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HIGHWAY BRIDGE LIVE LOADS BASED ON LAWS OF CHANCE*

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SYNOPSIS

A new method, based on elementary probability theory, has been developed for estimating live load frequencies on highway bridges that may be expected from various types and levels of heavy motor vehicle operation. The main objective of the method is to provide a relatively simple mathematical basis for estimating approximately how often any specified sequence or group of two or more vehicles might be expected to occur on any particular part or length of bridge as a result of given or anticipated compositions, volumes, and speeds of traffic. In addition to making use of the frequency distributions of heavy vehicle loads obtained from loadometer surveys, the new method provides the means for estimating the frequencies of various intensities of live load which result from the change grouping of two or more heavy vehicles on a given part or length of bridge at the same time.

INTRODUCTION

This paper is concerned with a study of live load frequencies on highway bridges which result from the chance grouping of vehicles in traffic. Its object is to present a new method for estimating the frequencies of various intensities of these loadings; and to show how such loadings for a given span may be related to their stress producing characteristics and effects. It deals with the problem of vehicle grouping from a mathematical standpoint, based on the same elementary laws of chance or probability that have already been used successfully for solving many types of frequency problems encountered in the various branches of science and engineering.(1) It presents:

1. A discussion of the factors in highway traffic which influence the spacings and frequencies of individual vehicles and vehicle groups.

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2. The development of the mathematical equations for estimating specified vehicle group frequencies and a discussion on their uses.

3. A few selected graphs and tables, covering typical problems relating to vehicle grouping, which illustrate how the method may be used for estimating approximately how often various sequences or groups of specified vehicles might be expected to occur within specified lengths of time, or distance along the highway.

Need of Method for Analyzing Bridge Loading Frequencies

The selection of the proper live load to be used for the design of various parts and types of highway bridges represents one of the most important as well as difficult problems encountered by those responsible for the planning of such structures. In large measure, the choice of a design live load not only determines the maximum sizes and weights of vehicles but also their speeds, spacings and other operating conditions necessary to insure that a given bridge will perform safely and economically the functions for which it was intended.

The successful planning of any particular bridge requires that the engineer have adequate information concerning the site and foundation conditions; a thorough knowledge of bridge design procedures and how they are related to the physical properties of the materials to be used in its construction; and last, but perhaps most important, he must somehow arrive at a design live load that will be commensurate with present as well as anticipated traffic conditions. Satisfactory procedures are presently available for performing each of the several operations involved in the planning of a bridge, except for that of determining the live load for which it should be designed. As a partial contribution toward the fulfillment of this need, this paper presents a new method for analyzing the frequencies of heavy vehicle loadings which provides a simple yet rational mathematical procedure for selecting a design live load consistent with the other requirements which may obtain for any given structure. The method also provides the means for investigating the adequacy of existing bridges of given design designation.

Mathematical Basis for Study of Vehicle Group Frequencies

1. General Discussion

The proper design live load for highway bridges is not only a function of the sizes, weights and frequencies of individual heavy vehicles found on the highways, but also of the frequencies of various intensities of loading that might be expected to occur on a given part or length of bridge, as a result of the chance grouping of two or more of these heavy vehicles in traffic. Fortunately it is only necessary to make a few simplifying assumptions concerning the behavior of highway traffic in order to apply the theory of probability to the chance grouping of vehicles and the frequency of specified vehicle groups. These assumptions may be stated as follows:

a) That vehicles, both individually and by types, are distributed at random in ordinary highway traffic.

b) That the average composition, volume and speed of traffic remain constant during the time period under consideration.

The first assumption means that the time and distance spacings of vehicles occur entirely by chance and not as a result of artificial control. Similarly, it means that the various vehicle types—such as automobiles, busses and trucks—occur entirely by chance throughout the traffic stream. The second assumption merely means that the time period under consideration must be of short enough duration to insure that the average composition, volume and speed of the traffic remain constant during that time. At certain times this time period could be several hours; but at others when the characteristics of the traffic are changing rapidly, the time period may be only one half or quarter hour.

Numerous studies by the author and others have demonstrated that the above assumptions approximate the actual behavior of ordinary highway traffic sufficiently close for solving many types of traffic problems now thought to be incapable of solution by mathematical means. Moreover these studies have shown that the time and distance spacings of vehicles—both individually and by groups—in ordinary traffic agree rather closely with the distributions given by the Poisson frequency distribution formula; also known as Poisson's law. This means therefore that the probability of vehicle groups of unspecified types occurring within specified lengths of time or distance can be estimated mathematically by use of Poisson's law. Once this probability has been determined, the probability that the group consists of certain specified vehicles or that they are arranged in some particular order may be found by use of the basic theorems for calculating simple and compound probabilities. It should be mentioned also that Poisson's law has also been found to provide a very good estimate of the frequency distribution of various intensities of heavy vehicle loads measured in terms of their H truck loading equivalencies on a given span. (See pages 427-438, Ref. 2)

2. Basic Theorems for Calculating Simple and Compound Probabilities

The fundamental theorems for calculating simple and compound probabilities are fully explained in almost any book on college algebra. For this reason it will only be necessary here to state these theorems and illustrate how they may be applied to a few simple situations to show how they lead more or less automatically to the Binomial and Poisson frequency distributions. Special emphasis is placed on the Poisson frequency distribution because it is the limit of the Binomial distribution and also because it is the simpler of the two to use in many cases.

Fundamental Theorems

Events of a set are usually classified as being independent, dependent, or mutually exclusive. The theorems corresponding with these classifications are, respectively:

Theorem 1 - The probability that all of a set of independent events will happen on a given occasion when each of them is possible is the product of their separate probabilities of occurrence.

Theorem 2 - If the probability of a first event is P_1, and if, after this has happened, the probability of a second event is P_2; then the probability that both events will happen in the order specified is P_1P_2 (the obvious extension of this to m events would result in the probability, $P_1P_2 \ldots P_m$).

Theorem 3 - The probability that one or the other of a set of mutually exclusive events will occur is the sum of the probabilities of occurrence for the separate events.

3. The Binomial Distribution

The binomial distribution is given by the successive terms of the expansion of the binomial:

$$(q+p)^m = C_m^m q^m p^0 + C_m^{(m-1)} q^{(m-1)} p^1 + C_m^{(m-2)} q^{(m-2)} p^2 + \cdots + C_m^0 q^0 p^m \quad (1)$$

in which p = probability of success on any one trial
q = probability of failure on any one trial
and m = number of trials (sample size or lot size)
also $p < 1$, and $q = 1-p$

In this binomial expansion, the symbol C_m^n means the number of combinations of m things taken n at a time. This may be expressed algebraically as follows:

$$C_m^n = \frac{m!}{n!(m-n)!} \quad (2)$$

This may be illustrated by inquiring the number of 3 letter combinations that can be obtained from the 4 letters; a, b, c, and d. This may be done in the following 4 ways:

abc, abd, acd, and bcd

and by the above algebraic expression, this would be determined as follows:

$$C_m^n = C_4^3 = \frac{4\cdot3\cdot2\cdot1}{3\cdot2\cdot1(1)} = 4 \quad (3)$$

With this in mind, it may now be explained that each term in the above binomial expansion gives the probability of exactly n successes in a set of m trials and each term may be written thus:

$$P_m(n) = C_m^n q^{(m-n)} p^n \quad (4)$$

in which the symbol $P_m(n)$ means the probability of n successes in a given sample of m trials, where n = 0,1,2,3,. . . ,m. In other words, the first term gives the probability of no successes in m trials; the second term, the probability of 1 success in m trials; and so on to the last term which gives the probability of m successes in m trials. In this connection, it should be noted that any given sequence or set of m trials each may be thought of as a sample of size m or a lot of size m.

Perhaps the simplest way to explain the meaning of the binomial distribution is to apply it to the tossing of one or more coins. On a single toss of a coin it can fall in 2 ways, either a head or a tail, each of which is equally likely. Now if 2 coins are tossed at the same time (or one coin tossed twice in succession) they may fall in any one of the following 4 equally likely ways: TT, TH, HT, HH. Here, it will be noted that 1 of the 4 ways is favorable to 2 tails (no heads); 2 of the 4 ways are favorable to 1 head and 1 tail (one head); and 1 of the 4 ways is favorable to 2 heads.

Now if the tossing of a head is considered a success and a tail considered a failure, then according to the above nomenclature: p = .5 and q = .5, from which it will be seen that the binomial expansion

$$(q+p)^2 = q^2 + 2qp + p^2 \tag{5}$$

gives the same results as were obtained by enumerating all the different combinations that could be obtained from the tossing of a single coin twice in succession (or the tossing of 2 coins simultaneously). The first term of this expansion means that the probability of no successes (2 tails) is q^2; the probability of 1 success (1 head and 1 tail) is 2pq; and the probability of 2 successes (no tails) is p^2.

Similarly the probabilities of obtaining no heads, 1 head, 2 heads, and 3 heads in any 3 tosses of a single coin (or a single toss of 3 coins) would be given by the 4 respective terms of the binomial expansion for 3 trials per sample or sample size of m = 3, thus:

$$(q+p)^3 = q^3 + 3q^2p + 3qp^2 + p^3 \tag{6}$$
$$= .125 + .375 + .375 + .125 \tag{6a}$$

This means that the probability of getting no heads (3 tails) is .125; the probability of getting 1 head is .375; the probability of 2 heads is .375; and the probability of getting 3 heads is .125.

From this it will be seen that the calculation of values for the successive terms in a binomial becomes quite laborious when m is large. A better appreciation of the time required to make such calculations may be obtained by examining the binomial expansion for m = 5, which is as follows:

$$(q+p)^5 = q^5 + 5q^4p + 10q^3p^2 + 10q^2p^3 + 5qp^4 + p^5 \tag{7}$$

Now if the number of trials or sample size, m, were increased, to say 100, it will be seen that the time required to evaluate the 101 terms of such a binomial distribution would be considerable to say the least. It is for this reason that resort is made to approximations of the binomial distribution in many practical problems where the number of trials per sample or sample size is large.

The Poisson distribution, for example, is used in many practical situations to approximate the values of a specific binomial distribution, particularly in cases where the sample size is large. The agreement between the binomial and the Poisson distributions, however, increases as the sample size increases. In fact, the binomial distribution tends to approach the Poisson distribution as a limit as the number of trials or sample size becomes very large.

Use of Binomial Distribution for Sampling

In order to simulate a continuous process, suppose that a large bin is continuously being supplied or filled as needed with balls which are identical in every respect except that 80 per cent of them are white and 20 per cent of them are black. Now, if these balls are withdrawn at random from the bin and put into boxes containing 5 balls each, what proportion of the boxes would be expected to contain n black balls, where n = 0,1,2,3,4, and 5, respectively?

If a single ball is withdrawn, the probability of its being black would be p =.2, and similarly the probability of its being white would be q = .8. Under these conditions, the expected frequency of appearance of 0,1,2,3,4, and 5 black balls among the boxes of 5 balls each (sample size m = 5) can be calculated by evaluating the successive terms of the expansion of the binomial.

$$(.8 + .2)^5 = .3277 + .4096 + .2048 + .0512 + .0064 + .0003 \tag{8}$$

This means that 32.77 per cent of the boxes would be expected to contain no black balls; 40.96 per cent, 1 black ball; 20.48 per cent, 2 black balls; 5.12 per cent, 3 black balls; 0.64 per cent, 4 black balls; and only about 3 of each 10,000 boxes would be expected to contain 5 black balls.

Comment

If the drawing of a black ball is considered a success, and the letter K is used to indicate the average number of successes per sample or box of 5 balls each, then

$$K = mp \tag{9}$$

$$= (5)(0.2) = 1 \tag{9a}$$

which means that the average number of successes (black balls) per sample would be 1. In general, this means that the average number of successes, K, expected per sample is equal to the probability of success on a single trial, p, times the number of trials per sample or sample size m.

4. Development of the Poisson Distribution

In the preceding discussion it was explained that each term in the binomial expansion gives the probability of exactly n successes in a set of m trials and may be written thus:

$$P_m(n) = C_m^n \, q^{(m-n)} p^n \tag{4}$$

in which the symbol P_m (n) means the probability of n successes in a given sample of m trials where n = 0,1,2, . . . , m.

In the case of the binomial law, it was shown that the average number of successes, K, expected per sample (expectation of n) is equal to K = mp.

With this information, it can now be shown that the binomial distribution approaches the Poisson distribution as a limit as the number of trials m become very large. This development is accomplished by first noting that the probability p may be determined thus:

$$p = \frac{K}{m} \tag{10}$$

and if the value of p is now substituted in the above equation, it becomes:

$$P_m(n) = C_m^n \left(\frac{K}{m}\right)^n \left(1 - \frac{K}{m}\right)^{(m-n)} \tag{11}$$

Now if the operations indicated in this equation are carried out and the intermediate steps are omitted, it can be shown (see page 214, Ref. 1 or page 372, Ref. 2) that:

$$P_m(n) = \left[\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdot\cdot\cdot\left(1-\frac{n-1}{m}\right)\right] \left[\left(1-\frac{K}{m}\right)^{-n}\right]\left[\left(1-\frac{K}{m}\right)^{m}\right]\left(\frac{K^n}{n!}\right) \tag{12}$$

By remembering that p is rather small, it is obvious that only those values of n are of consequence which are very small as compared to m which is very large. On this basis, therefore, each of the factors enclosed within the first set of brackets becomes approximately equal to unity, as m becomes larger and larger compared with n. The same is true of the quantity 1 - (K/m) which occurs in the secondand third brackets, because K/m, or p, is very small. Therefore, since there are comparatively few of these factors in the first 2 sets of brackets, it follows that their product is also not greatly different from unity and actually approaches unity as m becomes very large compared with n.

The same line of reasoning cannot be applied to the factor within the third bracket, however, owing to the fact that the quantity 1 - (K/m) is raised to a very large power. By consulting almost any text on algebra or calculus, it will be found that the expression in the third bracket is equal to e^{-K}, or

$$\left(1-\frac{K}{m}\right)^m = e^{-K} \tag{13}$$

in which e = 2.71828 (Base of Napierian or natural logarithms).

On the basis of this line of reasoning, therefore, one would be justified in concluding that in the limit

$$P_m(n) = \frac{K^n e^{-K}}{n!} \tag{14}$$

which is known as the Poisson distribution or Poisson's law. The important thing to note here is that the binomial law approaches the Poisson law as a limit as m becomes very large. The successive terms of the binomical expansion therefore have as their limits the corresponding terms in the Poisson distribution, as follows:

$$P(n) = e^{-K} + Ke^{-K} + \frac{K^2 e^{-K}}{2!} + \frac{K^3 e^{-K}}{3!} + \cdots = 1 \qquad (15)$$

for n = 0,1,2,3. . .

The successive terms in this series may be interpreted as the proportion of samples in which 0,1,2,3, . . . of some specified event would be expected to occur when the average number of occurrences per sample is K = mp.

Comment

One of the principal advantages of using the Poisson distribution as an approximation to a specific binomial distribution is the comparative ease with which the successive terms of the Poisson series may be evaluated. Actually, though, there is rarely ever any occasion for making such calculations since tables (See Ref. 3, or pages 380-384, Ref. 2) are available that cover a wide range of values for K = mp.

Frequency of Specified Vehicle Groups Occurring Within Specified Lengths

1. General Discussion

Assuming the average composition, volume and speed of traffic remains constant during the time period under consideration, the problem of estimating the frequency of specified vehicle groups occurring within specified lengths of time or distance is most conveniently handled by breaking it down into the following three parts:

a) First Part - Perhaps the most common situation requiring consideration in the first part consists of calculating the probability of n vehicles, unspecified as to type, occurring at a given location in any manner in either or both directions of travel, within a specified interval of t seconds or a specified length of X feet (such as a bridge) along the highway. The next most important situation no doubt consists of calculating the probability of n unspecified vehicles occurring simultaneously in each direction of travel, within a specified interval of t seconds or x feet; when the traffic volume and speed is the same in each direction. Many other situations could be defined involving different vehicle group sizes n, as well as different average volumes and speeds of traffic in each direction; but these will not be considered here owing to space limitations. It should be added, though, that these situations can be calculated quite as easily and in the same way as the more important situations mentioned above. Once these probabilities have been found, the frequencies of the events under consideration can be readily determined.

b) Second Part - The second part consists of calculating the probability of and group of n unspecified vehicles, selected in a random manner from the traffic stream (such as the group of n unspecified vehicles in the preceding step) occurring as to type or arrangement as previously specified. Once this probability has been found, the frequency of the event can be easily determined.

c) Third Part - The third part consists of calculating the combined probabilities or frequencies from those found in the first and second parts. This gives the desired information concerning the frequency of specified vehicle groups occurring within specified lengths of time or distance.

2. First Part of Problem

a) Occurrence of n Unspecified Vehicles in Either or Both Directions

At any particular location on a highway and for any given average composition, volume and speed of traffic, the probability of n vehicles, unspecified as to type, occurring in any manner in either or both directions of travel within a time interval of t seconds or a distance of X feet is given by Poisson's formula as follows:

$$P(n, X; \frac{a}{2}) = \frac{K^n e^{-K}}{n!} \tag{16}$$

in which K is the average number of vehicles expected within the distance X, based on the total number of vehicles per hour (both directions) at the given location. Thus,

$$K = \left(\frac{\text{Number of vehicles per hour}}{\text{Average speed in miles per hour}}\right)\left(\frac{X}{5280}\right) = \left(\frac{R}{D}\right)\left(\frac{X}{5280}\right) = \frac{RX}{5280D} \tag{17}$$

If time instead of distance is used to measure the interval in which n vehicles is to occur in any manner in either or both directions, the probability that they will occur within t seconds is also given by Poisson's formula as follows:

$$P(n, t; \frac{a}{2}) = \frac{K^n e^{-K}}{n!} \tag{18}$$

This equation would be read: The probability of n vehicles occurring in any manner in either or both directions in t seconds is given by the Poisson formula in which K is the average number of vehicles per time interval of t seconds. Equation (16) would be read similarly using distance X instead of t seconds; and using K as the average number of vehicles expected in X distance; or the average number of vehicles per cell of X feet in length along the highway.

To illustrate the use of Eqs. (16) and (18) suppose a traffic volume, R, of 500 vehicles per hour (12,000 vehicles per day) with an average speed, D, of 39.457 m.p.h. is used and it is desired to know the probability of n vehicles occurring within a length, X of 125 ft. or in a time interval corresponding to the number of seconds required to travel 125 ft. at 39.457 m.p.h., or a time, t, of 2.16 seconds. Thus

$$\text{for } X = 125 \text{ ft.} \qquad K = \frac{500 \times 125}{5280 \times 39.457} = .3$$

$$\text{and for } t = 2.16 \text{ sec.} \quad K = (500 \times 2.16)/3600 = .3$$

The average number of vehicles expected within the 125 ft. length, therefore, is 0.3, and the average number of vehicles expected in each 2.16 sec. interval is also 0.3.

Suppose it is desired to know the probability that no vehicles will occur within any 2.16 sec. interval. This is given by Eq. (18)

$$P(0, 2.16 \text{ sec.}; a/2) = (.3^0 \times e^{-.3})/0! = e^{-.3} = .7408182$$

This means that 74.08% of the 2.16 sec. time intervals will contain no vehicles. Solved for other values of n, Eq. (18) gives the following results for K = .3.

Table 1

for K = .3

n	Individual Terms	Cumulative Terms
0	.7408182	1.0000000
1	.2222455	.2591818
2	.0333368	.0369363
3	.0033337	.0035995
4	.0002500	.0002658
5	.0000150	.0000158
6	.0000008	.0000008
Total	1.0000000	

From the individual terms it will be seen that 74.08% of these time intervals would contain no vehicles; 22.22% of them would contain one vehicle; 3.33% would contain 2 vehicles and so on. If distance instead of time were considered, the probability of n vehicles occurring in X distance of 125 ft. (K = .3) would be given by Eq. (16) and would be the same as shown for t = 2.16 sec. (K = .3).

The cumulative terms on the right are also informative. They show that 100% of the time intervals (t = 2.16 sec.) will contain none or more vehicles; 25.92% of them will contain one or more; 3.69% will contain 2 vehicles or more, and so on.

If the above frequency distribution for K = .3 were applied to a very large number of intervals (observations, samples, or trials)—say ten million—the total number of vehicles involved would be 10,000,000 x .3 = 3,000,000. The distribution of these 3,000,000 vehicles among the 10,000,000 intervals would be as follows:

Table 2

Distribution of Three Million Vehicles among Ten Million Intervals for K = .3

No. of Vehicles n	No. of Intervals with n Vehicles	Total No. of Vehicles
0	7,408,182	0
1	2,222,455	2,222,455
2	333,368	666,736
3	33,337	100,011
4	2,500	10,000
5	150	750
6	8	48
	10,000,000	3,000,000

From these figures it will be seen that 2,222,455/3,000,000 = 74.08 per cent of the vehicles occur on the length X = 125 feet (or t = 2.16 sec.) one at a time; and, similarly, 22.22 per cent are on the 125 ft. length 2 at a time; and 3.33 per cent are on it 3 at a time and so on.

Note, for example, that 33,337 of the intervals contained 3 vehicles each; and since there are a total of three million vehicles in all the intervals, this means that on the average 3,000,000/33,337 = 90 vehicles would pass for each time that 3 vehicles occurred simultaneously within the interval. In this case 90 is the vehicle interval, V.

In the 2nd column of Table 2, it will be noted that if the decimal point is moved 7 places to the left, the numbers will be the same as the probability values given by the individual terms of the distribution in the 2nd column of Table 1. Also if the decimal in the 3rd column of Table 2 is moved 7 places to the left, it will be noted that the total number of vehicles would be 0.3, which is the same as K. This shows that the vehicle interval, V, required for each occurrence of n vehicles within the defined interval is found by dividing the average number, K, per interval by the probability of n occurring.

This means that the number of vehicles, on the average, that would be required to pass for each occurrence of a given event—that is, the vehicle interval would be calculated thus

$$V(n, X; a/2) = K/P(n, X; a/2)$$
$$\text{or} \quad V(n, t; a/2) = K/P(n, t; a/2)$$

depending on whether the length is measured in time or distance.

For example, with a total traffic volume of 500 vehicles per hour, the vehicle interval required, on the average, between occurrences of 3 vehicles within the 125 ft. length (K = .3) would be

$$V(3, 125; a/2) = 0.3/.0033337 = 90$$

The time interval, on the average, between the occurrences of 3 vehicles within X = 125 ft. for the traffic conditions defined above, would be the vehicle interval, V, divided by the rate, R, thus:

$$T(n, X; a/2) = V(n, X; a/2)/R$$
$$= 90/500 = .18 \text{ hrs.}$$

For given averages volumes and speeds of traffic, the above procedure provides the means for determining the probabilities, vehicle intervals and time intervals associated with the occurrence of n unspecified vehicles, in any manner in either or both directions of travel, within a specified time interval of t seconds or X feet.

b) Occurrence of n Unspecified Vehicles in Each Direction When Average Volumes and Speeds of Traffic are the Same in Each Direction

The determination of probabilities, vehicle intervals, and time intervals, for events involving the occurrence of n unspecified vehicles in each of the two directions of travel, is very similar to that given above for events involving the occurrence of n unspecified vehicles in any manner in either or both directions of travel. In the case of n vehicles occurring in each direction, it is only necessary to determine the average number of vehicles per time or distance interval for each direction individually. And if the traffic in each direction is the same then k_1 in direction 1 is equal to k_2 in direction 2; and if they were different k_1 would not be the same as k_2. The present discussion though is confined to situations where $k_1 = k_2$.

Therefore the probability of n unspecified vehicles occurring in each direction within an interval of t seconds or X feet is given by the product of the separate probabilities indicated by Poisson's formula for each direction, individually, as follows:

$$P(n, X; 2) = (k^n e^{-k}/n!)^2$$

In this case k is the average number of vehicles per interval or cell in each of the two directions.

And as previously explained, the vehicle interval will be

$$V(n, X; 2) = 2k/(k^n e^{-k}/n!)^2$$

in which 2 k is the average number of vehicles per cell for total traffic.

Also as previously shown, the time interval would be the vehicle interval divided by the number of vehicles per unit length of time, or

$$T(n, X; 2) = V(n, X; 2)/R$$

3. Second Part of Problem

The probability of n unspecified vehicles, selected in a random manner from the traffic stream, occurring as to type or arrangement as previously specified is found by use of the simple and compound probability theorems given in Part IV of this paper. A more complete discussion of them though may be found in almost any book on college algebra.

For example, consider a traffic composition consisting of 75% M, 20% L and 5% H. If a group of 2 vehicles are selected at random, what is the probability that they both are H, or heavy vehicles? It would be calculated thus

$$P(2H) = (.05)(.05) = .0025$$

The frequency with which this event will occur is the number of trials required on the average for each success. In this case

$$E(2H) = 1.0/P(2H) = 1.0/.0025 = 400$$

If it were desired to investigate groups of vehicles containing two or more types, such as H and L vehicles, the probabilities associated with various permutations and combinations can be calculated without great difficulty but space does not permit a discussion of them here. This type of calculation can also be found in almost any college algebra.

So, for present purposes it is believed that the above illustrations will suffice to calculate the probabilities and frequencies pertaining to specified vehicle groups.

4. Third Part of Problem

For given traffic conditions, the probability of n specified vehicles occurring within specified intervals of time or distance is merely the product of the two separate probabilities calculated in the two preceding parts of the problem, respectively.

For example, suppose the traffic conditions are as follows: 500 vehicles per hour (12,000 vehicles per day) equally divided between the two directions; average speed of 39.46 mph; and traffic composition 75% M, 20% L, and 5% H vehicles.

For these conditions, suppose it is desired to know how often 4 heavy vehicles will occur on a 125 ft. span; with K = .3 and k - .15. From Table 1 it will be found that the probability of 4 unspecified vehicles occurring on the 125 ft. span is

$$P(4,\ 500;\ a/2) = .00025$$

from which the vehicle interval is determined, thus

$$V(4,\ 500;\ a/2) = .3/.00025 = 1200$$

which results in the time interval

$$T(4,\ 500;\ a/2) = 1200/500 = 2.4 \text{ hours}$$

Then the probability that these 4 vehicles will be heavy vehicles is calculated thus

$$P(4H) = (.05)^4 = .000,006,25$$

and the number of trials or events required for each success is given by

$$E(4H) = 1.0/.000,006,25 = 160,000$$

Therefore the time interval required for each occurrence of 4H on the 125 ft. span would be

$$T(4H,\ 125;\ a/2) = 2.4 \text{ hrs. x } 160,000 = 384,000 \text{ hrs.} = 44.9 \text{ yrs.}$$

These illustrations will suffice to indicate how the frequencies, vehicle intervals and time intervals for the occurrence of specified vehicle groups within specified intervals of time or distance may be evaluated.

For the same traffic conditions, if it were now desired to know how often 2 heavy vehicles will occur in each direction simultaneously on this 125 ft. span, the results for the unspecified vehicles (omitting the detail calculations) would be:

$$P(2, 125; 2) = .000,093,76$$
$$V(2, 125; 2) = 3200$$
$$T(2, 125; 2) = 3200/500 = 6.4 \text{ hours}$$

and for the specified vehicle groups the time interval would be

$$T(2H, 125; 2) = 6.4 \times 160,000 = 1,025,000 \text{ hrs.} = 117 \text{ yrs.}$$

Other time intervals for the same traffic composition and varying numbers of unspecified and specified vehicle groups on spans from 10 to 500 feet in length are given by the graphs in Figures 1 and 3 for 250 vehicles per hour (6000 vehicles per day) and Figures 2 and 4 for 500 vehicles per hour (12,000 vehicles per day).

Heavy Vehicle Frequencies Related to Design Stresses in Bridges

If one considers the simple situation of an ordinary bridge on a main rural highway where the traffic may be considered distributed at random, it will be found that two or more heavy vehicles (those weighing in excess of about 13 tons) in each of the two directions of travel would occur so seldom on bridges of 500 feet or less in length that the effects of such loadings might be neglected in so far as their effects on design stresses are concerned.

For ordinary highway bridges, therefore, the most severe loading condition that need be considered (at normal service load allowable stresses) is for one heavy vehicle to occur in each of the two directions of travel at the same time. For example, if one considered a traffic volume of 500 vehicles per hour or 12,000 vehicles per day containing 5 per cent heavy vehicles, it will be found from Figure 4 that one heavy vehicle would occur in each of the two directions of a 50 ft. span, within a critical 10 or 12 ft. length at or near the mid-span about 80 times per year; and, for this same traffic, one heavy vehicle would occur in each of the two directions of a 100 ft. span, within a critical 20 to 25 ft. length at or near the mid-span, about 120 times per year.

Similarly, if a traffic volume of 250 vehicles per hour or 6000 vehicles per day containing 5 per cent heavy vehicles is considered (which is a very high volume for main rural roads and also an extremely high concentration of heavy vehicles) it will be found from Figure 3 that one heavy vehicle would occur in each of the two directions on a 50 ft. span, within a critical 10 or 12 ft. length at or near the mid-span, about 20 times a year. And, for this same traffic, one heavy vehicle would occur in each of the two directions of a 100 ft. span, within a critical 20 or 25 ft. length at or near the mid-span, 35 times a year.

But even though two heavy vehicles do occur within a critical distance, at or near the mid-span of a given bridge, several times a year, the probability that both vehicles would either be the least or the greatest H-equivalency encountered in such traffic is so remote that it may be neglected. In fact, it can be shown that the two heaviest vehicles likely to occur on a 50 ft. bridge at the same time would produce less stress than a single vehicle with one of the higher H-equivalencies.

In order to illustrate some of the implications of the above discussion, the loading and stress frequencies, resulting from a traffic volume of 500 vehicles per hour (12,000 vehicles per day) with 5 per cent heavy trucks, will be considered in both a 50 ft. and a 100 ft. simple span bridge. And since the stress effects of overload are greater in bridges with the smaller ratios of dead load to design load stresses, the lightest type of ordinary construction will be considered; namely, bridges with a minimum thickness concrete deck supported by longitudinal steel beams or girders. The ratios of dead load stress to total design stress, and live load plus impact stress to total design stress for spans of various lengths are presently available (see page 17, Ref. 4) for bridges of this type of H 15-44 design.

If the dead load and live load plus impact stress ratios given on page 17 of Ref. 4 are used, and it is further assumed that a single vehicle in one lane will produce a maximum bending stress equal to about 75 per cent of that produced by identical vehicles in each lane, one can then develop some rather interesting stress frequency relationships. For these purposes it will be assumed, for example, that a single H 15 truck in one lane will ordinarily produce about 75 per cent as much live load moment in a stringer or girder as an H 15 truck in each of the two adjacent lanes. Another way of saying this is that two H trucks of given designation (one in each lane) will produce about 4/3 as much live load stress in the most critically stressed interior stringer as a single H truck of the same designation.

Based on these assumptions, together with the dead and live load plus impact ratios given in Reference 4, and the frequency distribution of H-equivalencies as found from the national loadometer survey of 1942 (see p. 409, Ref. 2), the frequencies of stress repetitions in a 50 ft. bridge and a 100 ft. bridge for an assumed life of 50 years would be approximately as shown in Fig. 5 and Fig. 6, respectively. The amazing thing about these figures is that, even with full allowance for impact, there is such a small number of stress repetitions in excess of the allowable design stresses that would result from a continuously flowing traffic volume of 500 vehicles per hour or 12,000 vehicles per day containing 5 per cent heavy trucks for the full 50 years useful life of each bridge.

Much more could be said about Figures 5 and 6, of course, but it is believed that the implications are sufficiently clear without burdening the reader with further explanation or discussion. It might be pointed out in closing though that in no case do the maximum bending stresses produced by legal loads approach values that would be considered critical.

CONCLUSIONS

Based on the previously substantiated fact that vehicles, both individually and by types, are distributed at random in ordinary highway traffic, the paper shows that the frequencies with which specified heavy vehicle groups might be expected to occur on various parts or lengths of bridge can be analyzed mathematically. This together with highway loading frequencies, measured in terms of equivalent H or H-S standard design trucks or any other convenient equivalent design loads (see pages 390-438, Ref. 2) and the stress producing effects of such loads (see Ref. 4) provides the means for estimating the number of repetitions of various intensities of stress that might be expected at any point in a given bridge during its useful life. Two typical examples of this kind are shown in Figure 5 and Figure 6, respectively. Although the mathematics involved in the method presented is quite simple, most of the situations that would be of interest to the engineer could be reduced to charts similar to Figures 1-4 inclusive. At a glance, for 6000 and 12000 vehicles per day, these charts will give the time interval for both unspecified vehicle groups occurring within specified lengths on a 2-directional highway. For example, Figure 4 shows that for 12000 vehicles per day with 5% heavy trucks, 2 heavy trucks in each direction on a 100 ft. span would occur about once in each 250 years. Time intervals for many other situations can be found in a similar manner from Figures 1-4, inclusive.

NOMENCLATURE AND DEFINITIONS

M represents one miscellaneous vehicle (automobile or bus).

L represents one light freight vehicle.

H represents one heavy freight vehicle (weight in excess of 13 tons).

X = length of section or distance in feet along highway (distance interval) which the grouping of vehicles is to occur.

t = length of time in seconds (time interval) within which the grouping of vehicles is to occur.

R = average number of vehicles per hour in any one designated direction or total traffic in both directions as may be specified.

D = average speed of traffic in any designated direction.

n = number of vehicles in a group or sequence but unassigned as to class or type.

n! = factorial n. For example, factorial 4 = 4! = 1 x 2 x 3 x 4 = 24

e = exponential base, 2.718,281 . . .

K = average number of vehicles expected within a specified length of X feet or a specified time of t seconds, based on total traffic in both directions. For a specified length of X feet; K = RX/5280D and for a specified time of t seconds; K = Rt/3600

k = average number of vehicles expected within a length of X feet or a time of t seconds in one designated lane, based on the number of vehicles per hour, (R_1), and average speed of vehicles, (D), in that lane.

P represents a general term used to indicate the probability that an event (to be defined) will occur as specified.

E = number of events or trials between occurrences of vehicle groups as may be defined.

V = vehicle interval between occurrences of certain specified events to be defined.

T = time interval between occurrences of certain specified events to be defined.

P(4H, X; 2) = probability of the group, 4H, occurring within X feet in each of the 2 directions.

P(G, X; a/2) = probability of the group, G, occurring within X feet in any manner in either or both directions.

E(n, X; 2) = number of events between occurrences of n vehicles in each of 2 lanes within X feet.

V(G, X; a/2) = vehicle interval between occurrences of the group, G, in any manner in either or both directions within X feet.

T(G, X; a/2) = time interval between occurrences of the group, G, within X feet in either or both directions.

The terms given above do not show all the possible combinations of symbols for describing conditions associated with vehicle groups on a two or more lane highway. Those shown, however, are typical: other combinations can be selected suitable for describing the particular operation under consideration.

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TIME INTERVAL FOR TYPICAL UNSPECIFIED VEHICLE GROUPS OCCURRING WITHIN SPECIFIED LENGTHS
Based on 250 Vehicles Per Hour (6000 Per Day) at Average Speed of 39.5 M.P.H.

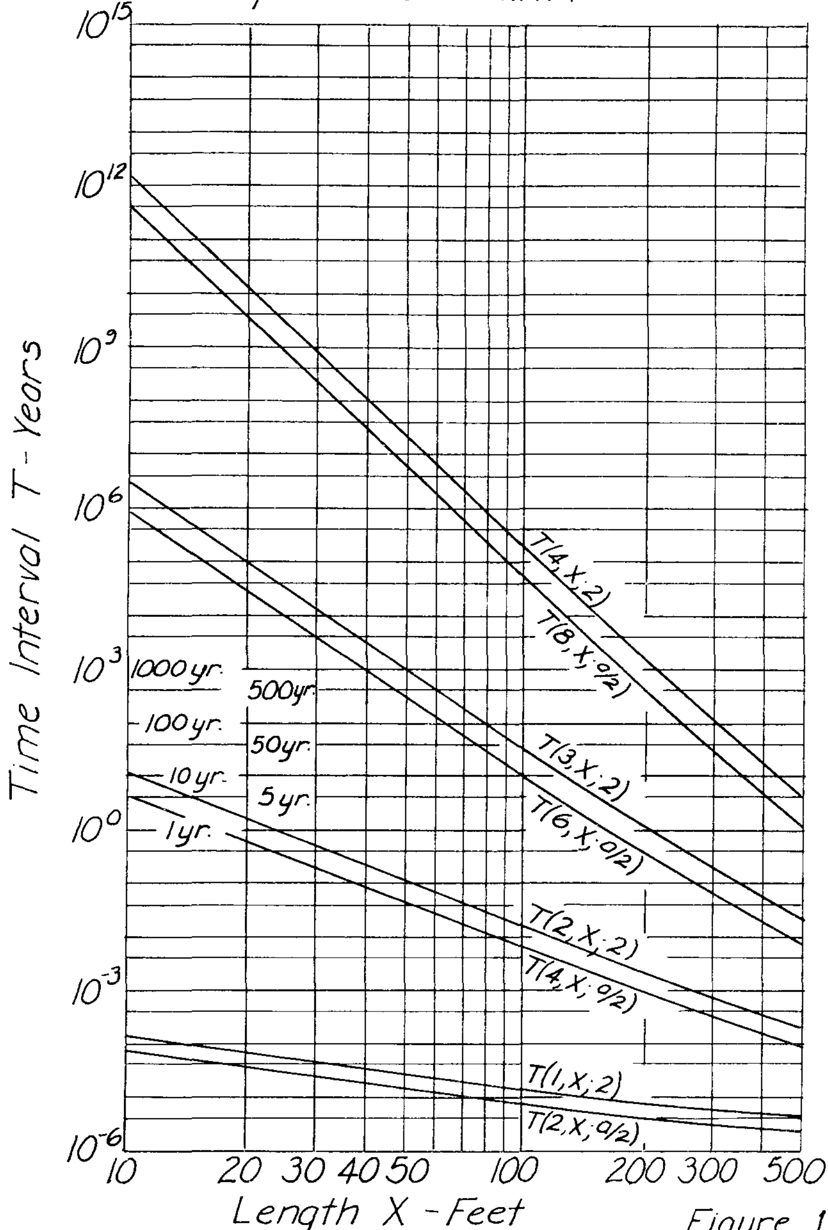

Figure 1

TIME INTERVAL FOR TYPICAL UNSPECIFIED VEHICLE GROUPS OCCURRING WITHIN SPECIFIED LENGTHS

Based on 500 Vehicles Per Hour (12,000 Per Day) at Average Speed of 39.5 M.P.H.

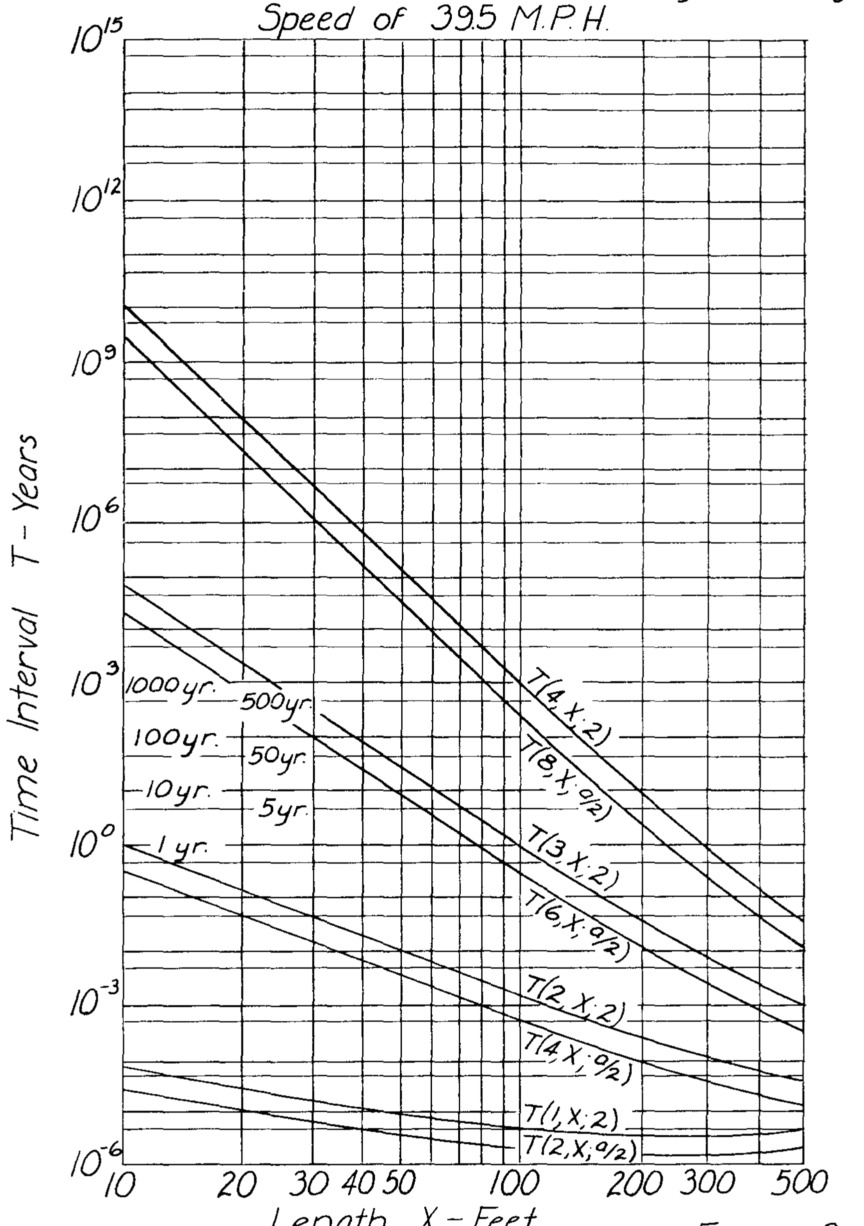

Figure 2

TIME INTERVAL FOR TYPICAL SPECIFIED VEHICLE GROUPS OCCURRING WITHIN SPECIFIED LENGTHS
Based on 250 Vehicles Per Hour (6000 Per Day) at Average Speed of 39.5 M.P.H. Consisting of 75% M, 20% L, and 5% H

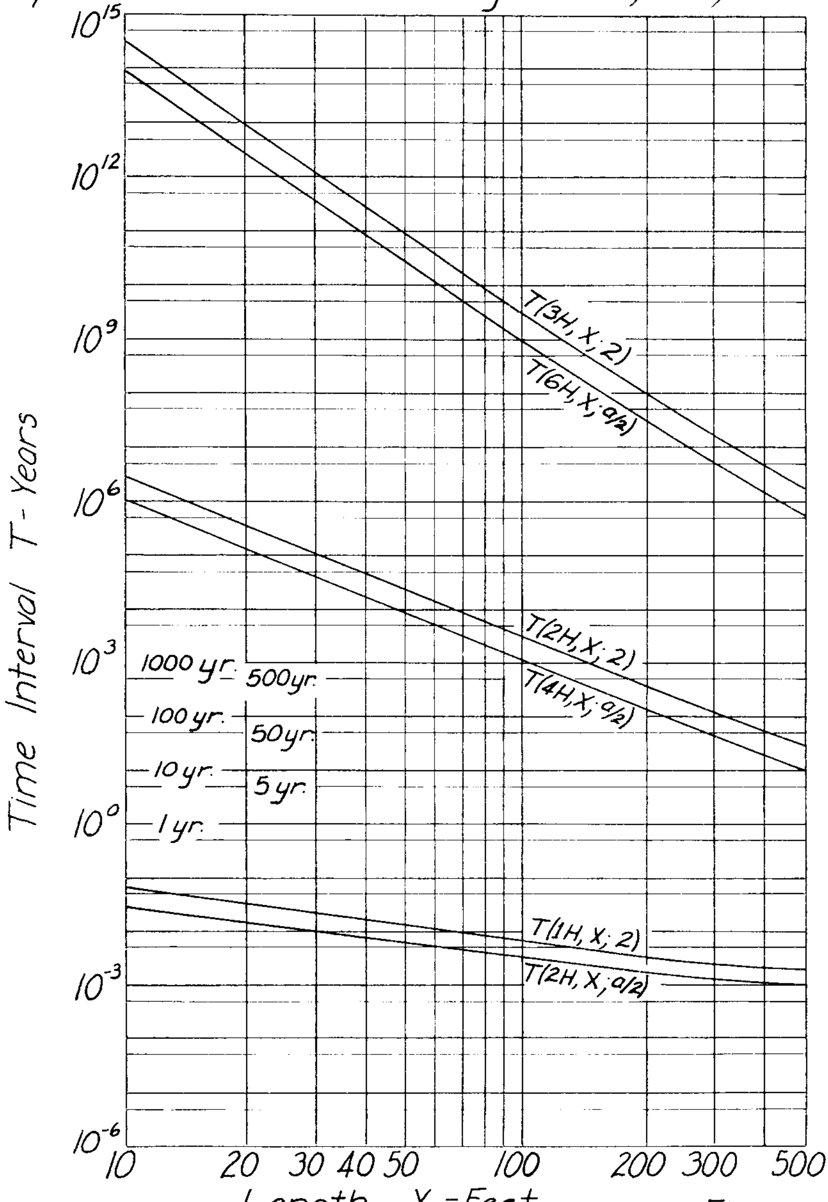

Figure 3

TIME INTERVAL FOR TYPICAL SPECIFIED VEHICLE GROUPS OCCURRING WITHIN SPECIFIED LENGTHS

Based on 500 Vehicles Per Hour (12,000 Per Day) at Average Speed of 39.5 M.P.H. Consisting of 75% M, 20% L, and 5% H

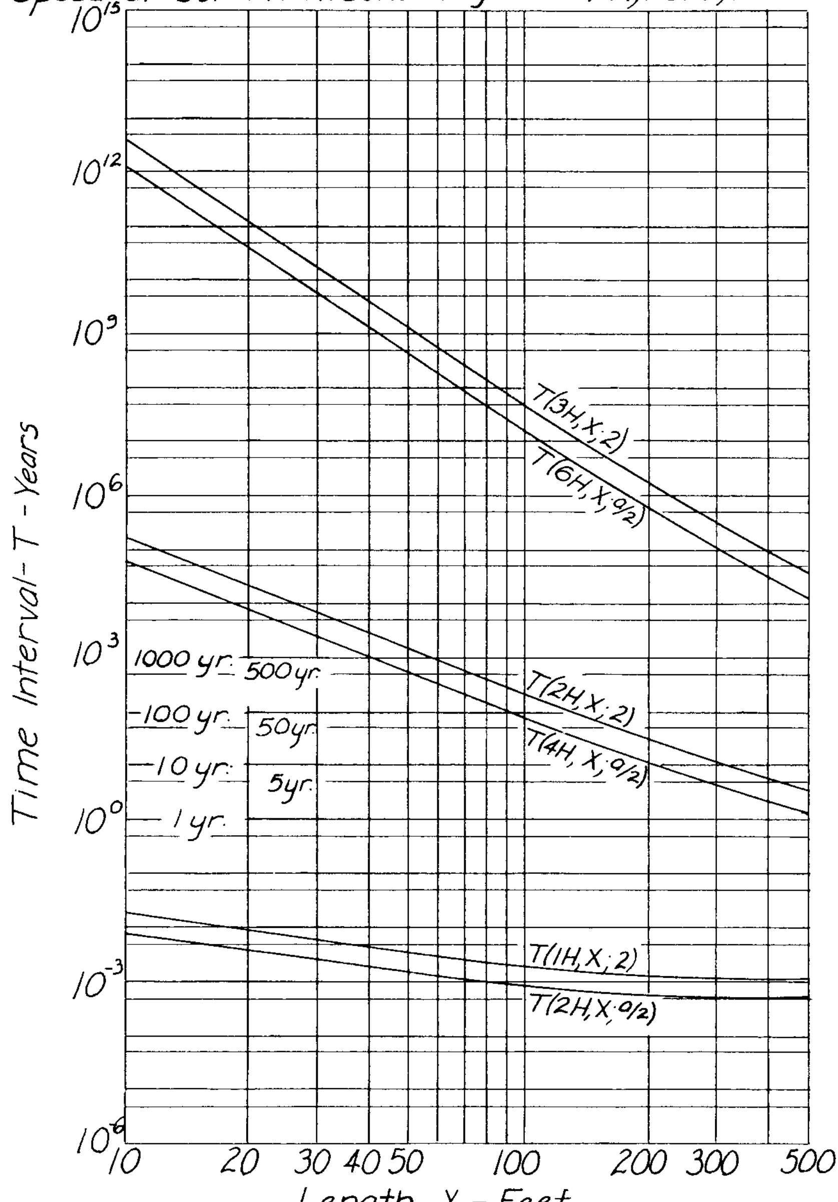

Figure 4

NUMBER OF STRESS REPETITIONS PRODUCED IN A 50 FOOT SIMPLE SPAN BRIDGE OF H 15-44 DESIGN DURING AN ASSUMED USEFUL LIFE OF 50 YEARS

Stress Effects Are Based on Continuous Traffic Volume of 500 Vehicles Per Hour (12,000 Per Day) Containing 5% Heavy Vehicles. Heavy Vehicles by Definition Are Those in Excess of 13 Tons Gross Weight.

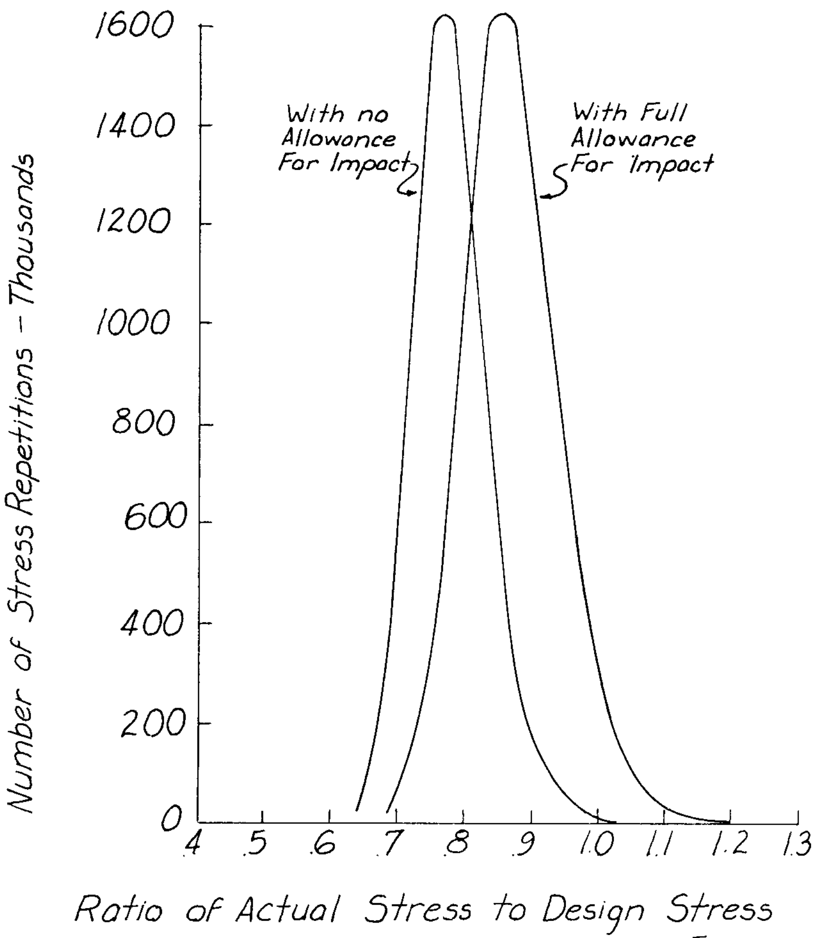

Figure 5

NUMBER OF STRESS REPETITIONS PRODUCED IN A 100 FOOT SIMPLE SPAN BRIDGE OF H15-44 DESIGN DURING AN ASSUMED USEFUL LIFE OF 50 YEARS

Stress Effects Are Based on Continuous Traffic Volume of 500 Vehicles Per Hour (12,000 Per Day) Containing 5% Heavy Vehicles. Heavy Vehicles by Definition Are Those in Excess of 13 Tons Gross Weight.

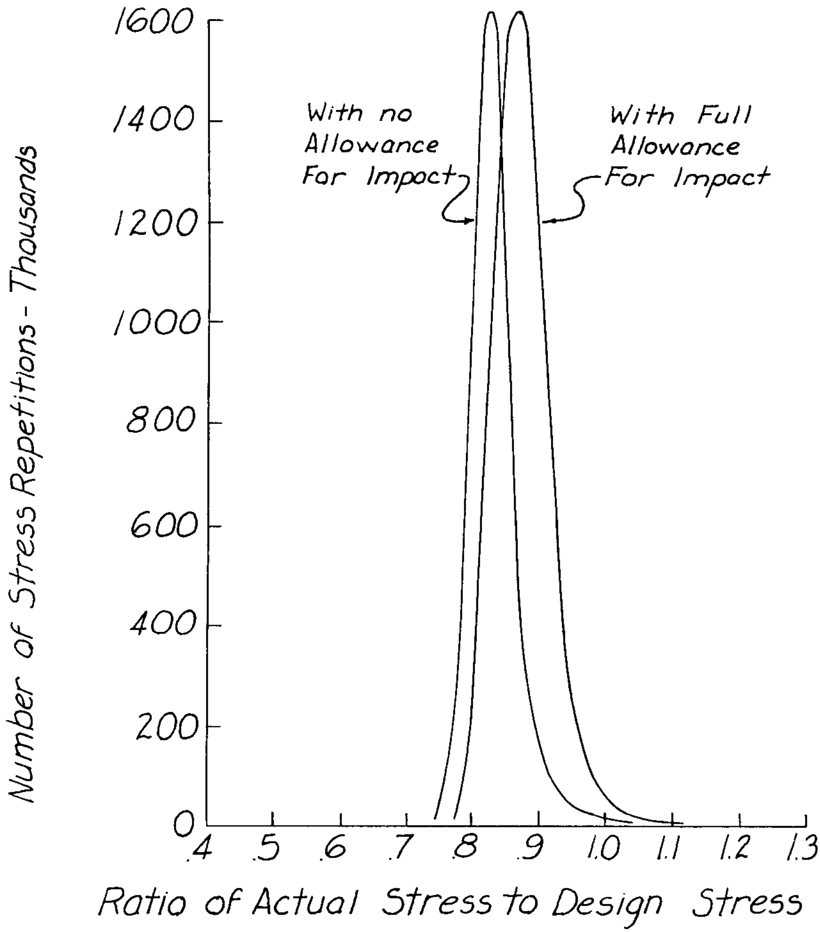

Figure 6