1} \right] + d_j \frac{h_x^4}{h_t^4} \left[ - w_{j,k} + w_{j,k+1} \right] = h_x^3 Q_{j,k} \] (Al.7)

For an implicit formula, the preceding equation becomes

\[0.5 \left[ F_{j-1} - 0.25 \ h_x (R_{j-1} + h_x P_{j-1}) \right] w_{j-2,k+1} - \left[ F_{j-1} + F_j \right] w_{j-1,k+1} + \left\{ 0.5 \left[ F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j + 0.25 \ h_x (R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1}) \right] \right\} \frac{h_x^4}{h_t^2} \rho_j + \frac{h_x^4}{h_t^2} d_j \left[ F_j + F_{j+1} \right] w_{j+1,k+1} + 0.5 \left[ F_{j+1} - 0.25 \ h_x (R_{j+1} + h_x P_{j+1}) \right] w_{j+2,k+1} = h_x^3 Q_{j,k} + 2 \frac{h_x^4}{h_t^2} \rho_j w_{j,k} - \frac{h_x^4}{h_t^2} \rho_j w_{j,k-1} + \frac{h_x^4}{h_t^2} d_j w_{j,k} - 0.5 \left[ F_{j-1} - 0.25 \ h_x (R_{j-1} + h_x P_{j-1}) \right] w_{j-2,k-1} + \left[ F_{j-1} + F_j \right] w_{j-1,k-1} - 0.5 \left[ F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j + 0.25 \ h_x (R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1}) \right] w_{j,k-1} \]

(equation continued)
\[ + \left[ F_j + F_{j+1} \right] w_{j+1, k-1} - 0.5 \left[ F_{j+1} - 0.25 h_x \left( R_{j+1} + \right. \right. \]
\[ + h_x P_{j+1} \left. \right) \right] w_{j+2, k-1} \]  

The foregoing equation corresponds to Eq 3.8 in the text.
APPENDIX 2

STATIC PLATE COEFFICIENTS
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APPENDIX 2. STATIC PLATE COEFFICIENTS

\[ x_1 = \frac{1}{h_x^4} D_{i-1,j} \]

\[ x_2 = -\frac{2}{h_x^3} \left[ D_{i-1,j} + D_{i+1,j} \right] + \frac{2\nu}{h_x^3 h_y^3} D_{i-1,j} \]

\[ x_3 = \frac{1}{h_x^3} \left[ D_{i-1,j} + 4D_{i,j} + D_{i+1,j} \right] + \frac{4\nu}{h_x^3 h_y^3} D_{i,j} \]

\[ x_4 = -\frac{2}{h_x^3} \left[ D_{i,j} + D_{i+1,j} \right] - \frac{2\nu}{h_x^3 h_y^3} D_{i+1,j} \]

\[ x_5 = \frac{1}{h_x^3} D_{i+1,j} \]

\[ x_6 = \frac{\nu}{h_x^3 h_y^3} D_{i-1,j} \]

\[ x_7 = -\frac{2\nu}{h_x^3 h_y^3} D_{i,j} \]

\[ x_8 = \frac{\nu}{h_x^3 h_y^3} D_{i+1,j} \]

\[ x_9 = x_6 \]

\[ x_{10} = x_7 \]

\[ x_{11} = x_8 \]

\[ y_1 = \frac{1}{h_y^3} D_{i,j-1} \]
\[
Y_2 = -\frac{2}{h_y} \left[ D_{i,j-1} + \frac{2\nu}{h_x h_y} D_{i,j-1} \right] - \frac{2\nu}{h_x h_y} D_{i,j-1}
\]

\[
Y_3 = \frac{1}{h_x h_y} \left[ D_{i,j-1} + \frac{4\nu}{h_x h_y} D_{i,j} + D_{i,j+1} \right] + \frac{4\nu}{h_x h_y} D_{i,j}
\]

\[
Y_4 = -\frac{2}{h_y} \left[ D_{i,j} + D_{i,j+1} \right] - \frac{2\nu}{h_x h_y} D_{i,j+1}
\]

\[
Y_5 = \frac{1}{h_x} D_{i,j+1}
\]

\[
Y_6 = \frac{\nu}{h_x h_y} D_{i,j-1}
\]

\[
Y_7 = -\frac{2\nu}{h_x h_y} D_{i,j}
\]

\[
Y_8 = \frac{\nu}{h_x h_y} D_{i,j+1}
\]

\[
Y_9 = Y_6
\]

\[
Y_{10} = Y_7
\]

\[
Y_{11} = Y_8
\]

Let

\[
T_{i,j} = (1-\nu) \ D_{i-1/2, \ j-1/2}
\]

\[
T_{i,j+1} = (1-\nu) \ D_{i-1/2, \ j+1/2}
\]

\[
T_{i+1,j} = (1-\nu) \ D_{i+1/2, \ j-1/2}
\]
\[ T_{i+1, j+1} = (1 - \nu) D_{i+\frac{1}{2}, j+\frac{1}{2}} \]

Then

\[ Z_1 = \frac{1}{h_x h_y} T_{i, j} \]

\[ Z_2 = -\frac{1}{h_x h_y} \left[ T_{i, j} + T_{i, j+1} \right] \]

\[ Z_3 = \frac{1}{h_x h_y} T_{i, j+1} \]

\[ Z_4 = -\frac{1}{h_x h_y} \left[ T_{i, j} + T_{i+1, j} \right] \]

\[ Z_5 = \frac{1}{h_x h_y} \left[ T_{i, j} + T_{i, j+1} + T_{i+1, j} + T_{i+1, j+1} \right] \]

\[ Z_6 = -\frac{1}{h_x h_y} \left[ T_{i, j+1} + T_{i+1, j+1} \right] \]

\[ Z_7 = \frac{1}{h_x h_y} T_{i+1, j} \]

\[ Z_8 = -\frac{1}{h_x h_y} \left[ T_{i+1, j} + T_{i+1, j+1} \right] \]

\[ Z_9 = \frac{1}{h_x h_y} T_{i+1, j+1} \]
APPENDIX 3

DYNAMIC PLATE EQUATION
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The partial differential equation for a transversely vibrating plate is

\[ \frac{\partial^2 w}{\partial x^2} \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right] + \frac{\partial^2 w}{\partial y^2} \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right] \]

\[ + 2 \frac{\partial^2}{\partial x \partial y} \left[ D (1-\nu) \frac{\partial w}{\partial x \partial y} \right] + sw + \rho \frac{\partial^2 w}{\partial t^2} = q(x,y,t) \] (A3.1)

The finite difference form of Eq A3.1 is derived in the following manner.

An expansion of Eq A3.1 establishes

\[ \frac{1}{h_x^2} \left\{ \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right]_{i-1, j, k} - 2 \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right]_{i, j, k} + \left[ D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \right]_{i+1, j, k} \right\} \]

\[ + \nu \frac{\partial^2 w}{\partial x ^2} \right\}_{i, j-1, k} - 2 \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right]_{i, j, k} + \left[ D \left( \frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2} \right) \right]_{i, j+1, k} \]

\[ + \frac{2(1-\nu)}{h_x h_y} \left\{ \left( D \frac{\partial^2 w}{\partial x \partial y} \right)_{i-\gamma_2, j, k} - \left( D \frac{\partial^2 w}{\partial x \partial y} \right)_{i, j, k} - \left( D \frac{\partial^2 w}{\partial x \partial y} \right)_{i-\gamma_2, j, k} + \left( D \frac{\partial^2 w}{\partial x \partial y} \right)_{i, j+\gamma, k} \right\} + (sw)_{i, j, k} \]

\[ + \frac{\rho h_x}{h_y} \left[ w_{i,j,k-1} - 2w_{i,j,k} + w_{i,j,k+1} \right] = q_{i,j,k} \] (A3.2)

and
\[ \frac{D_{i-1,j+1}}{h_x} \left\{ \frac{1}{h_x} \left[ w_{i-2,j,k} - 2w_{i-1,j,k} + w_{i,j,k} \right] \right\} \\
+ \frac{\nu}{h_y} \left[ w_{i-1,j-1,k} - 2w_{i-1,j,k} + w_{i-1,j+1,k} \right] \}

- 2 \left( \frac{D_{i+2,j}}{h_x} \right) \left\{ \frac{1}{h_x} \left[ w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k} \right] \right\} \\
+ \frac{\nu}{h_y} \left[ w_{i+1,j-1,k} - 2w_{i+1,j,k} + w_{i+1,j+1,k} \right] \}

+ \frac{D_{i+1,j-1}}{h_x} \left\{ \frac{1}{h_x} \left[ w_{i,j-2,k} - 2w_{i,j-1,k} + w_{i,j,k} \right] \right\} \\
+ \frac{\nu}{h_y} \left[ w_{i+1,j-1,k} - 2w_{i+1,j-1,k} + w_{i+1,j+1,k} \right] \}

+ \frac{D_{i,j-1}}{h_y} \left\{ \frac{1}{h_y} \left[ w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k} \right] \right\} \\
- 2 \left( \frac{D_{i,j}}{h_y} \right) \left\{ \frac{1}{h_y} \left[ w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k} \right] \right\} \\
+ \frac{\nu}{h_x} \left[ w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k} \right] \}

\text{(equation continued)}
\[ + \frac{D_{i,j+1}}{h_y} \left\{ \frac{1}{h_y} \left[ w_{i,j,k} - 2w_{i,j+1,k} + w_{i,j+2,k} \right] \right\} \]
\[ + \frac{\gamma}{h_x} \left[ w_{i-1,j+1,k} - 2w_{i,j+1,k} + w_{i+1,j+1,k} \right] \]
\[ + \frac{2(1-\nu)}{h_x h_y} \left\{ \frac{D_{i-1,j-1,k}}{h_x h_y} \left[ w_{i-1,j-1,k} - w_{i-1,j,k} - w_{i,j-1,k} \right] \right\} \]
\[ + w_{i,j,k} \right] - \frac{D_{i-j} + \gamma}{h_x h_y} \left[ w_{i-1,j,k} - w_{i,j+1,k} - w_{i,j,k} \right] \]
\[ + \frac{D_{i+1,j+1,k}}{h_x h_y} \left[ w_{i,j,k} - w_{i,j+1,k} - w_{i+1,j,k} \right] \]
\[ + w_{i+1,j,k} \right] \right\} \right\} + s_{i,j} w_{i,j,k} \]
\[ + \frac{\rho_{i,j}}{h_y} \left[ w_{i,j+k-1} - 2w_{i,j,k} + w_{i,j,k+1} \right] = q_{i,j,k} \] (A3.3)

Let
\[ s_{i,j} = h_x h_y s_{i,j} \] (A3.4)
\[ q_{i,j,k} = h_x h_y q_{i,j,k} \] (A3.5)
\[ c_{i,j} = (1-\nu) D_{i-j}, j-\gamma \] (A3.6)
\[ C_{i, j+1} = (1-\nu) D_{i-\frac{1}{2}, j+\frac{1}{2}} \quad (A3.7) \]
\[ C_{i+1, j} = (1-\nu) D_{i+\frac{1}{2}, j-\frac{1}{2}} \quad (A3.8) \]

and

\[ C_{i+1, j+1} = (1-\nu) D_{i+\frac{1}{2}, j+\frac{1}{2}} \quad (3.9) \]

Equations A3.3 through A3.9 are combined to yield

\[
\begin{align*}
\frac{D_{i-1, j}}{h_x} & \left\{ \frac{1}{h_x} \left[ w_{i-2, j, k} - 2w_{i-1, j, k} + w_{i, j, k} \right] \\
& + \frac{\nu}{h_y} \left[ w_{i-1, j-1, k} - 2w_{i-1, j, k} + w_{i-1, j+1, k} \right] \right\} \\
& - 2 \frac{D_{i, j}}{h_x} \left\{ \frac{1}{h_x} \left[ w_{i-1, j, k} - 2w_{i, j, k} + w_{i+1, j, k} \right] \\
& + \frac{\nu}{h_y} \left[ w_{i, j-1, k} - 2w_{i, j, k} + w_{i, j+1, k} \right] \right\} \\
& + \frac{D_{i+1, j}}{h_x} \left\{ \frac{1}{h_x} \left[ w_{i, j, k} - 2w_{i+1, j, k} + w_{i+2, j, k} \right] \\
& + \frac{\nu}{h_y} \left[ w_{i+1, j-1, k} - 2w_{i+1, j, k} + w_{i+1, j+1, k} \right] \right\} \\
& + \frac{D_{i, j-1}}{h_y} \left\{ \frac{1}{h_y} \left[ w_{i, j-2, k} - 2w_{i, j-1, k} + w_{i, j, k} \right] \right\}
\end{align*}
\]

(equation continued)
\[ + \frac{\nu}{h_x^2} \left[ \frac{w_{i-1,j-1,k} - 2w_{i,j-1,k} + w_{i+1,j-1,k}}{h_x^2} \right] \]

\[ - \frac{2D_{i,j}}{h_y^2} \left[ \frac{1}{h_y} \left[ \frac{w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k}}{h_y} \right] \right] \]

\[ + \frac{\nu}{h_x^2} \left[ \frac{w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k}}{h_x^2} \right] \]

\[ + \frac{D_{i,j+1}}{h_y^2} \left[ \frac{1}{h_y} \left[ \frac{w_{i,j,k} - 2w_{i,j+1,k} + w_{i,j+2,k}}{h_y} \right] \right] \]

\[ + \frac{\nu}{h_x^2} \left[ \frac{w_{i-1,j+1,k} - 2w_{i,j+1,k} + w_{i+1,j+1,k}}{h_x^2} \right] \]

\[ + \frac{2}{h_x h_y} \left[ \frac{C_{i,j}}{h_x} \left[ \frac{w_{i-1,j-1,k} - w_{i-1,j,k} - w_{i,j-1,k} + w_{i,j,k}}{h_x} \right] \right] \]

\[ - C_{i,j+1} \left[ \frac{w_{i-1,j,k} - w_{i-1,j+1,k} - w_{i,j,k} + w_{i,j+1,k}}{h_x} \right] \]

\[ - C_{i+1,j} \left[ \frac{w_{i,j-1,k} - w_{i+1,j-1,k} - w_{i,j,k} + w_{i,j+1,k}}{h_x} \right] \]

\[ + C_{i+1,j+1} \left[ \frac{w_{i,j,k} - w_{i,j+1,k} - w_{i+1,j,k} + w_{i+1,j+1,k}}{h_x} \right] \]

\[ + \frac{S_{i,j}}{h_x h_y} w_{i,j,k} + \frac{\rho_{i,j}}{h_x h_y} \left[ \frac{w_{i,j,k-1} - 2w_{i,j,k} + w_{i,j,k+1}}{h_x^2} \right] \]

\[ = \frac{Q_{i,j,k}}{h_x h_y} \]  

(A3.10)
The static plate coefficients, which are defined in Appendix 2, are substituted
into Eq A3.10. For an implicit formula, Eq A3.10 becomes

\[
\frac{1}{2} (X_1) \left[ w_{j-2, k-1} + w_{j-2, k+1} \right] + \frac{1}{2} (X_2 + Y + 2Z_2) \left[ w_{j-1, k-1} + w_{j-1, k+1} \right] + \frac{1}{2} (X_3 + Y + 2Z_5 + S_{i,j}) + \rho_{i,j} \left[ w_{i,j,k-1} + w_{i,j,k+1} \right]
\]

\[
+ \frac{1}{2} (X_4 + Y + 2Z_8) \left[ w_{i+1,j, k-1} + w_{i+1,j,k+1} \right] + \frac{1}{2} (Y_1) \left[ w_{i,j-2, k-1} + w_{i,j-2, k+1} \right] + \frac{1}{2} (Y_2 + X + 2Z_4) \left[ w_{i,j-1, k-1} + w_{i,j-1,k+1} \right] + \frac{1}{2} (Y_3 + X + 2Z_6) \left[ w_{i,j+1, k-1} + w_{i,j+1,k+1} \right]
\]

\[
+ \frac{1}{2} (Y_4 + X + 2Z_8) \left[ w_{i+1,j+1, k-1} + w_{i+1,j+1,k+1} \right] + \frac{1}{2} (Y_5) \left[ w_{i,j+2, k-1} + w_{i,j+2,k+1} \right] + \frac{1}{2} (X_9 + Y_6 + 2Z_1) \left[ w_{i-1,j-1, k-1} + w_{i-1,j-1,k+1} \right] + \frac{1}{2} (X_6 + Y_8 + 2Z_3) \left[ w_{i-1,j+1, k-1} + w_{i-1,j+1,k+1} \right] + \frac{1}{2} (X_11 + Y_9 + 2Z_7) \left[ w_{i+1,j-1, k-1} + w_{i+1,j-1,k+1} \right] + \frac{1}{2} (X_8 + Y_{11} + 2Z_9) \left[ w_{i+1,j+1, k-1} + w_{i+1,j+1,k+1} \right] - \frac{2\rho_{i,j,k}}{h_x h_y} w_{i,j,k}
\]

\[
= \frac{Q_{i,j,k}}{h_x h_y}.
\]
APPENDIX 4

SUMMARY FLOW DIAGRAM, GUIDE FOR DATA INPUT, AND LISTING FOR PROGRAM DBC1
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SUMMARY FLOW DIAGRAM - DBCI

READ problem identification

Is prob num zero?

Yes

9999

STOP

No

PRINT problem identification

READ and PRINT
Table 2. Constants
Table 3. Specified slopes and deflections
Table 4. Beam data
Table 5. Initial velocities
Table 6. Time dependent loading

DO for each time K from 2 to MTP2

DO for each station J from 3 to MP5

- If K - 3 + 0

7001 Static equations
7002 Dynamic equations with initial velocity
7003 Dynamic equations

7004 CONTINUE
6060 CONTINUE
7000 CONTINUE

PRINT Table 7. Deflections

702 CONTINUE
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GUIDE FOR DATA INPUT FOR PROGRAM DBC1 (BEAM)

with Supplementary Notes
IDENTIFICATION OF PROGRAM AND RUN (2 alphanumeric cards per run)

IDENTIFICATION OF PROBLEM (one card each problem)

TABLE 2. CONSTANTS (one card)

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<th>TIME</th>
<th>INCR</th>
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TABLE 3. SPECIFIED DEFLECTIONS AND SLOPES (number of cards according to TABLE 2)

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<th>DEFLECTION</th>
<th>SLOPE</th>
<th>CASE = 1 for deflection only, 2 for slope only, 3 for both</th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>10</td>
<td>16</td>
<td>20 30 40</td>
</tr>
</tbody>
</table>
**TABLE 4** BEAM DATA AND STATIC LOADING (number of cards according to TABLE 2). Data added to storage as lumped quantities per increment length, linearly interpolated between values input at indicated end stations, with 1/2-values at each end station. Concentrated effects are established as full values at single stations by setting final station equal to initial station. (2 cards per set of data required)

<table>
<thead>
<tr>
<th>FROM BEAM STA</th>
<th>TO BEAM STA</th>
<th>ENTER 1 IF CONT'D ON NEXT SET OF CARDS</th>
<th>F BENDING STIFFNESS</th>
<th>RHO MASS DENSITY</th>
<th>DF DAMPING COEF</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>30</td>
<td>40</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Q TRANSVERSE LOAD</th>
<th>S SPRING SUPPORT</th>
<th>P AXIAL LOAD</th>
<th>T TRANSVERSE COUPLE</th>
<th>R ROTATIONAL RESTRAINT</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>15</td>
<td>25</td>
<td>35</td>
<td>45</td>
</tr>
</tbody>
</table>

**TABLE 5** INITIAL VELOCITIES (number of cards according to TABLE 2). Full values of velocity occur at each station and the input is not cumulative.

<table>
<thead>
<tr>
<th>FROM BEAM STA</th>
<th>TO BEAM STA</th>
<th>ENTER 1 IF CONT'D ON NEXT CARD</th>
<th>WV INITIAL VELOCITY</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
</tbody>
</table>

**TABLE 6** TIME DEPENDENT LOADING (number of cards according to TABLE 2). Full values of load occur at each station and the input is not cumulative.

<table>
<thead>
<tr>
<th>FROM BEAM STA</th>
<th>TO BEAM STA</th>
<th>FROM TIME STA</th>
<th>TO TIME STA</th>
<th>ENTER 1 IF CONT'D ON NEXT CARD</th>
<th>QT DEPENDENT TIME LOADING</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
</tr>
</tbody>
</table>

STOP CARD (one blank card at end of run)
The data cards must be stacked in proper order for the program to run.

A consistent system of units should be used for all input data; for example: pounds, inches, and seconds.

All 5-space words are understood to be integers ............................. 4321
All 10-space words are floating-point decimal numbers in an E format ........ 4.321E+03
All integer data words must be right justified in the field provided.

The calculated deflections for all beam stations are printed in tabular form for each station.

The program will adjust the number of time stations so that this value will be a multiple of five. Thus, the number of time stations input will be increased by the computer by one to four to accommodate the output format.

TABLE 2. CONSTANTS

Typical units for the beam and time increment lengths are inches and seconds.
The maximum number of beam increments into which the beam-column may be divided is 100.
There is no maximum number of time increments, except that dynamic loading may be specified for only the first 110 time increments.

TABLE 3. SPECIFIED DEFLECTIONS AND SLOPES

The maximum number of stations at which deflections and slopes may be specified is 20.
Cards must be arranged in order of station numbers.
A slope may not be specified closer than 3 increments from another specified slope.
A deflection may not be specified closer than 2 increments from a specified slope, except that both a deflection and a slope may be specified at the same station.

TABLE 4. BEAM DATA AND STATIC LOADING

Typical units:

<table>
<thead>
<tr>
<th>variables</th>
<th>F</th>
<th>RHO</th>
<th>DF</th>
<th>Q</th>
<th>S</th>
<th>P</th>
<th>T</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>values per station</td>
<td>lb-in²</td>
<td>lb/sec²/in²</td>
<td>lb-sec/in</td>
<td>lb</td>
<td>lb/in</td>
<td>lb</td>
<td>lb-in</td>
<td>lb-in-rad</td>
</tr>
</tbody>
</table>
Axial tension or compression values $P$ must be stated at each station in the same manner as any other distributed data; there is no provision in the program to automatically distribute the internal effects of an externally applied axial force.

For the interpolation and distribution process, there are four variations in the station numbering and in referencing for continuation to succeeding cards. These variations are explained and illustrated on the following page.

There are no restrictions on the order of cards in Table 4, except that within a distribution sequence the stations must be in regular order.

**TABLE 5. INITIAL VELOCITIES**

| Typical units: | variable: | $WV$ | values per station: | in/sec |

A linear variation in initial velocities may be specified for any interval of beam stations, including the two end stations. The sequential order of the stations must be observed.

Initial velocities are input in the same manner as distributed quantities in Table 4, except that full values occur at every beam station and the input is not cumulative.

**TABLE 6. TIME DEPENDENT LOADING**

| Typical units: | variable: | $QT$ | values per station: | lb |

The time dependent loading may be specified for any beam station and for a maximum of 110 time stations.

The program permits any continuous linear variation in loading with time; however, if the loading is input for an interval of beam stations, the timewise variation in loading must be the same for every station within the interval.

The sequential order of both beam and time stations must be observed.

Full values of load occur at each station and the input is not cumulative.
STATION NUMBERING AND REFERENCING FOR TABLE 4.

Fixed-position Data

<table>
<thead>
<tr>
<th>Individual-card Input</th>
<th>FROM</th>
<th>TO</th>
<th>CONT'D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case a.1. Data concentrated at one sta.</td>
<td>7 → 7</td>
<td>O = NO</td>
<td>3.0</td>
</tr>
<tr>
<td>Case a.2. Data uniformly distributed</td>
<td>5 → 15</td>
<td>O = NO</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>15 → 20</td>
<td>O = NO</td>
<td>4.0</td>
</tr>
<tr>
<td></td>
<td>10 → 20</td>
<td>O = NO</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Multiple-card Sequence

<table>
<thead>
<tr>
<th>Multiple-card Sequence</th>
<th>FROM</th>
<th>TO</th>
<th>CONT'D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case b. First-of-sequence</td>
<td>25</td>
<td>i = YES</td>
<td>0.0</td>
</tr>
<tr>
<td>Case c. Interior-of-sequence</td>
<td>30</td>
<td>i = YES</td>
<td>4.0</td>
</tr>
<tr>
<td></td>
<td>35</td>
<td>i = YES</td>
<td>2.0</td>
</tr>
<tr>
<td>Case d. End-of-sequence</td>
<td>40</td>
<td>O = NO</td>
<td>2.0</td>
</tr>
</tbody>
</table>

Resulting Distributions of Data

STIFFNESS F

LOAD Q
TABLE 6. TIME DEPENDENT LOADING (continued)

The variable QT is input at any beam station and time station by specifying j and k in the FROM and TO columns.

EXAMPLES OF PERMISSIBLE INPUT OF THE VARIABLE QT ARE SHOWN BELOW

<table>
<thead>
<tr>
<th>BEAM STATIONS</th>
<th>TIME STATIONS</th>
<th>CONT'D TO NEXT CARD?</th>
<th>$Q_{i,k}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>FROM</td>
<td>TO</td>
<td>FROM</td>
<td>TO</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>0</td>
<td>20</td>
</tr>
<tr>
<td>20</td>
<td>20</td>
<td>20</td>
<td>40</td>
</tr>
</tbody>
</table>
-COOP, CE=51118, MATLOCK = S/2S.
-FTN, ERN.

PROGRAM DBC1

1 FORMAT (5L5, 52H)

C-----SOLVES FOR THE DYNAMIC RESPONSE OF A BEAM BY AN IMPLICIT METHOD

C-----NOTATION FOR DBC 1

C AN1( ), AN2( ), ETC IDENTIFICATION AND REMARKS (ALPHA-NUM) 12JE3
C DF(J) DAMPING COEF 01JL5
C DWS( ) VALUE OF SPECIFIED SLOPE D/DX 04JE3
C ESM MULTIPLIER FOR HALF VALUES AT END STAS 07JE3
C FN1, FN2, F(J) FLEXURAL STIFFNESS (EI) (INPUT AND TOTAL) 12JE3
C H BEAM INCREMENT 09JL5
C J TIME INCREMENT 01JL5
C J1, J2 INITIAL AND FINAL STATIONS IN SEQUENCE 05JE3
C JS STA OF SPECIFIED DEFLECTION OR SLOPE 05JE3
C K TIME STATION 09JL5
C KASE CASE NUM FOR SPECIFIED CONDITIONS 07JE3
C KASE 1=DEFL, 2=SLOPE, 3=BOTH 01JL5
C M TOTAL NUMBER OF INCREMENTS OF BMCOL 12JE3
C M MAX NUM = 50 01JL5
C MT NUMBER TIME INCREMENTS 01JL5
C MT MAX NUM NOT SPECIFIED 09JL5
C NCT3, 4, 5 AND 6 NUM CARDS IN TABLES 3, 4, 5 AND 6 07JN6
C NPROB PROBLEM NUMBER (PROG STOPS IF ZERO) 25Y3
C NS INDEX NUM FOR SPECIFIED CONDITIONS 05JE3
C PN1, PN2, P(J) AXIAL TENSION OR COMPRESSION (INPUT, TOTAL) 12JE3
C QT(J,K) TIME DEPENDENT TRANSVERSE LOADING 01JL5
C QT(J) MAX NUM (50, 110) 09JL5
C RHO(J) MASS DENSITY OF BEAM 01JL5
C RNI1, RNI2, R(J) ROTATIONAL RESTRAINT (INPUT, TOTAL) 12JE3
C SN1, SN2, S(J) SPRING SUPPORT STIFFNESS (INPUT AND TOTAL) 23MR4
C TN1, TN2, T(J) TRANSVERSE TORQUE (INPUT, TOTAL) 12JE3
C W(J,K) LATERAL DEFL OF BEAM AT J, K 09JL5
C WSJS1 SPECIFIED VALUE OF DEFL AT STA JS 12JE3
C W(VJ) INITIAL VELOCITY 09JL5
C X(F,XB MULTIPLIER 01JL5

DIMENSION AN1(32), AN2(14), F(107), Q(107), S(107), T(107) 07JN6
1 R(107), P(107), A(107), B(107), C(107), W(107) 07JN6
2 KEY(107), WS(20), DWS(20), QT(107), RHO(107) 07JN6
3 W(107), DF(107) 07JN6

10 FORMAT (5H, 80X, 10HI----TRIM ) 27FE4 ID
11 FORMAT (5H1 , 80X, 10HI----TRIM ) 27FE4 ID
12 FORMAT (16A5 ) 04MY3 ID
13 FORMAT (5X, 16A5 ) 27FE4 ID
14 FORMAT ( A5, 5X, 14A5 ) 18FE5 ID
15 FORMAT (///10H PROB, /5X, A5, 5X, 14A5 ) 18FE5 ID
16 FORMAT (///7H PROB (CONT) , /5X, A5, 5X, 14A5 ) 18FE5 ID
19 FORMAT (///46H RETURN THIS PAGE TO TIME RECORD FILE -- HM ) 12MR5 ID
21 FORMAT (2( 5X, IS, E10.3 ), 4( 5X, IS ) ) 07JN6
31 FORMAT (2(5X, IS, 2E10.3 ) ) 23MR4
41 FORMAT (5X, 315, 3E10.3 ) 07JN6
TABLE 2: CONSTANTS

1 / 5X, 25H NUM BEAM INCRE 20X, E10.3, 01JL5
2 / 5X, 25H BEAM INCRE LENGTH 20X, E10.3, 01JL5
3 / 5X, 25H TIME INCRE 20X, E10.3, 01JL5
4 / 5X, 25H TIME INCRE LENGTH 20X, E10.3, 01JL5
5 / 5X, 25H NUM CARDS TABLE 3 20X, E10.3, 01JL5
6 / 5X, 25H NUM CARDS TABLE 4 20X, E10.3, 07JN6
7 / 5X, 25H NUM CARDS TABLE 5 20X, E10.3, 07JN6
8 / 5X, 25H NUM CARDS TABLE 6 20X, E10.3, 07JN6

TABLE 3 - SPECIFIED DEFLECTIONS AND SLOPES

1 / 5X, 25H STA CASE DEFLECTION 20X, E10.3, 07JN6
2 / 5X, 25H STA CASE SLOPE 20X, E10.3, 07JN6
3 / 5X, 25H STA CASE SLOPE 20X, E10.3, 07JN6
4 / 5X, 25H STA CASE SLOPE 20X, E10.3, 07JN6

TABLE 4: BEAM DATA AND STATIC LOADING

1 / 5X, 25H FROM TO CONTD 10X, E10.3, 07JN6
2 / 5X, 25H FROM TO CONTD 10X, E10.3, 07JN6
3 / 5X, 25H FROM TO CONTD 10X, E10.3, 07JN6
4 / 5X, 25H FROM TO CONTD 10X, E10.3, 07JN6

TABLE 7: DEFLECTIONS

J=BEAM AXIS, K=TIME AXIS 07JN6

TABLE 5: INITIAL VELOCITIES

BEAM STA TIME STA 07JN6

TABLE 6: TIME DEPENDENT LOADING

BEAM STA TIME STA 07JN6

TABLE 8: TIME DEPENDENT LOADING

BEAM STA TIME STA 07JN6
A4.19

619 FORMAT (30X, I4, 5X, I4, 2X, E12.3)
904 FORMAT (// 4UH TOO MUCH DATA FOR AVAILABLE STORAGE //)
907 FORMAT (// 4UH ERROR STOP -- STATIONS NOT IN ORDER !)
C-----START EXECUTION OF PROGRAM -- SEE GENERAL FLOW CHART
      ITEST = 5H
1000 PRINT 1U
      CALL TIME
C-----PROGRAM AND PROBLEM IDENTIFICATION
      READ 12. (AN1(N), N = 1, 32)
1-10 READ 14., NPROB. (AN2(N), N = 1, 14)
      IF (NPROB - ITEST I 1020. 9990. 1020)
      PRINT 11
      PRINT 1
      PRINT 13. (AN1(N), N = 1, 32)
      PRINT 15. (AN2(N), N = 1, 14)
C-----INPUT TABLE 2: CONSTANTS
1210 READ 21. M, H, MT, HT, NCT3, NCT4, NCT5, NCT6
      PRINT 201. M, H, MT, HT, NCT3, NCT4, NCT5, NCT6
C-----COMPUTE CONSTANTS AND INDEXES
1240      HT2 = H + H
      HTE2 = HT * HT
      HE2 = H * H
      HE3 = H * HE2
      HE4 = H * HE3
      MP1 = M + 1
      MP4 = M + 4
      MP5 = M + 5
      MP6 = M + 6
      MP7 = M + 7
      MTP2 = MT + 2
      MTP9 = MT + 9
      H41T = HE4
      H4T2 = HE4 * HTE2
      XF = 0.5
      XB = 0.5
C-----INPUT TABLE 3: SPECIFIED SLOPES AND DEFLECTIONS
1300 PRINT 310
1310 DO 1315 J = 3, MP5
      KEY(J) = 1
1315 CONTINUE
1325 IF (NCT3 - 20) 1327, 1327, 1326
1326 PRINT 904
      GO TO 1U1
1327 JS = 3
      DO 1350 N = 1, NCT3
      READ 31. IN1, KASE, WS(N), DWS(N)
      IF (IN1 + 4 - JS) 1328, 1328, 1329
1328 PRINT 907
      GO TO 9999
1329 JS = IN1 + 4
C-----SET INDEXES FOR FUTURE CONTROL OF SPECIFIED CONDITION ROUTINES
      GO TO (1330, 1335, 1340), KASE
1330 KEY(JS) = 2
      PRINT 311. IN1, KASE, WS(N)
      GO TO 1350
1335  KEY(JS-1) = 3
     KEY(JS+1) = 5
     PRINT 312, IN1, KASE, DWS(N)
     GO TO 1350
1340  KEY(JS-1) = 3
     KEY(JS) = 4
     KEY(JS+1) = 5
     PRINT 313, IN1, KASE, W5(N), DWS(N)
1350  CONTINUE
1399  CONTINUE
      CLEAR STORAGE
      DO 1402 J=1,MP7
           F(J) = 0.0
           Q(J) = 0.0
           S(J) = 0.0
           T(J) = 0.0
           R(J) = 0.0
           PI(J) = 0.0
           RHO(J) = 0.0
           DF(J) = 0.0
           WV(J) = 0.0
      DO 1403 K=1,110
           QT(J,K) = 0.0
      DO 1402 KD=1,8
           W(J,KD) = J
      CONTINUE
C-----INPUT TABLE 4, BEAM DATA
C                              NCH4 = NCT4 / 2
1400  PRINT 400
1406  KR2 = 0
      DO 1480 N=1, NCH4
           KR1 = KR2
           READ  41, IN1, IN2, KR2, FN2, RHON2, DFN2
           READ  606, GN2, SN2, PN2, TN2, RN2
           JN = IN1 + 4
           J2 = IN2 + 4
           KSW = 1 + KR2 * 2 + KR1
      GO TO ( 1407, 1410, 1415, 1415 ), KSW
1407  PRINT  411, IN1, IN2, KR2, FN2, RHON2, DFN2, GN2, SN2, PN2, TN2, RN2, 07JN6
           1
           GO TO 1420
1410  PRINT  412, IN1, KR2, FN2, RHON2, DFN2, GN2, SN2, PN2, TN2, RN2, 07JN6
           GO TO 1420
1415  PRINT  413, IN2, KR2, FN2, RHON2, DFN2, GN2, SN2, PN2, TN2, RN2, 07JN6
           GO TO 1435
1420  J1 = JN
1425  FN1 = FN2
     QN1 = GN2
     SN1 = SN2
     TN1 = TN2
     RN1 = RN2
     PN1 = PN2
     DFN1 = DFN2
     RHON1 = RHON2
     07JN6
A4.21

GO TO ( 1435, 1480, 9999, 1480 ), KSW

C------SEE FLOW CHART, TABLE INTERPOL AND DISTRIB
1435 JINCR = 1
ESM = 1.0
IF ( J2 - J1 ) 1437, 1450, 1440

1437 PRINT 907
GO TO 1010
1440 DENOM = J2 - J1
ISW = 1
GO TO 1455
1450 DENOM = 1.0
ISW = 0
1455 DO 1460 J = J1, J2, JINCR
DIFF = J - J1
PART = DIFF / DENOM
F(J) = F(J) + ( FN1 + PART * ( FN2 - FN1 ) ) * ESM
Q(J) = Q(J) + ( QN1 + PART * ( QN2 - QN1 ) ) * ESM
S(J) = S(J) + ( SN1 + PART * ( SN2 - SN1 ) ) * ESM
T(J) = T(J) + ( TN1 + PART * ( TN2 - TN1 ) ) * ESM
R(J) = R(J) + ( RN1 + PART * ( RN2 - RN1 ) ) * ESM
P(J) = P(J) + ( PN1 + PART * ( PN2 - PN1 ) ) * ESM
DF(J) = DF(J) + ( DFN1 + PART * ( DFN2 - DFN1 ) ) * ESM
RHO(J) = RHO(J) + ( RHON1 + PART * ( RHON2 - RHON1 ) ) * ESM
CONTINUE
1460 IF ( ISW ) 9999, 1470, 1465
1465 JINCR = J2 - J1
ESM = - 0.5
ISW = 0
GO TO 1455
1470 GO TO ( 1480, 9999, 1480, 1475 ), KSW
1475 J1 = J2
GO TO 1425
1480 CONTINUE
C------INPUT TABLE 5, INITIAL VELOCITIES
PRINT 605
KR2 = 0
DO 1493 N=1, NCT5
KR1 = KR2
READ 612, IN1, IN2, KR2, WV2
JN = IN1 + 4
J2 = IN2 + 4
KSW = 1 + KR2 + 2 * KR1
GO TO ( 1481, 1482, 1483, 1483 ), KSW
1481 PRINT 613, IN1, IN2, WV2
GO TO 1484
1482 PRINT 615, IN1, KR2, WV2
GO TO 1484
1483 PRINT 616, IN2, KR2, WV2
GO TO 1486
1484 J1 = JN
1485 WV1 = WV2
GO TO ( 1486, 1493, 9999, 1493 ), KSW
1486 IF ( J2 - J1 ) 1487, 1489, 1488
1487 PRINT 907
GO TO 1010
DENOM = J2 - J1
GO TO 1490
1489
DENOM = 1.0
1490
DO 1491 J = J1, J2
    DIFF = J - J1
    PART = DIFF / DENOM
    WV(J) = WV1 + PART * ( WV2 - WV1)
1491
CONTINUE
GO TO ( 1493, 9999, 1493, 1492 ), KSW
1492
    J1 = J2
GO TO 1485
1493
CONTINUE
C-----INPUT TABLE 6, TIME DEPENDENT LOADING
PRINT 607
    KR2 = 0
    DO 635 N = 1, NCT6
        KR1 = KR2
        READ 609, IN1, IN2, KN1, KN2, KR2, QTN
        J1 = IN1 + 4
        J2 = IN2 + 4
        KN = KN1 + 2
        K2 = KN2 + 2
        KSW = 1 + KR2 + 2 * KR1
        GO TO ( 620, 621, 622, 622 ), KSW
620
PRINT 608, IN1, IN2, KN1, KN2, KR2, QTN
GO TO 623
621
PRINT 611, IN1, IN2, KN1, KR2, QTN
GO TO 623
622
PRINT 619, KN2, KR2, QTN
GO TO 625
623
    K1 = KN
    QN1 = QTN
    GO TO ( 625, 635, 9999, 635 ), KSW
625
    IF ( J2 - J1 ) 626, 627
626
PRINT 907
GO TO 9999
627
    IF ( K2 - K1 ) 628, 629, 630
628
PRINT 907
GO TO 9999
629
    DENOM = 1.0
GO TO 631
630
    DENOM = K2 - K1
631
    DO 633 J = J1, J2
        DO 632 K = K1, K2
            DIFF = K - K1
            PART = DIFF / DENOM
            QT(J,K) = QN1 + PART * ( QTN - QN1 )
632
        CONTINUE
633
CONTINUE
GO TO ( 635, 635, 635, 634 ), KSW
634
    K1 = K2
GO TO 624
635
CONTINUE
C-----START OF BEAM-COLUMN SOLUTION
PRINT 11
A4.23

PRINT 1  18FE5 ID
PRINT 13, ( AN1(N), N = 1, 32 )  18FE5 ID
PRINT 16, NPROB, ( AN2(N), N = 1, 14 )  28AG3 ID
PRINT 500  23MR4

K = 1  01JL5
DO 7009 NOT= 8, MTP9, 5  01JL5
IF ( NOT = MT - 4 ) 7009, 7005, 7005  01JL5
7005 MTP = NOT  01JL5
GO TO 7006  01JL5
7009 CONTINUE  01JL5
7006 DO 7000 KD = 2, MTP  01JL5
K = K + 1  01JL5
6000 NS = 1  01JL5
DO 6060 J = 3, MTP5  04JE3
704 QTP = 0.0  01JL5
GO TO 706  01JL5
705 QTP = QT(J, KD-1)  01JL5
706 CONTINUE  01JL5
C------COMPUTE MATRIX COEFFS AT EACH STA J  10JE3
YA = F(J-1) - 0.25 * H * ( R(J-1) + H * P(J-1) )  01JL5
YB = - 2.0 * ( F(J-1) + F(J) )  01JL5
YC = F(J-1) + 4.0 * F(J) + F(J+1) + HE3 * S(J) +  01JL5
1 0.25 * H * ( ( R(J-1) + H * P(J-1) ) + ( R(J+1)  01JL5
2 + H * P(J+1) ) )  01JL5
YD = - 2.0 * ( F(J) + F(J+1) )  01JL5
YE = F(J+1) - 0.25 * H * ( R(J+1) + H * P(J+1) )  01JL5
IF (KD=3) 7001, 7002, 7003  01JL5
7001 AA = YA  01JL5
BB = YB  01JL5
CC = YC  01JL5
DD = YD  01JL5
EE = YE  01JL5
FF = HE3 * Q(J) - 0.5 * HE2 * ( T(J-1) - T(J+1) )  01JL5
GO TO 7004  01JL5
7002 AA = XF * YA  01JL5
BB = XF * YB  01JL5
CC = XF * YC + 4.0 * H4T2 * RH(J) + 2.0 * H41T * DF(J)  01JL5
DD = XF * YD  01JL5
EE = XF * YE  01JL5
FF = HE3 * QTP - XB * (YA * W(J-2,K-1) + YB * W(J-1,K-1)  01JL5
1 + YC * W(J,K-1) + YD * W(J+1,K-1) + YE * W(J+2,K-1) ) 01JL5
2 + 8.0 * H4T2 * RH(J) * (W(J,K-1) + 0.5 * HT * WV(J) 01JL5
3 - 4.0 * H4T2 * RH(J) * W(J,K-1) + 2.0 * H41T *  01JL5
4 DF(J) * (W(J,K-1) + 0.5 * HT * WV(J))  01JL5
GO TO 7004  01JL5
7003 AA = XF * YA  01JL5
BB = XF * YB  01JL5
CC = XF * YC + H4T2 * RH(J) + H41T * DF(J)  01JL5
DD = XF * YD  01JL5
EE = XF * YE  01JL5
FF = HE3 * QTP - XB * (YA * W(J-2,K-2) + YB * W(J-1,K-2)  01JL5
1 + YC * W(J,K-2) + YD * W(J+1,K-2) + YE * W(J+2,K-2) ) 01JL5
2 + 2.0 * H4T2 * RH(J) * W(J,K-1) - H4T2 * RH(J)  01JL5
3 * W(J,K-2) + H41T * DF(J) * W(J,K-1)  01JL5
C-----COMPUTE RECURSION OR CONTINUITY COEFFS AT EACH STA

7004 CONTINUE

E = AA * B(J-2) + BB
DENOM = E * B(J-1) + AA * C(J-2) + CC
IF (DENOM) 6010, 6015, 6010

C-----NOTE IF DENOM IS ZERO, BEAM DOES NOT EXIST, D = 0 SETS DEFL = 0.

6005 D = 0.0
GO TO 6015

6010 D = -1.0 / DENOM
6015 C(J) = D * EE
B(J) = D * (E * C(J-1) + DD)
A(J) = D * (E * A(J-1) + AA * A(J-2) - FF)

C-----CONTROL RESET ROUTINES FOR SPECIFIED CONDITIONS

KEYJ = KEY(J)
GO TO (6060, 6020, 6030, 6020, 6050, KEYJ)

C-----RESET FOR SPECIFIED DEFLECTION

6020 C(J) = 0.0
B(J) = J.U
A(J) = WS(NS)
IF (KEYJ = 3) 6059, 6030, 6060

C-----RESET FOR SPECIFIED SLOPE AT NEXT STA

6030 DTEMP = D
CTEMP = C(J)
BTEMP = B(J)
ATEMP = A(J)
C(J) = 1.0
B(J) = 0.0
A(J) = -HT2 * DWS(NS)

C-----RESET FOR SPECIFIED SLOPE AT PRECEDING STATION

6050 DREV = 1.0 / (1.0 - (BTEMP * B(J-1) + CTEMP - 1.0) * D / DTEMP)
CREV = DREV * C(J)
BREV = DREV * (B(J) + (BTEMP * C(J-1)) * D / DTEMP)
AREV = DREV * (A(J) + (HT2 * DWS(NS) + ATEMP + BTEMP)

C-----C01'U4PUTE DEFLECTIONS

DO 6100 L = 3, MP5
J = M + 8 - L
W(J+K) = A(J) + B(J) * W(J+1+K) + C(J) * W(J+2+K)

6100 CONTINUE

IF (8 - K) 7007, 7007, 7000

7007 KSA = KD - 8
KSB = KD - 7
KSC = KD - 6
KSD = KD - 5
KSE = KD - 4

PRINT 617, KSA, KSB, KSC, KSD, KSE
DO 7008 J = 3, MP5
JSTA = J - 4

7008 10JE3
PRINT 618, JSTA, W(J+2), W(J+3), W(J+4), W(J+5), W(J+6)  
7008 CONTINUE  
K = 3  
DO 7010 J = 3, MP5  
W(J+2) = W(J+7)  
W(J+3) = W(J+8)  
7010 CONTINUE  
7000 CONTINUE  
CALL TIME  
GO TO 1010  
9990 CONTINUE  
9999 CONTINUE  
PRINT 11  
PRINT 13, ( AN1(N), N = 1, 32 )  
PRINT 19  
END  
FINISH  
-EXECUTE.
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-- CTR Library Digitization Team
APPENDIX 5

SUMMARY FLOW DIAGRAM, GUIDE FOR DATA INPUT, AND LISTING FOR PROGRAM DII1
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-- CTR Library Digitization Team
READ problem identification

Is prob num zero?

Yes

9999

STOP

No

PRINT problem identification

READ and PRINT

Table 1. Program control data
Table 2. Constants
Table 3. Stiffnesses and static loads
Table 4. Initial velocities and densities
Table 5. Dynamic loads
Table 6. Closure parameters

DO for each time KT from 2 to MTP2

DO specified num of iterations

Solve x system

DO for each station J from 4 to MYP4
I from 3 to MXP5

CONTINUE

Solve y system

DO for each station I from 4 to MXP4
J from 3 to MYP5

CONTINUE

PRINT monitor data

Closure of WX and WY?

Yes

No

CONTINUE

PRINT deflections

CONTINUE
This page replaces an intentionally blank page in the original.

-- CTR Library Digitization Team
GUIDE FOR DATA INPUT FOR PROGRAM DFII (PLATE)

with Supplementary Notes
DPII  GUIDE FOR DATA INPUT -- Card forms

IDENTIFICATION OF PROGRAM AND RUN (2 alphanumeric cards per run)

<table>
<thead>
<tr>
<th>NPROB</th>
<th>DESCRIPTION OF PROBLEM (alphanumeric)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>11</td>
</tr>
</tbody>
</table>

TABLE 1  CONTROL DATA (One card)

<table>
<thead>
<tr>
<th>NUMBER CARDS IN TABLE</th>
<th>MAX NUM ITER</th>
<th>CLOSURE TOLERANCE</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>6</td>
<td>60</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>65</td>
</tr>
<tr>
<td>5</td>
<td>26</td>
<td>60</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
<td>65</td>
</tr>
</tbody>
</table>

MONITOR MESH POINTS (specify the I and J stations for three mesh points)

<table>
<thead>
<tr>
<th>I</th>
<th>J</th>
<th>I</th>
<th>J</th>
<th>I</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>16</td>
<td>20</td>
<td>26</td>
<td>30</td>
</tr>
<tr>
<td>36</td>
<td>40</td>
<td>46</td>
<td>50</td>
<td>56</td>
<td>60</td>
</tr>
</tbody>
</table>

TABLE 2  CONSTANTS (One card)

<table>
<thead>
<tr>
<th>NUM X INCRS</th>
<th>NUM Y INCRS</th>
<th>NUM TIME INCRS</th>
<th>X INCRS LENGTH</th>
<th>Y INCRS LENGTH</th>
<th>TIME INCRS LENGTH</th>
<th>POISSON'S RATIO</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>16</td>
<td>20</td>
<td>26</td>
<td>30</td>
<td>40</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
<td>26</td>
<td>30</td>
<td>36</td>
<td>40</td>
<td>46</td>
</tr>
<tr>
<td>5</td>
<td>36</td>
<td>46</td>
<td>50</td>
<td>56</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>6</td>
<td>60</td>
<td>65</td>
<td>70</td>
<td>60</td>
<td>65</td>
<td>70</td>
</tr>
</tbody>
</table>
TABLE 3. STIFFNESS AND STATIC LOADING (number of cards according to TABLE 1)

<table>
<thead>
<tr>
<th>FROM</th>
<th>TO</th>
<th>D</th>
<th>T</th>
<th>S</th>
<th>Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>35</td>
</tr>
<tr>
<td>45</td>
<td>55</td>
<td>65</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| BENDING STIFFNESS | TORSIONAL STIFFNESS | SPRING SUPPORT | TRANSVERSE FORCE |

TABLE 4. INITIAL VELOCITIES AND DENSITIES (number of cards according to TABLE 1)

<table>
<thead>
<tr>
<th>FROM</th>
<th>TO</th>
<th>Wv</th>
<th>RHO</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
<td>20</td>
</tr>
<tr>
<td>25</td>
<td>35</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

| VELOCITY | DENSITY |

TABLE 5. DYNAMIC LOADING (Number of cards according to TABLE 1)

<table>
<thead>
<tr>
<th>FROM</th>
<th>TO</th>
<th>QT</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>20</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>35</td>
<td>45</td>
<td></td>
</tr>
</tbody>
</table>

| LOAD |

TABLE 6. CLOSURE PARAMETERS (Number of cards according to TABLE 1) Use one card for each parameter.

<table>
<thead>
<tr>
<th>CLOSURE PARAMETER</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
</tr>
<tr>
<td>15</td>
</tr>
</tbody>
</table>

STOP CARD (One blank card at end of each run)
GENERAL PROGRAM NOTES

Two cards containing any desired alphanumeric information are required (for identification purposes only) at the beginning of the data for each new run.

The data cards must be stacked in proper order for the program to run.

A consistent system of units must be used for all input data; for example, pounds, inches, and seconds.

All integer data words must be right justified in the field provided.

All data words of 5 spaces or less are integers .......................................................... -1 2 3 4

All data words of 10 spaces are to be entered as floating-point decimal numbers in an E format

-1 . 2 3 4 E + 0 3

Blank data fields are interpreted as zeros in an integer or floating point mode.

One card with a problem number in columns 1-5 is required as the first card of each problem. This number may be alphanumeric. The remainder of the card may contain any information desired.

Any number of problems may be stacked in one run.

One card with problem number blank is required to stop the run.

The calculated deflections for the monitor mesh points are printed after each iteration.

When the closure tolerance is satisfied at all mesh points, or when the maximum number of iterations is reached, the calculated deflections for all mesh points are printed.

TABLE 1. CONTROL DATA

The maximum number of iterations is 999.

A closure tolerance of $1.0 \times 10^{-6}$ in. is usually adequate.
TABLE 2. CONSTANTS

The maximum number of x and y plate increments is 15.

There is no maximum number of time increments.

TABLE 3. STIFFNESSES AND STATIC LOADING

Typical units:

variables: D T S Q

values per station: 1b-in 1b-in 1b/in 1b

In the foregoing, \( D = \frac{E h^3}{12 (1 - \nu^2)} \), wherein \( h \) is the thickness of the plate, and \( T = D (1 - \nu) \).

The remaining symbols have been previously defined.

For a rectangular plate that is divided into an \( M \times N \) grid, \( i = 0, 1, \ldots, M \) and \( j = 0, 1, \ldots, N \). The variables \( D, S, \) and \( Q \) are input at any grid or mesh point by specifying \( i \) and \( j \) in the FROM and TO columns. However, the variable \( T \) defines the torsional stiffness which is assumed to be concentrated at the center of each rectangular grid. In the program, \( T \) is numbered according to the mesh point that is located in the upper right corner of each grid, and it is assumed that the \( i \) station numbers increase from left to right and the \( j \) station numbers increase from bottom to top. Thus, for an \( M \times N \) grid, \( T \) is specified from \( i=1, j=1 \) to \( i=M, j=N \).

There are no restrictions on the sequential order of the cards. The input is cumulative with full values at each mesh point.

TABLE 4. INITIAL VELOCITIES AND DENSITIES

Typical units:

variables: WV RHO

values per station: in/sec 1b sec^2/in^3

The variables \( WV \) and \( RHO \) are input at any mesh point by specifying \( i \) and \( j \) in the FROM and TO columns.

A zero initial velocity is automatically established in the program. Thus only non-zero velocities must be specified.
TABLE 4. Continued

There are no restrictions on the sequential order of the cards. The input is cumulative with full values at each mesh point.

TABLE 5. DYNAMIC LOADING

Typical units:
variable: QT
values per station: 1b

The variable QT is input at any mesh point and time station by specifying i, j, and k in the FROM and TO columns.

There are no restrictions on the sequential order of the cards. The input is cumulative with full values at each station.

The loading may be specified for any mesh point and for a maximum of 28 time stations. Therefore, k maximum is 28.

TABLE 6. CLOSURE PARAMETERS

Typical units:
variable: RP
values per station: 1b/in³

The maximum number of parameters that may be input is nine.

The parameters are used in the cyclic order in which they are input.

The parameters are calculated on the basis of an average stiffness D and the increment length h in the x-direction from the equation.

\[
(RP)_m = \frac{4D}{h^4} \left(1 - \cos \frac{\pi m}{M}\right) \left(2 - \cos \frac{2\pi m}{M}\right) ; \quad m = 1, 2, 3 \ldots M - 1.
\]

The parameters for the y system are calculated internally in the program.
9COOP,CEO51015, MATLOCK-SALANI, S/25, 10, 6000. DPII
9FTN,E.

PROGRAM DPII
1 FORMAT (5X,52HPROGRAM DPII - MASTER DECK - HJ SALANI, H MATLOCK22JLS ID
1 28H REVIEW DATE 22 JUL 65)------ •
C-----SOLVES FOR THE DYNAMIC RESPONSE OF A PLATE BY AN IMPLICIT METHOD 01JLS
C-----NOTATION 01JLS
C
C ANA(N), ANB(N) ALPHA NUMERIC IDENTIFICATION 01JLS
C A(N),B(N), C(N) COEFFICIENTS 01JLS
C CTOL CLOSURE TOLERANCE 01JLS
C D(I,J) PLATE STIFFNESS PER UNIT AREA 01JLS
C D(I,J) (EHMM)/(112) (1-VV) 01JLS
C DN,RHON,TN,QN,QTN TEMP VALUES OF D,RHO,T,Q,QT 01JLS
C HX, HY, HT INCREMENT LENGTHS IN X,Y AND Z DIRECTIONS 01JLS
C ITEST BLANK FIELD FOR ALPHANUMERIC ZERO 22JLS
C ITMAX MAX NUM ITERATIONS 01JLS
C I X PLATE AXIS 01JLS
C J Y PLATE AXIS 01JLS
C K TIME AXIS 01JLS
C IM1, JM1 ETC MONITOR STAS FOR DEFL 01JLS
C MX, MY, MT NUMBER OF INCREMENTS IN X,Y AND Z 01JLS
C MX, MY, MT DIRECTIONS. MAX MX=MAX MY= 15, NO MAX MT 01JLS
C NCT3...NCT6 NUMBER CARDS IN TABLES 3 THRU 6 01JLS
C NPROB PROBLEM NUMBER, ZERO TO EXIT 01JLS
C PR POISSON'S RATIO 01JLS
C S(I,J) SPRING SUPPORT PER MESH POINT 01JLS
C T(I,J) STIFFNESS PER UNIT AREA, (1-V)(D) 01JLS
C WV(I,J) INITIAL VELOCITY 01JLS
C WY(I,J,K) TRANSVERSE DEFLECTION FOR Y SYSTEM 01JLS
C WX(I,J,K) TRANSVERSE DEFLECTION FOR X SYSTEM 01JLS

DIMENSION ANI(32), AN2(14), 18FE5 ID
1 Q122,22, W122,22, 01JLS
2 S122,22, R122,22, OD122,22,30, A122, B122, C122, 01JLS
3 RP(9), W122,22,4, W122,22,4, JSTA(25) 01JLS
COMMON /D122,22, T122,22,2/X11,2X3,2X4,5,6,7,8,9,10, 16MR5
1 X11, Y1, Y2, Y3, Y4, Y5, Y6, Y7, Y8, Y9, Y10, Y11, X1, X2, X3, 16MR5
2 X4, X5, X6, X7, X8, X9, X10, J, JA, HB, HC, HD, HYX, 16MR5
3 HYX, HXY, HXY 16MR5
10 FORMAT (5H , 80X , 10H1----TRIM ) 27FE4 ID
11 FORMAT (5H1 , 80X , 10H1----TRIM ) 27FE4 ID
12 FORMAT (16A5 ) 04MY3 ID
13 FORMAT (5X, 16A5 ) 27FE4 ID
14 FORMAT ( A5, 5X, 14A5 ) 18FE5 ID
15 FORMAT (///10H PROB , /5X, A5, 5X, 14A5 ) 18FE5 ID
16 FORMAT (///17H PROB (CONTD) /5X, A5, 5X, 16A5 ) 18FE5 ID
19 FORMAT (///48H RETURN THIS PAGE TO TIME RECORD FILE -- HM ) 12MR5 ID
20 FORMAT (5(2X,13),5X,E10.3) 01JLS
21 FORMAT (///30H TABLE 1. CONTROL DATA , / 01JLS
1 30H NUM CARDS TABLE 3 , 40X, I5, / 01JLS
2 30H NUM CARDS TABLE 4 , 40X, I5, / 01JLS
3 30H NUM CARDS TABLE 5 , 40X, I5, / 01JLS
4 30H NUM CARDS TABLE 6 , 40X, I5, / 01JLS
5 30H MAX NUM ITERATIONS , 40X, I5, / 01JLS
6 30H CLOSURE TOLERANCE , 35X,E10.3 ) 01JLS
22 FORMAT ( 815) 01JLS
23 FORMAT (30H MONITOR STAS I,J , 20X, 3(12,2X,12,4X) 01JLS
24 FORMAT (3110,4E10.3) 01JLS
25 FORMAT (///30H TABLE 2. CONSTANTS ,/ 01JL5
1 30H NUM INCREMENTS MX , 40X,15, / 01JL5
2 30H NUM INCREMENTS MY , 40X,15, / 01JL5
3 30H NUM INCREMENTS MT , 40X,15, / 01JL5
4 30H INCR LENGTH HX , 35X,E10.3, / 01JL5
5 30H INCR LENGTH HY , 35X,E10.3, / 01JL5
6 30H INCR LENGTH HT , 35X,E10.3, / 01JL5
7 30H POISSON'S RATIO , 35X,E10.3 ) 01JL5
26 FORMAT (///45H TABLE 3. STIFFNESSES AND STATIC LOADING ,/ 01JL5
1 29H FROM(I,J) THRU(I,J),6X,1HD,9X,1HT,10X , 01JL5
2 2HS ,10X,3HQ ) 01JL5
28 FORMAT (415,4E10.3) 01JL5
29 FORMAT ( 10X,J2,2X,I2,12,2X,12,4X,2X,12,4X,4(E10.3,2X) ) 01JL5
33 FORMAT (///50H TABLE 4. INITIAL VELOCITIES AND DENSITIES ,/ 01JL5
1 34H FROM(I,J) THRU(I,J),20X,2HWV 01JL5
2 12X, 3HRHO ) 01JL5
35 FORMAT ( 10X,J2,2X,I2,12,2X,12,9X,12,2X,12,4X,4IE10.3,2X ) 01JL5
1II150H TABLE 5. DYNAMIC LOADING, 01JL5
1 36H FROM(I,J,K) THRU(I,J,K),18X,2HQT 01JL5
1II150H TABLE 6. CLOSURE PARAMETERS 01JL5
104 FORMAT ( ) 01JL5
1000 PRINT 10 19MRS ID
1000 CALL TIME 18FE5 ID
C-----PROGRAM AND PROBLEM IDENTIFICATION
READ 12, ( AN1(N), N = 1, 32 ) 04MY3 ID
1010 READ 16, NPROB, ( AN2(N), N = 1, 14 ) 18FE5 ID
1020 PRINT 11 26A3G ID
1020 PRINT 1 18FE5 ID
1020 PRINT 13, ( AN1(N), N = 1, 32 ) 18FE5 ID
1020 PRINT 15, NPROB, ( AN2(N), N = 1, 14 ) 26A3G ID
C-----INPUT TABLE 1, CONTROL DATA
READ 20, NCT3,NCT4,NCT5,NCT6,ITMAX,CTOL 01JL5
READ 21, NCT3,NCT4,NCT5,NCT6,ITMAX,CTOL 01JL5
READ 22, IM1, JM1, IM2, JM2, IM3, JM3 01JL5
READ 23, IM1, JM1, IM2, JM2, IM3, JM3 01JL5
C-----INPUT TABLE 2, CONSTANTS
READ 24, MX, MY, MT, HX, HY, HT, PR 01JL5
PRINT 25, MX, MY, MT, HX, HY, HT, PR 01JL5
PRINT 25, MX, MY, MT, HX, HY, HT, PR 01JL5
PRINT 25, MX, MY, MT, HX, HY, HT, PR 01JL5
MXP3=MX+3 $ MYP3=MY+3 $ MXP2=MX+2 $ MYP2=MY+2 01JL5
MTP2=MT+2 $ MTP7=MX+7 $ MYP7=MY+7 $ MXP4=MX+4 01JL5
MYP4=MYP+4 $ MXP3=MXP+5 $ MYP5=MYP+5 01JL5
HXE4=HX**4 $ HXE4=HY**4 $ HTE2=HT**2 16MRS
HP=HX*HY $ HXY=HP*HP $ HT2=2.0*HT 16MRS
HA=1.0/HXE4 $ HB=2.0*HA $ HXY1=1.0/HXY 16MRS
C----CLEAR STORAGE

```
DO 30 I=1, MXP7
   A(I)=B(I)=C(I) = 0.0
DO 30 J=1, MYP7
   D(I,J)=T(I,J)=Q(I,J)=WV(I,J)=S(I,J)=RHO(I,J) = 0.0
DO 31 K=1,4
   WX(I,J,K)=WY(I,J,K) = 0.0
31 CONTINUE
DO 32 KK=1, 30
   QT(I,J,KK) = 0.0
32 CONTINUE
30 CONTINUE
```

C----INPUT TABLE 3, STIFFNESSES AND STATIC LOADING

```
PRINT 26
DO 27 N=1, NCT3
   READ 28, IN1, JN1, IN2, JN2, DN, TN, SN, QN
   I1=IN1+4 $ J1=JN1+4 $ I2=IN2+4 $ J2=JN2+4
   DO 27 I=I1, I2
   DO 27 J=J1, J2
      D(I,J)=D(I,J)+DN $ T(I,J)=T(I,J)+TN
      Q(I,J)=Q(I,J)+QN $ S(I,J)=S(I,J)+SN
   27 CONTINUE
```

C----INPUT TABLE 4. INITIAL VELOCITIES AND DENSITY

```
PRINT 33
DO 34 N=1, NCT4
   READ 28, IN1, JN1, IN2, JN2, WVN, RHON
   I1=IN1+4 $ J1=JN1+4 $ I2=IN2+4 $ J2=JN2+4
   DO 34 I=I1, I2
   DO 34 J=J1, J2
      WV(I,J)=WV(I,J)+WVN
      RHO(I,J)=RHO(I,J)+RHON
   34 CONTINUE
```

C----INPUT TABLE 5. DYNAMIC LOADING

```
PRINT 36
DO 37 N=1, NCT5
   READ 38, IN1, JN1, KN1, IN2, JN2, KN2, QTN
   I1=IN1+4 $ J1=JN1+4 $ K1=KN1+2
   I2=IN2+4 $ J2=JN2+4 $ K2=KN2+2
   DO 37 I=I1, I2
   DO 37 J=J1, J2
   DO 37 K=K1, K2
      QT(I,J,K) = QT(I,J,K) + QTN
   37 CONTINUE
```

C----INPUT TABLE 6. CLOSURE PARAMETERS

```
PRINT 40
DO 41 N = 1, NCT6
   READ 42, RP(N)
   PRINT 43, RP(N)
41 CONTINUE
```

C----SET ERRONEOUSLY STORED DATA TO ZERO

```
DO 44 I=3, MXP5
   DO 44 J=3, MYP5
      D(I,J,MYP5) = 0.0 $ D(MXP5,J) = 0.0
   44 CONTINUE
```

C----CALCULATE JSTAIN
C-----SOLUTION OF PROBLEM-----------------------------------------------01JL5

K=1
DO 46 KT=2,MTP2
  KSTA=KT-2
  IF(4-K) 82,83,83
  IF(KT-3) 78,79,79
  K=3
  DO 84 I=3,MXP5
  DO 84 J=3,MYP5
    WX(I,J,1)= WX(I,J,3) $ WY(I,J,1)= WY(I,J,3)
    WX(I,J,2)= WX(I,J,4) $ WY(I,J,2)= WY(I,J,4)
  CONTINUE
  CONTINUE
  ITER=O $ N=O
  PRINT 45 , IM1,JM1,M2,J,M3,JM3
  DO 47 NIT=1,ITMAX
    KCTOl =0
    ITER=ITER + 1
    N=N+l
    IF(NCTOl-NI 78,79,79
    N=I
    CONTINUE
  IF(KT-3)50,51,51
  QTP=O.O
  GO TO 52
  OTP=QT(I,J,KT-l)
  CALL COXY
  IF(KT-3)53,54,55
  AA = X1
  BB = X2 + XY2
  CC = X3 + XY5 + S(I,J) / HP + SF
  DD = X4 + XY8
  EE = X5
  FL = Q(I,J) / HP + SF * WY(I,J,K) - X6 * WX(I-1,J+1,K)
  F2 = X7 * WX(I,J+1,K) - X8 * WX(I+1,J+1,K) - X9 * WX(I-1,J-1,01JL5
  2 K) - X10 * WX(I,J-1,K) - X11 * WX(I+1,J-1,K) - Yl * WY(I,J)
  3 -2,K) - Y2 * WY(I,J-1,K) - Y3 * WY(I,J,K) - Y4 * WY(I+1,J,K)
  4 - Y5 * WY(I,J+1,K) - Y6 * WY(I-1,J-1,K) - Y7 * WY(I-1,J,K)
  5 - Y8 * WY(I-1,J+1,K) - Y9 * WY(I+1,J-1,K) - Y10 * WY(I+1,J,K)
  6 J,K) - YII * WY(I+1,J+1,K)
  FF = Fl + F2
  GO TO 56
  54 AA = 0.5 * X1
  BB = 0.5 * (X2 + XY2)
  CC = 0.5 * (X3 + XY5 + S(I,J) / HP + SF) + 4.0 * (RHD(I,J)
  1 I,J) / HTE2)
  DD = 0.5 * (X4 + XY8)
  EE = 0.5 * X5
  F1 = QTP / HP + 0.5 * SF * WY(I,J,K) + 4.0 * (RHD(I,J)
  10JL5
C----SOLVE Y SYSTEM

206 DO 61 I=4,MXP4
   DO 62 J=3,MPY5
   IF (D(I,J)) 97,97,98
   SF=0.0
   GO TO 100
97   SF=SF*P(N)
   IF(28-KT)63,64,64
   QTP=0.0
   GO TO 65
64   QTP=Q(T(I,J,KT-1))
   CALL COXY
   IF(KT-3)66,67,68
   AA = Y1
   BB = Y2 + XY4
   CC = Y3 + XY5 + S(I,J) / HP + SF
   DD = Y4 + XY6
   EE = Y5
   IF (I,J) / HP + SF * WX(I,J,K) - Y6 * WY(I-1,J-1,K)101JL5
   - Y7 * WY(I-1,J,K) - Y8 * WY(I-1,J+1,K) - Y9 * WY(I+1,J-1,01JL5
   K) - Y10 * WY(I+1,J,K) - Y11 * WY(I+1,J+1,K) - X1 * WX(I-2,01JL5
   J,K) - X2 * WX(I-1,J,K) - X3 * WX(I,J,K) - X4 * WX(I+1,J,K) - 01JL5
   X5 * WX(I+2,J,K) - X6 * WX(I+1,J+1,K) - X7 * WX(I+1,J,K) - 01JL5
   X8 * WX(I+1,J+1,K) - X9 * WX(I-1,J-1,K) - X10 * WX(I-1,J,K)101JL5
   - X11 * WX(I+1,J,K) 01JL5
   F1 = Q(I,J) / HP + SF * WX(I,J,K) - X12 * WY(I-1,J-1,K)01JL5
   F2 = -XY1 * (WX(I-1,J-1,K) + WY(I-1,J-1,K)) - XY2 * 01JL5
   1 (WX(I-1,J,K) + WY(I-1,J,K)) - XY3 * (WX(I-1,J,K) + WY(I-1,J,K)) - 01JL5
   2 (I-1,J+1,K) - XY4 * WX(I,J-1,K) - XY5 * WX(I,J,K) - XY6 * 01JL5
   3 WX(I,J,1,K) - XY7 * (WX(I+1,J-1,K) + WY(I+1,J,K)) - 01JL5
   4 XY8 * (WX(I+1,J,K) + WY(I+1,J,K)) - XY9 * (WX(I+1,J+1,K)101JL5
   5 + WY(I+1,J+1,K)) 01JL5
   FF = F1 + F2
   GO TO 69
69   AA = 0.5 * Y1
   BB = 0.5 * (Y2 + XY4)
   CC = 0.5 * Y3 + XY5 + S(I,J) / HP + SF + 4.0 * (RHO101JL5
   1 (I,J) / HTE2) 01JL5
   DD = 0.5 * (Y4 + XY6)
   EE = 0.5 * Y5
   F1 = QTP / HP + 0.5 * SF * WX(I,J,K) + 4.0 * (RHO(I,J) / 01JL5
   1 HT) * WY(I,J) + 4.0 * (RHO(I,J) / HTE2) * WY(I,J,K-1)01JL5
   F2 = -0.5 * (Y1 * WY(I,J-2,K-1) + (Y2 + XY4) * WY(I,J-1,01AP5
   K-1) + (Y3 + XY5 + S(I,J)) / HP) * WY(I,J,K-1) + (Y4 + 01AP5
   XY6) 01AP5
   1 (Y9 + XY7) * (WY(I,J+1,K-1) + WY(I+1,J,K-1) + (Y10 + 01AP5
   XY8) * (WY(I+1,J,K-1) + WY(I+1,J,K)) + (Y11 + XY9) * (WY(I+1,J+1,01AP5
   J,K-1) + WY(I+1,J+1,K))) 01JL5
   F3 = -0.5 * (X1 * (WX(I-2,J,K-1) + WX(I-2,J,K)) + (X2 + 01AP5
   XY2) * (WX(I-1,J,K-1) + WX(I-1,J,K)) + (X3 + XY5) * (WX(01AP5
   I,J,K-1) + WX(I,J,K)) + X4 + XY8) * (WX(I+1,J,K-1)101JL5
   + WX(I+1,J,K) + X5 * (WX(I+2,J,K-1) + WX(I+2,J,K))01JL5
   + (X6 + XY3) * (WX(I-1,J+1,K-1) + WX(I-1,J+1,K)) + (X7 + XY6)101JL5
   * (WX(I+1,J+1,K-1) + WX(I+1,J+1,K)) + (X8 + XY9) * (WX(I+1,J+1,K)101JL5
   -1) + WX(I+1,J+1,K) + (X9 + XY1) * (WX(I-1,J-1,K-1) + 01JL5
   WX(I-1,J-1,K)) + (X10 + XY4) * (WX(I-1,J-1,K-1) + WX(I-1,J-1,K))01AP5
   + (X11 + XY7) * (WX(I+1,J-1,K-1) + WX(I+1,J-1,K)) 01JL5
   FF = F1 + F2 + F3
   GO TO 69
69   AA = 0.5 * Y1
   BB = 0.5 * (Y2 + XY4)01AP5
107 DO 92 JKE=1,JTEST
     PRINT 87,(JSTA(N),N=JI,JF )
     DO 86 I=3,MXP5
     ISTA= I-4
     PRINT 88, ISTA , (WX(I,J,K), J=JI,JF)
     PRINT 91, (WX(I,J,K), J=JI,JF)
     86 CONTINUE
     JI=JI+5 $ JF=JF+5
     92 CONTINUE
     IF ( MYP5 - JI 93, 108, 108
     JF = MYP5
     JTEST = 1
     GO TO 107
     93 CONTINUE
     DO 101 I= 4,MXP4
     DO 101 J= 4,MVP4
     WY(I,J,K) = 0.5* ( WX(I,J,K) + WY(I,J,K) )
     WX(I,J,K) = WY(I,J,K)
     101 CONTINUE
     CALL TIME
     46 CONTINUE
     CALL TIME
     GO TO 1010
     9990 CONTINUE
     9999 CONTINUE
     PRINT 11
     PRINT 13, ( ANI(N), N = 1, 32 )
     PRINT 19
     END
     C-----SUBROUTINE
     SUBROUTINE COXY
     COMMON/1/D(22,22),T(22,22)/2/X1,X2,X3,X4,X5,X6,X7,X8,X9,X10,
     1 X11,Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9,Y10,Y11,Y12,Y13,
     2 X4,X5,X6,X7,X8,X9,X10,X11,X12,X13,
     3 HXYB,HXYC,HXYD
     X1 = HA * D ( I-1,J)
     X2 = - HB * ( D ( I-1,J) + D ( I,J ) - HXYB * D ( I-1,J) )
     X3 = HA * ( D ( I-1,J) + 4.0 * D ( I,J ) + D ( I+1,J ) )
     1
     X4 = - HB * ( D ( I,J ) + D ( I+1,J ) - HXYB * D ( I+1,J) )
     X5 = HA * D ( I+1,J)
     X6 = HXYB * D ( I-1,J)
     X7 = - HXYB * D ( I,J)
     X8 = HXYA * D ( I+1,J)
     X9 = X6
     X10 = X7
     X11 = X8
     Y1 = HC * D ( I,J-1)
     Y2 = - HD * ( D ( I,J-1) + D ( I,J ) ) - HXYB * D ( I,J-1)
     Y3 = HC * ( D ( I,J-1) + 4.0 * D ( I,J ) + D ( I+1,J ) )
     1
     Y4 = - HD * ( D ( I,J ) + D ( I,J+1 ) - HXYB * D ( I,J+1)
     Y5 = HC * D ( I,J+1)
     Y6 = HXYA * D ( I,J-1)
     Y7 = X7
     Y8 = HXYA * D ( I,J+1)
     Y9 = Y6
     Y10 = Y7
     Y11 = Y8
     XY1 = HXY1 * T(I,J)
     XY2 = - HXY1 * ( T ( I,J ) + T(I,J+1) )
XY3 = HXY1 * T(I,J+1)
XY4 = - HXY1 * ( T(I,J) + T(I+1,J) )
XY5 = HXY1 * ( T(I,J) + T(I,J+1) + T(I+1,J) + T(I+1,J+1) )
XY6 = - HXY1 * ( T(I,J+1) + T(I+1,J+1) )
XY7 = HXY1 * T(I+1,J)
XY8 = -HXY1 * ( T(I+1,J) + T(I+1,J+1) )
XY9 = HXY1 * T(I+1,J+1)

9EXECUTE.