# CONTINUUM THEORY OF MOISTURE MOVEMENT AND SWELL IN EXPANSIVE CLAYS

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# PREFACE

This report is the second in a series of reports from Research Project 3-8-68-118 entitled "Study of Expansive Clays in Roadway Structural Systems." The report is a theoretical study of the phenomena of expansive clay which is viewed as a macroscopically continuous material and is treated with the recently developed and mathematically powerful mixture theory of continuum mechanics. The subject matter is necessarily abstract but is considered essential to the formation of a firm analytical foundation for future developments in the project, including the computer programs to be presented in later reports and the envisioned field and laboratory experiments which are essential to a thorough understanding of the mechanics of expansive clay.

The theory of mixtures was devised for studying the properties of materials with different constituents. Clay is a mixture composed of clay mineral, water, and a gas which is itself a mixture of air and water vapor. The theory of mixtures uses gross mechanical properties of mixtures such as bulk compressibility rather than trying to deal with the properties of particles and minerals. The theory of mixtures looks at soil as a continuous material just as the engineer regards a steel beam as being continuous rather than as being composed of a collection of randomly connected grains and flakes of iron and carbon. Modulus of elasticity and Poisson's ratio are sufficient data for the engineer to make rather accurate computations on the deflections of beams.

The theory of mixtures has been used in this report to obtain a set of simultaneous nonlinear differential equations which represent the clay mixture. One of the more significant results of obtaining these equations has been in identifying the material functions needed to describe the behavior of expansive clay and in outlining laboratory tests needed to determine these material functions. Shear modulus, bulk modulus, and permeability are some of the material functions required, and they are termed "functions" because they are dependent on water content.

This project is a part of the cooperative highway research program of the Center for Highway Research, The University of Texas at Austin, and the Texas Highway Department in cooperation with the U. S. Department of Transportation, Bureau of Public Roads. The Texas Highway Department contact representative is Larry J. Buttler.

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# LIST OF REPORTS

Report No. 118-1, "Theory of Moisture Movement in Expansive Clay" by Robert L. Lytton, presents a theoretical discussion of moisture movement in clay soil.

Report No. 118-2, "Continuum Theory of Moisture Movement and Swell in Expansive Clays" by R. Ray Nachlinger and Robert L. Lytton, presents a theoretical study of the phenomena of expansive clay.

# ABSTRACT

This report presents a theoretical study of the phenomenon of expansive clay using the mixture theory of continuum mechanics. The laws concerning the balance of energy, mass, momentum, and angular momentum are written together with the general form of constitutive relations. Restrictions on the independent variables and explicit assumptions about the nature of the swelling clay phenomenon are applied to the balance laws and constitutive equations to give some simple, coupled, nonlinear differential equations. Two coupled equations are required in the isothermal case and three are required in the nonisothermal case.

In considering the isothermal case, it is found that six material functions are required to describe the behavior of the soil under these conditions. These six functions may be determined using a series of four experiments which are outlined in this report. An alternate set of four experiments is also described.

Boundary conditions are discussed and some remarks are made on one-dimensional problems. The differential equations for nonisothermal conditions are presented and it is shown that 13 material functions are required to describe the behavior of soil under these conditions.

KEY WORDS: soil mechanics, engineering mechanics, permeability, pore water pressure, mixtures, constitutive equations, derivation, theoretical (soil) mechanics, water, soil suction, soil science.

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# PARTIAL LIST OF SYMBOLS

| Symbol | Definition |
|---|---|
| $b_\alpha$ | Body force on the $\alpha^{th}$ constituent |
| E | Infinitesimal strain tensor |
| E* | Deviatoric strain |
| $F_\alpha$ | Deformation gradient, i.e., measure of the deformation |
| g | Grad $\theta$ |
| G | Gas |
| h | Heat supply of the mixture |
| H | F - I |
| L | Liquid |
| $\hat{p}_\alpha$ | Rate of mass supply |
| P | A point in the mixture |
| $\hat{P}_\alpha$ | Momentum supply |
| q | Heat flux of the mixture |
| s | Soil |
| T | Total stress |
| $T_\alpha$ | Stress on the $\alpha^{th}$ constituent |
| $u_\alpha$ | Diffusion velocity |
| U | Displacement vector |
| $v_\alpha$ | Velocity of the $\alpha^{th}$ constituent |
| $\hat{W}_\alpha$ | Supply of angular momentum |

| Symbol | Definition |
|---|---|
| $x$ | Position of print at some time later |
| $X$ | Position of a point at initial time |
| $X_\alpha$ | Position of a point P of the $\alpha^{th}$ constituent of the mixture |
| $\alpha^{th}$ | The $\alpha^{th}$ constituent of the mixture |
| $\beta$ | Porosity |
| $\epsilon$ | Density of the mixture |
| $\theta$ | Temperature |
| $\rho$ | Density of the mixture |
| $\rho_\alpha$ | Mass density for each constituent |
| $\rho^*_\alpha$ | Density of the free substance |
| $\phi$ | Function of either $\rho_G$ or $\rho_L$ |
| $\psi$ | Average or mean value of a quantity $\psi_\alpha$ over the mixture |
| $\dot{\psi}$ | Material derivative of a quantity $\psi$ following the path defined by the mean velocity $v$ |

Note: Capital Latin characters represent tensors; small Latin characters represent scalars. Greek subscripts on Latin characters signify the constituent of the mixture.

# CHAPTER 1. INTRODUCTION

This report is a theoretical examination of the problem of the movement of moisture in soil and the resulting change in volume. Results of this examination may be used to solve boundary value problems which in turn will be used in determining better design criteria for structures built on expansive clays.

No attempt will be made to give a historical introduction to this subject. This report presents a unified mathematical formulation that will give quantitative information about the movement of moisture and swell. In general, previous investigators have relied on empirical results and intuition. These results have been of value, but the system of differential equations proposed in this report can be solved to yield quantitative results which would eliminate the need of much experimental investigation and at the same time lead to better design criteria. This is one of the end products toward which this report is aimed.

The primary goal of this report is to develop rigorously the simplest system that will still describe the phenomenon in question so that the ramifications of the theory can be seen without being obscured by details. Once the phenomenon is understood, the complicating factors can be incorporated readily.

The means used to achieve this goal in this report are a departure from the normal techniques used in soil mechanics. Instead of reasoning from molecules, grains, and pores and trying to obtain quantitative information from models, which of necessity are inadequate, the view of continuum mechanics is adopted. In this view, soil, water, and air are considered to be a mixture of three constituents. At each point in space, it is assumed that there is a particle of <u>each</u> constituent.

It is also assumed that the dependent variables are continuous functions of the independent variables. To appreciate the type of results sought, consider an analogy between the work of this report and elasticity theory as compared with previous work in soils and metallurgy. The metallurgist considers the microscopic structure of metals to determine how and why the various properties vary from material to material. The science has not, however, progressed to the point where properties can be calculated directly. When a person is faced with the problem of analyzing a structure, however, he does not worry about the microscopic structure of the material; he relies on the theory of elasticity or some other suitable but simple continuum notion. In this report, an attempt is made to provide a comparable theory for use in design of soil-structure systems. A satisfying advantage of this approach is that, because the development is based on the principles of mechanics, the dependent variables have physical significance instead of the physically elusive quantities of thermodynamics.

This report contains five chapters in addition to this introduction. Chapter 2 contains the theoretical development of the governing equations and, of necessity, is quite abstract. For this reason, the remainder of the report is written so that Chapter 2 can be overlooked, if one is willing to accept the equations without development.

Chapter 3 presents a discussion of the general equations and gives all of the assumptions required to obtain the simpler differential equations used subsequently in the report. Chapter 4 suggests two systems of four experiments that will provide values of the material functions required to describe the behavior of expansive soil. Chapter 5 discusses boundary conditions and makes some particular remarks on one-dimensional problems and on differential equations for the nonisothermal soil condition. Chapter 6 presents conclusions drawn from this theoretical examination of the mechanics of swelling soil.

# CHAPTER 2. DEVELOPMENT OF THE EQUATIONS FOR MOISTURE MOVEMENT AND SWELL IN SOILS

In this chapter, a general theory of flow through a porous medium is presented. This theory will then be simpified for several classes of flow. The basis of this theory is the principle of superposed continua, which postulates that each point of a mixture is occupied by a particle of each constituent of the mixture.

## Notation and Kinematics

The notation that is used is direct notation where, in general, capital Latin characters represent tensors, small Latin characters represent vectors, and Greek characters represent scalars. Greek subscripts on Latin characters will signify the constituent of the mixture.

Consider a point P of the $\alpha^{th}$ constituent of the mixture which is at the position $X_\alpha$, when the mixture is in some reference configuration. Some time later, the point will be in a different position $x$ which can be expressed as

$$x_\alpha = x_\alpha(X_\alpha, t) \tag{2.1}$$

The deformation gradient, which is a measure of the deformation, is defined as

$$F_\alpha = \frac{\partial x_\alpha}{\partial X_\alpha} \equiv \nabla x_\alpha \tag{2.2}$$

The velocity of the $\alpha^{th}$ constituent is also defined as

$$v_\alpha = \frac{\partial x_\alpha}{\partial t}(X_\alpha, t) \equiv x'_\alpha(X_\alpha, t) \tag{2.3}$$

It is assumed that there exists a mass density for each constituent given by $\rho_\alpha$ so that the density of the mixture can be defined by

$$\rho = \Sigma\rho_\alpha \tag{2.4}$$

With this definition, the average or mean value $\psi$ of a quantity $\psi_\alpha$ over the mixture can be defined by

$$\rho\psi = \Sigma\rho_\alpha\psi_\alpha \tag{2.5}$$

As an example, the average velocity of the mixture $v$ is defined as

$$\rho v = \Sigma\rho_\alpha v_\alpha \tag{2.6}$$

If the diffusion velocity is defined as

$$u_\alpha = v_\alpha - v \tag{2.7}$$

then by using Eq 2.6, it is seen that

$$\Sigma\rho_\alpha u_\alpha = 0 \tag{2.8}$$

The material derivative of a quantity $\psi$ following the path defined by the mean velocity $v$ is

$$\dot{\psi} = \frac{\partial\psi}{\partial t}(x_\alpha,t) + [\ \mathrm{grad}\ \psi(x_\alpha,t)\ ] \cdot v \tag{2.9}$$

While following the motion $x_\alpha$

$$\psi'_\alpha = \frac{\partial\psi}{\partial t}(x_\alpha,t) + [\ \mathrm{grad}\ \psi(x_\alpha,t)\ ] \cdot v_\alpha \tag{2.10}$$

The difference of Eqs 2.9 and 2.10 is

$$\psi'_\alpha - \dot{\psi} = [\, \mathrm{grad}\, \psi(x_\alpha, t)\, ] \cdot u_\alpha \tag{2.11}$$

or

$$\psi'_\alpha - \dot{\psi} = [\, \nabla\psi(X_\alpha, t)\, ]\, F_\alpha^{-1} \cdot u_\alpha \tag{2.12}$$

where $F_\alpha^{-1}$ is the inverse of $F_\alpha$ .

## Balance Laws

In this section, the balance laws for the constituents of a mixture will be developed. The basis of the development will be the postulation of supply terms, which account for the fact that quantities like momentum and mass need not be balanced for each constituent, but these quantities can be interchanged among the constituents. This development is similar to that in Ref 1. These quantities are then restricted by the requirement that the quantities for the total mixture must be balanced.

Balance of Mass. It is assumed that the only mechanism that can effect a mass transfer between the constituents is chemical reaction. This leads to a postulation of a rate of mass supply $\hat{\rho}_\alpha$ which is defined by

$$\frac{\partial \rho_\alpha}{\partial t} + \mathrm{DIV}(\rho_\alpha v_\alpha) = \hat{\rho}_\alpha \tag{2.13}$$

Using Eqs 2.4 and 2.6, it is seen that

$$\frac{\partial \rho}{\partial t} + \mathrm{DIV}(\rho v) = \Sigma\hat{\rho}_\alpha \tag{2.14}$$

and thus, for the mass of the mixture to be balanced, the following restriction is obtained:

$$\Sigma\hat{\rho}_\alpha = 0 \tag{2.15}$$

By using Eqs 2.13, 2.9, and 2.10, it is possible to obtain the useful identity (for a derivation, see Ref 1)

$$\rho\dot{v} = \Sigma\ [\ \rho_\alpha v'_\alpha - \mathrm{DIV}(\rho_\alpha u_\alpha \otimes u_\alpha) + \hat{\rho}_\alpha u_\alpha\ ] \tag{2.16}$$

where $u_\alpha \otimes u_\alpha$ denotes the tensor product of two vectors.

Balance of Momentum. Following the same procedure as above, the momentum supply $\hat{P}_\alpha$ is defined by Eq 2.17.

$$\mathrm{DIV}\ T_\alpha - \rho_\alpha(-b_\alpha + v'_\alpha) = -\hat{P}_\alpha \tag{2.17}$$

where $T_\alpha$ and $b_\alpha$ are the stress and body force on the $\alpha^{th}$ constituent, respectively. Summing Eq 2.17 over all constituents, Eq 2.18 is found.

$$\mathrm{DIV}\ \Sigma T_\alpha + \Sigma\rho_\alpha b_\alpha - \Sigma\rho_\alpha v'_\alpha = -\Sigma\hat{P}_\alpha \tag{2.18}$$

or using Eq 2.16

$$\mathrm{DIV}\ \Sigma(T_\alpha - \rho_\alpha u_\alpha \otimes u_\alpha) + \Sigma\rho_\alpha b_\alpha - \rho\dot{v} = -\Sigma(\hat{P}_\alpha + \hat{\rho}_\alpha u_\alpha) \tag{2.19}$$

By defining

$$T = \Sigma(T_\alpha - \rho_\alpha u_\alpha \otimes u_\alpha) \tag{2.20}$$

$$\rho b = \Sigma\rho_\alpha b_\alpha$$

and requiring that momentum be balanced for the entire mixture, the following restriction is obtained:

$$\Sigma(\hat{P}_\alpha + \hat{\rho}_\alpha u_\alpha) = 0 \tag{2.21}$$

Balance of Angular Momentum. The supply of angular momentum $\hat{W}_\alpha$ is defined following the same procedure used in Eq 2.17, and the case where no body couples act is considered in obtaining Eq 2.22.

$$T_\alpha - T_\alpha^T = \hat{W}_\alpha \qquad (2.22)$$

where $T_\alpha^T$ denotes the transpose of $T_\alpha$. Summing Eq 2.22 over the constituents:

$$\Sigma(T_\alpha - T_\alpha^T) = \Sigma\hat{W}_\alpha \qquad (2.23)$$

But $\Sigma(T_\alpha - T_\alpha^T)$ is symmetric, since $u_\alpha \otimes u_\alpha$ is symmetric and $T$ must be symmetric. Therefore

$$\Sigma\hat{W}_\alpha = 0 \qquad (2.24)$$

Balance of Energy. It is possible to proceed as before and postulate the existence of a temperature, an internal energy, and a heat flux for each constituent of the mixture, but in this report the properties of these constituents will not be used individually, since the quantities that are most useful are those of the mixture as a whole.

$$0 = \rho\dot{\epsilon} - \rho h - TR(TL) - DIV\ q \qquad (2.25)$$

where $L = grad\ v$

and $q$, $\epsilon$, and $h$ are respectively the heat flux, internal energy, and the heat supply of the mixture.

To obtain a complete set of balance laws, an entropy production inequality should be developed. This will not be done, however, since there is some question as to the proper form of this inequality, and the results which are obtained will hardly be restricted by this inequality.

## Constitutive Equations

After examining the balance laws, it is noted that they are not a determinate system, and this makes it necessary that equations describing material properties must be obtained. Constitutive equations are needed which relate the independent variables to the following dependent variables: $T_\alpha$, $q$, $\epsilon$,

$\hat{\rho}_\alpha$ , $\hat{P}_\alpha$ , and $\hat{W}_\alpha$ . Because history dependence will not be considered, the general form of the required constitutive relations are as follows:

$$
\begin{aligned}
T_\beta &= T_\beta(F_\alpha , L_\alpha , v_\alpha , \rho_\alpha , \theta) \\
q &= q(F_\alpha , L_\alpha , v_\alpha , \rho_\alpha , \theta , g) \\
\varepsilon &= \varepsilon(F_\alpha , L_\alpha , v_\alpha , \rho_\alpha , \theta) \\
\hat{\rho}_\beta &= \hat{\rho}_\beta(F_\alpha , L_\alpha , v_\alpha , \rho_\alpha , \theta) \\
\hat{P}_\beta &= \hat{P}_\beta(F_\alpha , L_\alpha , v_\alpha , \rho_\alpha , \theta) \\
\hat{W}_\beta &= \hat{W}_\beta(F_\alpha , L_\alpha , v_\alpha , \rho_\alpha , \theta)
\end{aligned}
\tag{2.26}
$$

where $\theta$ is the temperature and $g = \text{grad}\ \theta$ . Note that $\hat{\rho}_\alpha$ and $\hat{P}_\alpha$ are subject to the restrictions of Eqs 2.14 and 2.21, respectively.

The forms of Eq 2.26 are not arbitrary, since they are subject to the principle of objectivity. This principle states that the constitutive equations must be invariant under a transformation of the form

$$x_\alpha = Qx_\alpha + x_0 \tag{2.27}$$

for any $Q$ such that $QQ^T = I$ (for any orthogonal $Q$ ).

Physically, the principle of objectivity states that a constitutive equation should be invariant under rigid body motions.

Thus by using Eq 2.27, the constitutive Eq 2.26 can be shown to be

$$
\begin{aligned}
T_\alpha &= T_\alpha(C_\beta , D_\beta , v_{\alpha\beta} , W_{\alpha\beta} , G_{\alpha\beta} , \rho_\alpha , \theta) \\
q &= q(C_\beta , D_\beta , v_{\alpha\beta} , W_{\alpha\beta} , G_{\alpha\beta} , \rho_\alpha , \theta , g) \\
\varepsilon &= \varepsilon(C_\beta , D_\beta , v_{\alpha\beta} , W_{\alpha\beta} , G_{\alpha\beta} , \rho_\alpha , \theta) \\
\hat{\rho}_\alpha &= \hat{\rho}_\alpha(C_\beta , D_\beta , v_{\alpha\beta} , W_{\alpha\beta} , G_{\alpha\beta} , \rho_\alpha , \theta)
\end{aligned}
$$

$$\hat{P}_{\alpha} = \hat{P}_{\alpha}(C_{\beta}, D_{\beta}, v_{\alpha\beta}, W_{\alpha\beta}, G_{\alpha\beta}, \rho_{\alpha}, \theta)$$

$$\hat{W}_{\alpha} = \hat{W}_{\alpha}(C_{\beta}, D_{\beta}, v_{\alpha\beta}, W_{\alpha\beta}, G_{\alpha\beta}, \rho_{\alpha}, \theta) \qquad (2.28)$$

where

$$C_{\beta} = F_{\beta}^{T} F_{\beta}$$

$$D_{\beta} = \frac{1}{2}(L_{\beta} + L_{\beta}^{T})$$

$$v_{\alpha\beta} = v_{\alpha} - v_{\beta}$$

$$W_{\alpha\beta} = \frac{1}{2}(L_{\alpha} - L_{\alpha}^{T}) - \frac{1}{2}(L_{\beta} - L_{\beta}^{T})$$

$$G_{\alpha\beta} = D_{\alpha} - D_{\beta} \qquad (2.29)$$

If the above constitutive equations are used with the balance laws, a determinate system results.

## An Isotropic Theory of Small Deformations of a Mixture of Soil, Air, and Water

Although the equations developed previously yield a determinate system, the determination of the constitutive functions is a quite formidable task. To simplify the task of determining the material properties and the solution of problems, further consideration is restricted to a mixture of soil, water, and air using the following assumptions, which experimental evidence predicts are realistic:

(1) The deformation of the soil is small.
(2) The velocities are small.
(3) The velocity gradients can be neglected.
(4) The stress in the water and air is hydrostatic.
(5) The constituents are isotropic.
(6) There is no change in the mass of the soil.

The assumption of small deformations and velocities only allows consideration of constitutive equations that are linear in the velocities and deformation variables. The assumption of isotropy allows the use of the well-known representation theorems presented in (2) to express the results in a more usable fashion.

Under the above assumptions, the constitutive equations can be expressed after considering restrictions (Eqs 2.14 and 2.21) as

$$
\begin{aligned}
T_s &= [\phi_1\theta + \phi_2 \mathrm{TRE} + \phi_3\rho_L + \phi_4\rho_G]\, I + \phi_5 E \\
T_L &= [\phi_6\theta + \phi_7 \mathrm{TRE} + \phi_8\rho_L + \phi_9\rho_G]\, I \\
T_G &= [\phi_{10}\theta + \phi_{11} \mathrm{TRE} + \phi_{12}\rho_L + \phi_{13}\rho_G]\, I \\
\hat{\rho}_s &= 0 \\
\hat{\rho}_L &= \phi_{14}\mathrm{TRE} + \phi_{15}\rho_L + \phi_{16}\rho_G + \phi_{17}\theta \\
\hat{\rho}_G &= -[\phi_{14}\mathrm{TRE} + \phi_{15}\rho_L + \phi_{16}\rho_G + \phi_{17}\theta] \\
\hat{P}_s &= \phi_{18}v_{Ls} + \phi_{19}v_{Gs} \\
\hat{P}_L &= \phi_{19}v_{sL} + \phi_{20}v_{GL} \\
\hat{P}_G &= \phi_{19}v_{sG} + \phi_{20}v_{LG} \\
q &= \phi_{21}g + \phi_{22}v_{Ls} + \phi_{23}v_{LG} + \phi_{24}v_{Gs} \\
\epsilon &= \phi_{25}\mathrm{TRE} + \phi_{26}\rho_L + \phi_{27}\rho_G + \phi_{28}\theta
\end{aligned}
\qquad (2.30)
$$

where all the $\phi$'s are functions of $\rho_G$, $\rho_L$, and $\theta$; $E$ is the infinitesimal strain tensor

$$E = \frac{1}{2}(H + H^T)$$

where

$H = F - I$,

$s$ = soil,

$G$ = gas,

$L$ = liquid.

If tests can be run to measure the 28 $\phi$'s, Eq 2.30 along with the field equations forms a determinate system that can be solved to obtain the moisture movement and resulting swell.

# CHAPTER 3. A DISCUSSION OF THE PHYSICAL MEANING OF THE SYSTEM OF EQUATIONS

In Chapter 2, a system of equations for describing the movement of water and the resulting swell was derived from the principles of continuum mechanics. This marks a distinct departure from the present way of considering the problem. To make the difference clearer, the physical interpretation of the assumptions that were made will be considered in this chapter together with the physical interpretation of the equations.

## Ramifications of the Continuum Assumption

The two basic assumptions in mixture continuum mechanics are that the dependent variables depend continuously on the independent variables and that particles of each constituent exist at every point in the body. On a microscopic scale, it is obvious that these assumptions are not satisfied by a mixture of soil, water, and air. The question thus comes to mind as to whether or not this inconsistency makes useless the equations that were derived from mixture continuum theory. The answer to this question is that these equations are quite useful.

The reason that they are useful is that the phenomenon of interest is the macroscopic one of moisture movement and swell in masses of soil that are very large in comparison to the dimensions of the pores in which the water moves. Thus the equations are seen to yield averages of the microscopic quantities over a region of the material.

Now it becomes obvious that the philosophy of this approach is to replace the intricate (and intractable) problem of trying to describe how the water will flow through the pores of a solid with the much simpler problem of trying to to describe the gross phenomenon. This gives some insight into the physical meaning of the $\phi$'s . They are the things that take into account the average porosity, surface tension, viscosity, etc., and they must be measured on a gross scale.

This approach does not deny the importance of work that has been done on the mechanisms of moisture movement on the microscopic scale. This type work can predict qualitatively how the phenomenon will vary from material to material. However, to expect this work to give quantitative answers about the gross phenomenon is to expect too much of nature and of the ingenuity of investigators. It is always useful to recall the analogy between this work and elasticity. When analyzing a structure, dislocation theory is not used. Instead, measured values of the elastic properties are used with the elasticity theory to obtain an average of the response. The fact that these results rely on measured properties and do not take into account the motion of microscopic dislocations does not keep the elasticity results from accurately predicting the phenomenon.

## The Field Equations for Small Deformations and Velocities

In the previous chapter, the constitutive equations for small deformations and velocities were presented. In addition to the simplification that was effected there, the field equations (Eqs 2.13 and 2.17) can also be simplified. For this case, the field equations become

$$\frac{\partial \rho_L}{\partial t} + \mathrm{DIV}(\rho_L v_L) = \hat{\rho}$$

$$\frac{\partial \rho_G}{\partial t} + \mathrm{DIV}(\rho_G v_G) = -\hat{\rho}$$

$$\mathrm{DIV}\ T_s + \rho_s b_s = -\hat{P}_s$$

$$\mathrm{DIV}\ T_L + \rho_L b_L = -\hat{P}_L$$

$$\mathrm{DIV}\ T_G + \rho_G b_G = -\hat{P}_G \qquad (3.1)$$

where now the total stress $T$ is given by

$$T = T_s + T_G + T_L \qquad (3.2)$$

and the restriction on the momentum transfer terms $\hat{P}_\alpha$ can be expressed as

$$\hat{P}_s + \hat{P}_L + \hat{P}_G = 0 \tag{3.3}$$

Also the various divergences can be taken to be the divergence with respect to the same initial coordinate system.

When these equations are used with the following constitutive equations, a determinate system results.

$$T_s = [\,\phi_1\theta + \phi_2 \mathrm{TRE} + \phi_3\rho_L + \phi_4\rho_G\,]\, I + \phi_5 E$$

$$T_L = [\,\phi_6\theta + \phi_7 \mathrm{TRE} + \phi_8\rho_L + \phi_9\rho_G\,]\, I$$

$$T_G = [\,\phi_{10}\theta + \phi_{11} \mathrm{TRE} + \phi_{12}\rho_L + \phi_{13}\rho_G\,]\, I$$

$$\hat{\rho} = \phi_{14} \mathrm{TRE} + \phi_{15}\rho_L + \phi_{16}\rho_G + \phi_{17}\theta$$

$$\hat{P}_s = \phi_{18} v_{Ls} + \phi_{19} v_{Gs}$$

$$\hat{P}_L = \phi_{18} v_{sL} + \phi_{20} v_{GL}$$

$$\hat{P}_G = \phi_{19} v_{sG} + \phi_{20} v_{LG} \tag{3.4}$$

where again $v_{\alpha\beta} = v_\alpha - v_\beta$ and all the $\phi$'s are functions of $\rho_L$ and $\rho_G$.

It should be noted that although these equations are quite formidable, many assumptions have been made. Among these is that the soil behaves as an isotropic, linearly elastic solid. In general, soil is more likely to be a nonlinear viscoelastic solid, but the inclusion of these effects would make the equations even more complicated. As stated in Chapter 1, the goal of this report is to develop rigorously the simplest system of equations that will still describe the phenomenon and not obscure it by details. Once this is accomplished, the complicating factors can be incorporated into the problem without much trouble.

## A Simpler Constitutive Equation for the Isothermal Case

In order for the systems (Eqs 3.3 and 3.4) to be of use, one must be able to measure experimentally all 20 $\phi$'s . Not only is this a formidable task, but there is some doubt as to whether some of them can be determined at all. As an example of this, think of how one might attempt to determine the dependence of the mass transfer on the dilatation of the soil. Perhaps someone will perfect a technique for the determination of all 20 $\phi$'s , but there is one more simplification that will eliminate most of the experimental problems.

The simplification that is proposed is to neglect the mass and the momentum of the gas. If this is done, the equations are identical with those of a two-constituent mixture. However, this does not mean that the soil is saturated, because the condition has not been imposed that the volumes occupied by the soil and water equal the total volume. Also, because the remaining $\phi$'s are still functions of $\rho_L$ , phenomena of unsaturated soil, such as a variation of stiffness with water content, can still be accounted for.

With this last assumption, the system of equations that will be used in the remainder of this report is obtained:

$$\frac{\partial \rho_L}{\partial t} + \mathrm{DIV}(\rho_L v_L) = 0$$

$$\mathrm{DIV}\ T_s + \rho_s b_s = \phi_{18} v_{Ls}$$

$$\mathrm{GRAD}\ P + \rho_L b_L = -\phi_{18} v_{Ls}$$

$$T_s = [\ \phi_2 \mathrm{TRE} + \phi_3 \rho_L\ ]\ I + \phi_5 E$$

$$P = \phi_7 \mathrm{TRE} + \phi_8 \rho_L \tag{3.5}$$

It should be noted that the above is valid for a mixture of an isotropic, elastic soil and a perfect fluid for small deformations and velocities, and where velocity gradients are neglected. This last simplification has reduced the experimental problem from that of determining 20 functions to that of determining six.

## Relation to Existing Theories

To make comparisons of the system, Eq 3.5, with existing theories, the only body force is taken to be gravity and the velocity of the soil is assumed to be zero. Also, a new parameter K is defined and the following relations are written:

$$\phi_{18} = \rho_L / K$$

$$\rho_L b_L = \text{GRAD}\ \gamma_L Z$$

$$\rho_s b_s = \text{GRAD}\ \gamma_s Z \qquad (3.6)$$

Equations 3.5 can now be written

$$\frac{\partial \rho_L}{\partial t} + \text{DIV}(\rho_L v_L) = 0$$

$$K\ \text{GRAD}\ H = -\rho_L v_L$$

$$\text{DIV}\ T_s + \text{GRAD}(\gamma_s Z + H) = 0 \qquad (3.7)$$

where

$$T_s = [\ \phi_2 \text{TRE} + \phi_3 \rho_L\ ]\ I + \phi_5 E$$

$$H = P + \gamma_L Z = \phi_7 \text{TRE} + \phi_8 \rho_L + \gamma_L Z$$

Note that the second of Eqs 3.7 is the well-known Darcy's Law.

If the first two equations are combined, the well-known flow or diffusion equation is obtained.

$$\frac{\partial \rho_L}{\partial t} - \text{DIV}(K\ \text{GRAD}\ H) = 0 \qquad (3.8)$$

This equation cannot be solved alone, however, because it is coupled to the balance of momentum for the solid through $H$ . Also, by observing the second equation, one would be tempted to say that the presence of the liquid affects the solid like a body force which is the gradient of the head causing flow $H$ . This is not, in general, true, since the liquid also affects the stress in the solid through the constitutive equation as well as through the balance laws. Thus, these results show that that Eq 3.6 can be put into a form that is similar to that now used, but the interpretation of the terms is different from that which has been used previously.

# CHAPTER 4. A DISCUSSION OF EXPERIMENTS THAT WILL DETERMINE THE CONSTITUTIVE PARAMETERS

Before any of this work can be of use, the $\phi$'s must be determined experimentally. In this chapter, a set of experiments that will determine the $\phi$'s will be discussed. Although there are infinitely many experiments that can be run, the prime consideration in the choice of the ones that will be presented is simplicity. These experiments will require measurements of quantities such as strain, concentration, and total stress. These measurements can be made with reasonable accuracy and simplicity.

The constitutive equations for isothermal stress conditions given by Eqs 4.1 contain five functions. Note that the function numbering given here is, as a matter of convenience, different from that given in Eq 3.4.

$$T_S = [\,\phi_1' \mathrm{TRE} + \phi_2 \rho_L\,]\, I + \phi_3' E$$

$$T_L = [\,\phi_4 \mathrm{TRE} + \phi_5 \rho_L\,]\, I \qquad (4.1)$$

In addition, a permeability function defined by Eq 4.2 is to be determined.

$$\mathrm{DIV}\; T_L + \rho_L b_L = \rho_L \phi_6 v_{LS} \qquad (4.2)$$

Thus, there is a total of six functions of the density of the fluid to be determined. This will require a series of four tests, all four performed for various values of $\rho_L$. A graphic description of five experiments which can be used is found in Figs 1 through 5. The following discussion indicates how these tests may be used.

If Eq 4.2 is rewritten using

$$PI = T_L$$

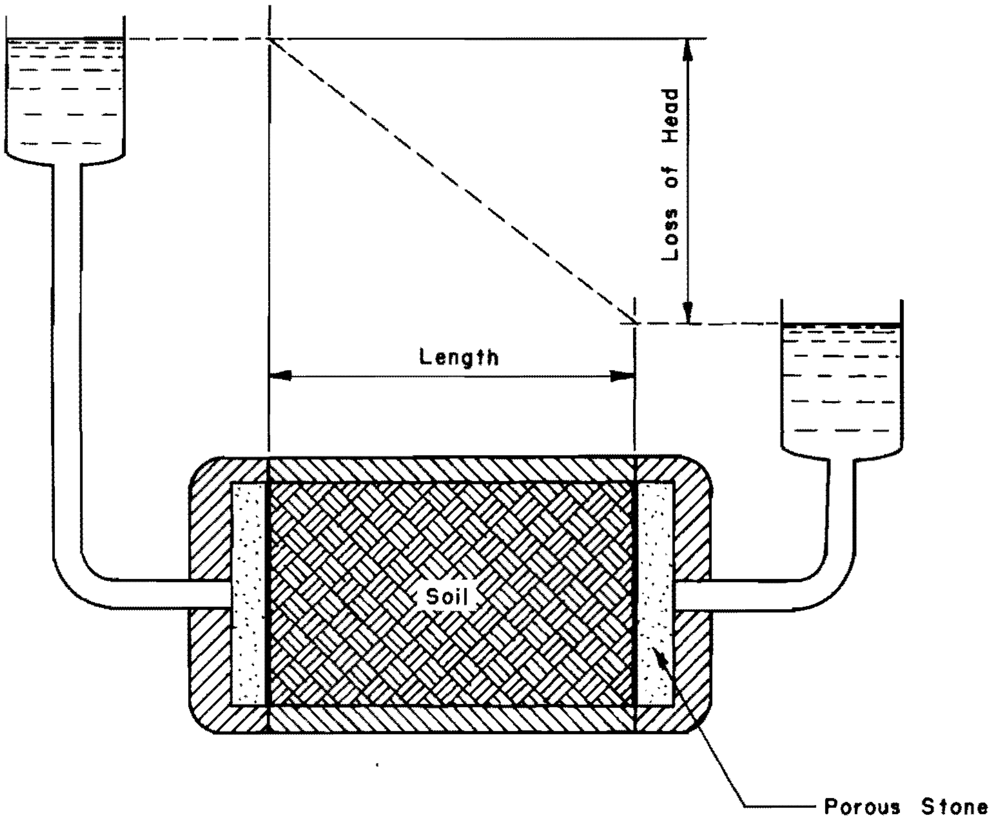

Fig 1. Schematic representation of Test No. 1, the permeability test, to measure K as a function of $\sigma_L$ .

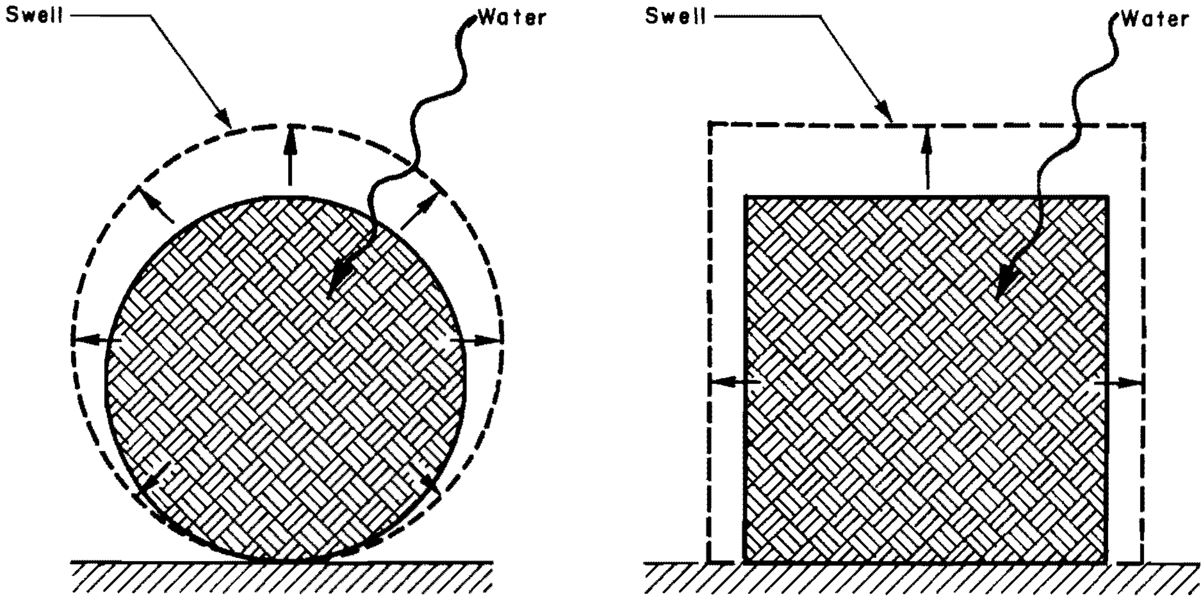

Fig 2. Schematic representation of Test No. 2 to determine $\frac{\phi_2}{\phi_1}$ and $\frac{\phi_5}{\phi_4}$.

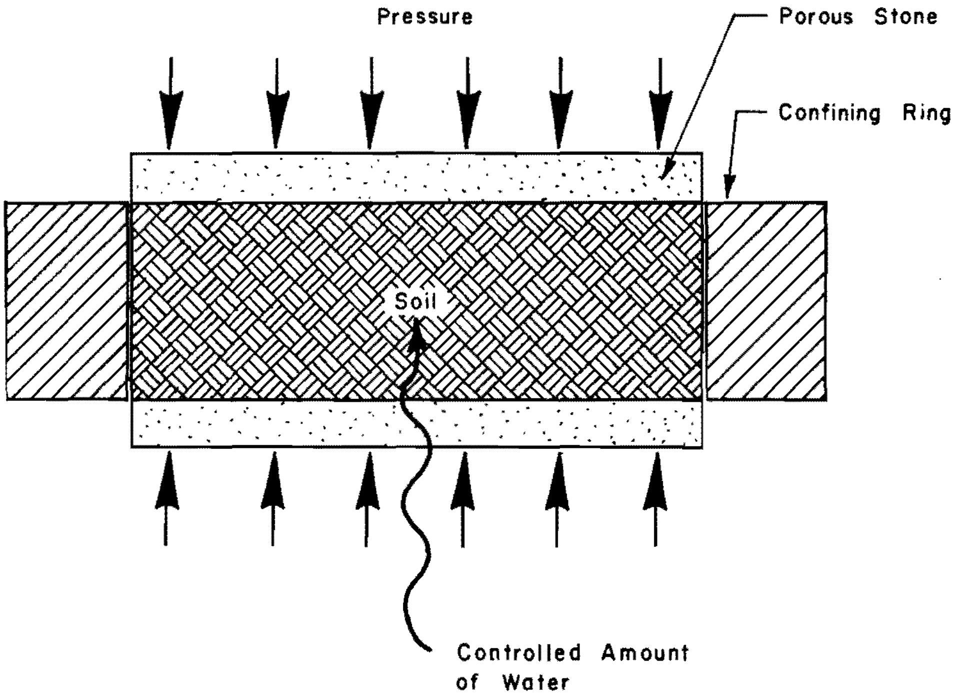

Fig 3. Schematic representation of Test No. 3, zero swell test, to measure $(\phi_2 + \phi_5)$ .

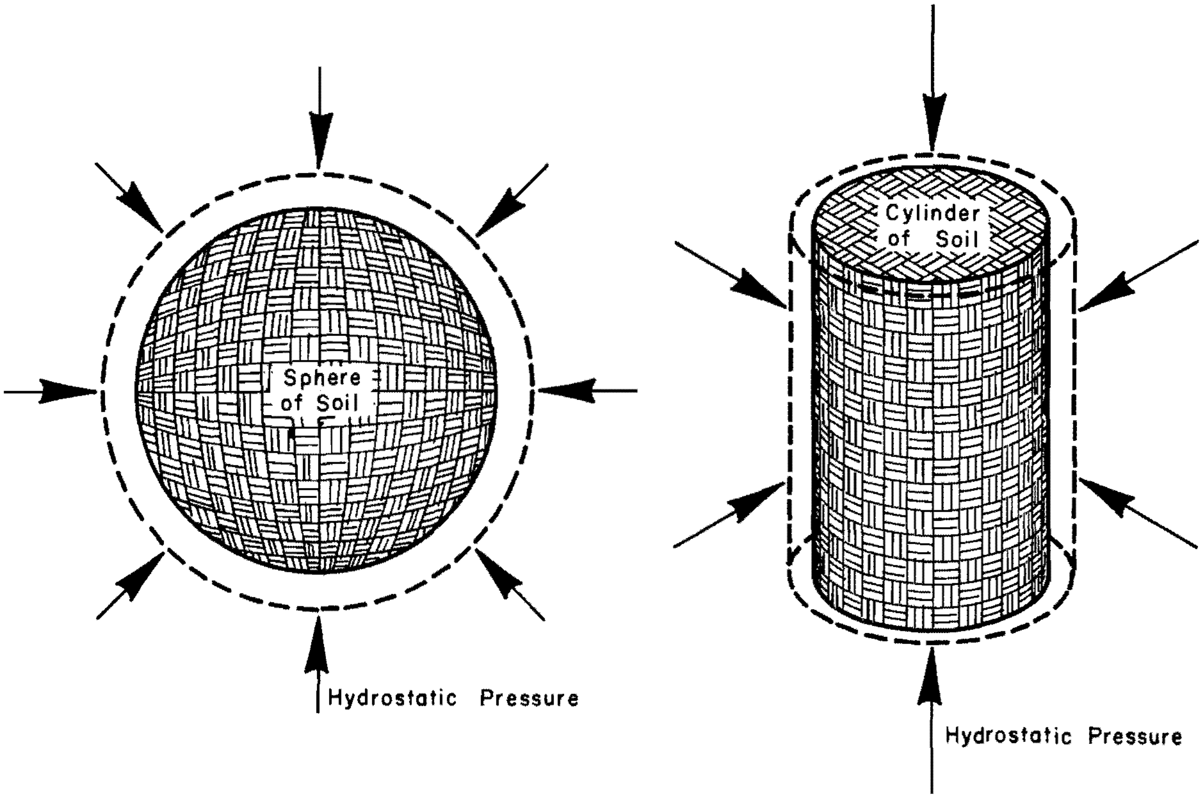

Fig 4. Schematic representation of Test No. 4, compression test, to determine the relations between $(\phi_2 , \phi_3)$ and $(\phi_4 , \phi_5)$ .

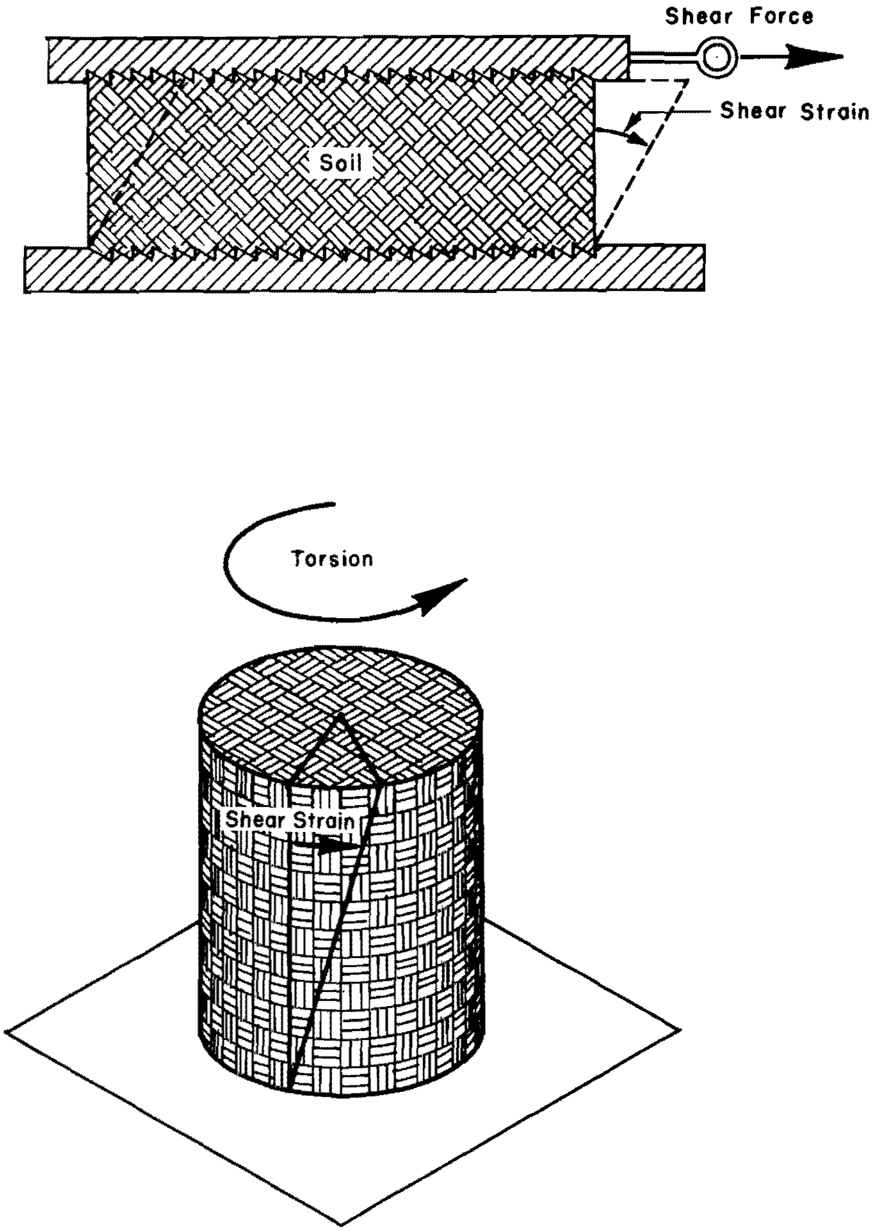

Fig 5. Schematic representation of Test No. 5 to determine shear modulus function $\phi_3$ .

$$\mathrm{GRAD}\gamma_L z = \rho_L b_L$$

$$\phi_6 = \frac{1}{\rho_L K} \tag{4.3}$$

We get

$$K\ \mathrm{GRAD}(P + \gamma_L z) = v_{LS} \tag{4.4}$$

which is well known as Darcy's Equation. The value of $K$ can be determined by any of several known experimental procedures. Once $K$ is known, $\phi_6$ follows immediately. This is Test No. 1, which is represented schematically in Fig 1.

The determination of the other parameters is greatly facilitated if the constitutive equations are rewritten using the deviatoric strain defined as

$$E^* = E - \frac{1}{3}\, I\ \mathrm{TR}E \tag{4.5}$$

Notice that

$$\mathrm{TR}E^* = 0 \tag{4.6}$$

so that if there is only a uniform volume change, $E^* = 0$ .

Using $E^*$ , the constitutive Eqs 4.1 can be written as

$$T_S = [\,\phi_1 \mathrm{TR}E + \phi_2 \rho_L\,]\, I + \phi_3 E^*$$

$$T_L = [\,\phi_4 \mathrm{TR}E + \phi_5 \rho_L\,]\, I \tag{4.7}$$

If a shear stress is applied to the material, the only parameter involved is $\phi_3$ . Thus $\phi_3$ is the shear modulus, and can be measured by a shear test on a sample at any given density of water. This is Test No. 5 shown in Fig 5.

The next type of experiment will yield two relations among the parameters. These two relations can be obtained because, in a soil and water specimen that is isolated from a source of water and has no external stress applied, the external stress in both the water and in the soil is zero. Thus if one starts with an initially dry spherical sample of soil and adds a known amount of water, the density of the water is known. If ample time is allowed for the water to move throughout the specimen, symmetry requires $E^*$ to be zero and the change in volume is easily measured, so that $E^*$ , TRE , $\rho_L$ , $T_L$ , and $T_S$ are all known. This is Test No. 2 shown in Fig 2 and it gives two relations among the remaining four unknown parameters.

The remaining two relations can be obtained from the knowledge of the total stress T . Remembering that the partial stress of a constituent $\alpha$ whose action on a surface is equivalent to the action of all constitutents exterior to the surface upon the material $\alpha$ interior to the surface, it is seen that at the boundary the partial stresses can be expressed as

$$T_S = \frac{\rho_S}{\rho_S^*} T$$

$$T_L = \frac{\rho_L}{\rho_L^*} T$$

$$T_G = \frac{\rho_G}{\rho_G^*} T \qquad (4.8)$$

where T is the total stress and $\rho_\alpha^*$ is the density of the free substance. Thus $\rho_S/\rho_S^*$ is the percentage of the volume occupied by the solid. Thus if $\beta$ is the porosity, $T_S$ can be expressed as

$$T_S = (1 - \beta) \frac{1}{3} \text{TRT I} + \left( T - \frac{1}{3} \text{TRT I} \right) \qquad (4.9)$$

The fourth test may consist of constraining an initially dry specimen so that it cannot change volume. If a known quantity of water is added, the density of the water is known. After waiting an ample time for the water to move throughout the material, the total stress required to constrain the material is easily measured. By using Eqs 4.8, the remaining two relations between the unknown parameters are found. This zero swell test is Test No. 3 and is shown in Fig 3.

Thus if these tests are performed for various values of the water density, the material parameters can be easily obtained as a function of the density.

The system of equations derived from these tests is given in Fig 6.

A compression test (Fig 4) may be used in place of the zero volume change test. In this test, a soil specimen of known water density is subjected to a known total stress and the volume strain is measured. A different set of equations is produced and these equations are shown in Fig 7.

$$\begin{bmatrix} A & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & A & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0 \\ B \\ C \\ 0 \\ D \end{bmatrix}$$

$A = \frac{TRE}{\rho_L}$ in Test No. 2

$B = \frac{(1-\beta)}{1-\beta(1-S)} \frac{\mathrm{tr}\,T}{3\rho_L}$ in Test No. 3

$C = \frac{T}{E^*}$ in Test No. 5

$D = \frac{S\beta}{1-\beta(1-S)} \frac{\mathrm{tr}\,T}{3\rho_L}$ in Test No. 3

$S$ = Degree of Saturation, Fraction Of Void Space Occupied by Water

Fig 6. Simultaneous equations to be solved for values of $\phi_1$ through $\phi_5$ for each level of $p_L$ using the swell test.

$$\begin{bmatrix} A & 1 & 0 & 0 & 0 \\ B & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & A & 1 \\ 0 & 0 & 0 & B & 1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \end{bmatrix} = \begin{bmatrix} 0 \\ C \\ D \\ 0 \\ E \end{bmatrix}$$

$A = \frac{TRE}{\rho_L}$ in Test No. 2

$B = \frac{TRE}{\rho_L}$ in Test No. 4

$C = \frac{1-\beta}{1-\beta(1-S)} \frac{tr\,T}{3\rho_L}$ in Test No. 4

$D = \frac{T}{E^*}$ in Test No. 5

$E = \frac{S\beta}{1-\beta(1-S)} \frac{trT}{3\rho_L}$ In Test No. 4

S = Degree of Saturation, Fraction
Of Void Space Occupied by Water

Fig 7. Simultaneous equations to be solved for values of $\phi_1$ through $\phi_5$ for each level of $\rho_L$ using the compression test.

# CHAPTER 5. ON THE BOUNDARY VALUE PROBLEMS

In addition to the equations that govern the system, the boundary and initial values must be known before a problem can be solved. In this chapter, discussion is presented on the various boundary conditions that will constitute a determinate system. In addition, a note on an interesting consequence for the one-dimensional problem is given. Finally, the system of equations for the nonisothermal case, which incorporates all of the other simplifying assumptions, is proposed.

## Boundary and Initial Conditions

Consider the governing isothermal system in the absence of body forces. Function numbering is as in Chapter 3, Eqs 3.4.

$$\frac{\partial \rho_L}{\partial t} + \mathrm{DIV}(\rho_L v_L) = 0$$

$$\mathrm{K\ GRAD\ P} = -\rho_L v_L$$

$$\mathrm{DIV\ T_S + GRAD\ P} = 0$$

$$P = \phi_7\ \mathrm{TRE} + \phi_8 \rho_L$$

$$T_S = [\phi_2\ \mathrm{TRE} + \phi_3 \rho_L\ ]\ I + \phi_5 E \tag{5.1}$$

This system can be rewritten as a system of two equations. When this is done, we obtain

$$\frac{\partial \rho_L}{\partial t} - \mathrm{DIV(K\ GRAD}\ \phi_8 \rho_L) - \mathrm{DIV(K\ GRAD}\ \phi_7\ \mathrm{TRE}) = 0$$

$$\text{GRAD}\ [\ (\phi_2 + \phi_7)\ \text{TRE}\ ] + \text{DIV}\ \phi_5\ \text{E} + \text{GRAD}\ [\ (\phi_3 + \phi_8)\ \rho_L\ ] = 0 \quad (5.2)$$

Notice that the second equation resembles that of elasticity, while the first resembles the diffusion equation.

In all the following, only the boundary conditions will be considered and the obvious fact that the initial distribution of the variables must be known is taken for granted. Because the first equation is a scalar one and the second is a vector equation, four pieces of information must be specified at each point on the boundary in a three-dimensional problem. Because the first equation is of the diffusion type and the second is like linear elasticity, a proper set of boundary conditions would be expected to consist of an appropriate boundary condition from each set. The conditions to be specified may be any of the following diffusion-type boundary conditions: $P$ , $\rho_L$ , or $v_L \cdot n$ (where $n$ is the normal to the surface), together with any of the following elasticity-type boundary conditions: $T_S n$ , $Tn$ , or $U$ (displacement vector). A linear combination such as $P + \alpha v \cdot n$ could also be specified as a diffusion-type boundary condition. Notice that a case where $P$ is specified is where the pores are open to the atmosphere and $P = 0$ . At an impermeable interface $v \cdot n = 0$ . Notice also that in most applications it is the total stress which will be specified instead of the stress in the solid.

One-Dimensional Case. An interesting phenomenon occurs in the one-dimensional case when all the constitutive parameters (the $\phi$'s and K) are constants. In this case

$$E = TRE = \frac{\partial U}{\partial X}$$

and Eqs 5.2 can be rewritten as

$$\frac{\partial \rho_L}{\partial t} - K\phi_8 \frac{\partial^2 \rho_L}{\partial X^2} - K\phi_7 \frac{\partial^3 U}{\partial X^3} = 0$$

$$(\phi_2 + \phi_5 + \phi_7) \frac{\partial^2 U}{\partial X^2} + (\phi_3 + \phi_8) \frac{\partial \rho_L}{\partial X} = 0 \quad (5.3)$$

Taking the derivative of the second with respect to X and eliminating $\partial^3 U/\partial X^3$ produces

$$\frac{\partial \rho_L}{\partial t} - \left[ K\phi_8 - \frac{(\phi_3 + \phi_8)\ K\ \phi_7}{(\phi_2 + \phi_5 + \phi_7)} \right] \frac{\partial^2 \rho_L}{\partial X^2} = 0 \qquad (5.4)$$

which is the diffusion equation. Here, an interesting phenomenon is observed: for the one-dimensional case, when the constitutive parameters are constant, the equations uncouple. As stated previously, two differential equations are said to be coupled if they must be solved simultaneously. This could account for the fact that the one-dimensional predictions based on a diffusion equation have been so successful.

## The Nonisothermal Case

Since the third chapter, this report has considered the isothermal case. This development led to two coupled equations with six constitutive parameters. The isothermal case was considered because the equations are simpler, and hence easier to interpret. This section presents the equations for the nonisothermal case, under the same assumptions as were used previously, and discusses them briefly.

The system of equations for the nonisothermal case is given in Eqs 5.5. For convenience, functions have been renumbered for these equations.

$$\mathrm{DIV}\ T_s + \rho_s b_s = \phi_1 v_{Ls}$$

$$\mathrm{DIV}\ T_L + \rho_L b_L = -\phi_1 v_{Ls}$$

$$\rho \dot{\varepsilon} = \rho h + \mathrm{DIV}\ q + (\phi_4 + \phi_8)\ \rho_L\ \mathrm{DIV}\ v$$

$$\frac{\partial \rho_L}{\partial t} + \mathrm{DIV}(\rho_L v_L) = 0$$

$$T_s = [\ \phi_2 \theta + \phi_3\ \mathrm{TRE} + \phi_4 \rho_L\ ]\ I + \phi_5 E$$

$$T_L = [\,\phi_6\theta + \phi_7 \text{ TRE} + \phi_8\rho_L\,]\, I$$

$$\varepsilon = \phi_9 \text{ TRE} + \phi_{10}\rho_L + \phi_{11}\theta$$

$$q = \phi_{12} \text{ GRAD } \theta + \phi_{13}v_{Ls} \tag{5.5}$$

Thus, instead of two coupled equations, there are three, and instead of six constitutive parameters, there are thirteen. The equation for the rate of internal energy production carries the assumption that changes of temperature will be relatively small.

One interesting thing can be observed from these equations, however. By defining $\phi_1$ as

$$\phi_1 = \frac{\rho_L}{K} \tag{5.6}$$

combining the second, fourth, and sixth lines of Eqs 5.5, and assuming $b_L = 0$, the following equation is obtained:

$$\frac{\partial\rho_L}{\partial t} - \text{DIV K GRAD } [\,\phi_6\theta + \phi_7 \text{ TRE} + \phi_8\rho_L\,] = 0 \tag{5.7}$$

This equation shows that a temperature gradient, as well as a volume change and a density gradient, will cause a flow of moisture. The magnitude of this effect, however, is directly proportional to $\phi_6$, i.e., if $\phi_6$ is much larger than $\phi_8$ or $\phi_7$, the temperature gradient will be the dominant factor in the movement of moisture.

# CHAPTER 6. CONCLUSIONS

As was stated in the introduction, the goal of this work is the simplest rigorous formulation of the problem that still describes the phenomenon. The system, Eq 3.5, appears to achieve this goal. However, the level of mathematical rigor used in this report needs to be discussed. This work was not planned to be a contribution to the basic formulation of mixture theory. The main contribution lies not in the development of the balance laws, since they are well known, but in the constitutive equations and the interpretation of the various results. Also, because this work was intended for use in soil mechanics, and in the interest of clarity, some mathematical rigor and precision has purposely been deleted.

The conspicuous absence of soil mechanics terminology and concepts such as pore pressure, suction, and effective stress is deliberate. A profusion of such terms exists because of the predominant microscopic viewpoint in soil mechanics. Many definitions and technical terms are required in order to adequately describe the different microscopic states of pressure and geometry in each of the soil constituents. In this report, no such terminology has been used in a deliberate attempt to formulate the problem in terms of well-known physical variables such as volume strain, shearing strain, volume concentration of soil and of water, and volume of water discharge. These quantities can be measured readily and no other assumptions are required.

A system of simultaneous differential equations has been proposed both for the isothermal and the nonisothermal case. Two coupled equations represent the isothermal case and three coupled equations represent the nonisothermal case. The equations are developed from the general field equations as a result of some explicit simplifying assumptions. These assumptions are as follows:

(1) The deformation of the soil is small.

(2) The velocities are small.

(3) The velocity gradients are small enough to be neglected.

(4) The stress in water and air is hydrostatic.

(5) The soil constituents are isotropic.

(6) There is no change in the mass of the soil.

(7) The changes of temperature will be small.

(8) The mass and momentum of the gas may be neglected.

In the isothermal case, assumption (7) becomes

(7a) There are no temperature changes or gradients.

Constitutive relations carrying the same assumptions require that a total of six material functions be found for the isothermal case. Thirteen material functions are required for the nonisothermal case.

A series of four different tests run at several water content levels is required for measuring the six isothermal material functions. Five simple tests are shown which may be used in determining these functions. Both suggested testing procedures use the following three tests:

(1) permeability test,

(2) shear test, and

(3) free swell test.

The first suggested method uses a swell pressure test in addition to these three. The second suggested method uses a constant water content compression test along with the three mentioned above. Methods of combining these test data to determine values of the material functions are demonstrated in Chapter 4.

The equations presented in this report represent a simple but comprehensive description of the mechanics of behavior of a soil mass which changes volume under the influence of water movement.

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