A SUMMARY OF DISCRETE-ELEMENT METHODS OF ANALYSIS FOR PAVEMENT SLABS

by

W. Ronald Hudson
Harvey J. Treybig
Adnan Abou-Ayyash

Research Report Number 56-27

Development of Methods for Computer Simulation
of Beam-Columns and Grid-Beam and Slab Systems

Research Project 3-5-63-56

conducted for

The Texas Highway Department

in cooperation with the
U. S. Department of Transportation
Federal Highway Administration

by the

CENTER FOR HIGHWAY RESEARCH

THE UNIVERSITY OF TEXAS AT AUSTIN

August 1972

PREFACE

This report describes a comprehensive summary of discrete-element methods of analysis for pavement slabs. It includes all the previous work related to pavement design and analysis, as well as other proposed applications.

This is the twenty-seventh in a series of reports that describes the work done in Research Project 3-5-63-56, entitled "Development of Methods for Computer Simulation of Beam-Columns and Grid-Beam and Slab Systems." The project is divided into two parts, one concerned primarily with bridge structures and the other with pavement slabs, and this is the ninth report in the series that deals directly with pavement slabs.

We are grateful to the entire staff of the Center for Highway Research, who provided support during the preparation of this report. The contributions herein of example work done by others which illustrate the wide applicability of SLAB methods are also appreciated. They, of course, did not involve expenditure of project funds.

This project is sponsored by the Texas Highway Department in cooperation with the U. S. Department of Transportation Federal Highway Administration.

W. Ronald Hudson
Harvey J. Treybig
Adnan Abou-Ayyash

August 1972

LIST OF REPORTS

Report No. 56-1, "A Finite-Element Method of Solution for Linearly Elastic Beam-Columns" by Hudson Matlock and T. Allan Haliburton, presents a solution for beam-columns that is a basic tool in subsequent reports. September 1966.

Report No. 56-2, "A Computer Program to Analyze Bending of Bent Caps" by Hudson Matlock and Wayne B. Ingram, describes the application of the beam-column solution to the particular problem of bridge bent caps. October 1966.

Report No. 56-3, "A Finite-Element Method of Solution for Structural Frames" by Hudson Matlock and Berry Ray Grubbs, describes a solution for frames with no sway. May 1967.

Report No. 56-4, "A Computer Program to Analyze Beam-Columns under Movable Loads" by Hudson Matlcok and Thomas P. Taylor, describes the application of the beam-column solution to problems with any configuration of movable non-dynamic loads. June 1968.

Report No. 56-5, "A Finite-Element Method for Bending Analysis of Layered Structural Systems" by Wayne B. Ingram and Hudson Matlock, describes an alternating-direction iteration method for solving two-dimensional systems of layered grids-over-beams and plates-over-beams. June 1967.

Report No. 56-6, "Discontinuous Orthotropic Plates and Pavement Slabs" by W. Ronald Hudson and Hudson Matlock, describes an alternating-direction iteration method for solving complex two-dimensional plate and slab problems with emphasis on pavement slabs. May 1966.

Report No. 56-7, "A Finite-Element Analysis of Structural Frames" by T. Allan Haliburton and Hudson Matlock, describes a method of analysis for rectangular plane frames with three degrees of freedom at each joint. July 1967.

Report No. 56-8, "A Finite-Element Method for Transverse Vibrations of Beams and Plates" by Harold Salani and Hudson Matlock, describes an implicit procedure for determining the transient and steady-state vibrations of beams and plates, including pavement slabs. June 1968.

Report No. 56-9, "A Direct Computer Solution for Plates and Pavement Slabs" by C. Fred Stelzer, Jr., and W. Ronald Hudson, describes a direct method for solving complex two-dimensional plate and slab problems. October 1967.

Report No. 56-10, "A Finite-Element Method of Analysis for Composite Beams" by Thomas P. Taylor and Hudson Matlock, describes a method of analysis for composite beams with any degree of horizontal shear interaction. January 1968.

Report No. 56-11, "A Discrete-Element Solution of Plates and Pavement Slabs Using a Variable-Increment-Length Model" by Charles M. Pearre, III, and W. Ronald Hudson, presents a method for solving freely discontinuous plates and pavement slabs subjected to a variety of loads. April 1969.

Report No. 56-12, "A Discrete-Element Method of Analysis for Combined Bending and Shear Deformations of a Beam" by David F. Tankersley and William P. Dawkins, presents a method of analysis for the combined effects of bending and shear deformations. December 1969.

Report No. 56-13, "A Discrete-Element Method of Multiple-Loading Analysis for Two-Way Bridge Floor Slabs" by John J. Panak and Hudson Matlock, includes a procedure for analysis of two-way bridge floor slabs continuous over many supports. January 1970.

Report No. 56-14, "A Direct Computer Solution for Plane Frames" by William P. Dawkins and John R. Ruser, Jr., presents a direct method of solution for the computer analysis of plane frame structures. May 1969.

Report No. 56-15, "Experimental Verification of Discrete-Element Solutions for Plates and Slabs" by Sohan L. Agarwal and W. Ronald Hudson, presents a comparison of discrete-element solutions with small-dimension test results for plates and slabs, including some cyclic data. April 1970.

Report No. 56-16, "Experimental Evaluation of Subgrade Modulus and Its Application in Model Slab Studies" by Qaiser S. Siddiqi and W. Ronald Hudson, describes a series of experiments to evaluate layered foundation coefficients of subgrade reaction for use in the discrete-element method. January 1970.

Report No. 56-17, "Dynamic Analysis of Discrete-Element Plates on Nonlinear Foundations" by Allen E. Kelly and Hudson Matlock, presents a numerical method for the dynamic analysis of plates on nonlinear foundations. July 1970.

Report No. 56-18, "A Discrete-Element Analysis for Anisotropic Skew Plates and Grids" by Mahendrakumar R. Vora and Hudson Matlock, describes a tridirectional model and a computer program for the analysis of anisotropic skew plates or slabs with grid-beams. August 1970.

Report No. 56-19, "An Algebraic Equation Solution Process Formulated in Anticipation of Banded Linear Equations" by Frank L. Endres and Hudson Matlock, describes a system of equation-solving routines that may be applied to a wide variety of problems by using them within appropriate programs. January 1971.

Report No. 56-20, "Finite-Element Method of Analysis for Plane Curved Girders" by William P. Dawkins, presents a method of analysis that may be applied to plane-curved highway bridge girders and other structural members composed of straight and curved sections. June 1971.

Report No. 56-21, "Linearly Elastic Analysis of Plane Frames Subjected to Complex Loading Conditions" by Clifford O. Hays and Hudson Matlock, presents a design-oriented computer solution for plane frames structures and trusses that can analyze with a large number of loading conditions. June 1971.

Report No. 56-22, "Analysis of Bending Stiffness Variation at Cracks in Continuous Pavements," by Adnan Abou-Ayyash and W. Ronald Hudson, describes an evaluation of the effect of transverse cracks on the longitudinal bending rigidity of continuously reinforced concrete pavements. April 1972.

Report No. 56-23, "A Nonlinear Analysis of Statically Loaded Plane Frames Using a Discrete Element Model" by Clifford O. Hays and Hudson Matlock, describes a method of analysis which considers support, material, and geometric nonlinearities for plane frames subjected to complex loads and restraints. May 1972.

Report No. 56-24, "A Discrete-Element Method for Transverse Vibrations of Beam-Columns Resting on Linearly Elastic or Inelastic Supports" by Jack Hsiao-Chieh Chan and Hudson Matlock, presents a new approach to predict the hysteretic behavior of inelastic supports in dynamic problems. June 1972.

Report No. 56-25, "A Discrete-Element Method of Analysis for Orthogonal Slab and Grid Bridge Floor Systems" by John J. Panak and Hudson Matlock, presents a computer program particularly suited to highway bridge structures composed of slabs with supporting beam-diaphragm systems. May 1972.

Report No. 56-26, "Application of Slab Analysis Methods to Rigid Pavement Problems" by Harvey J. Treybig, W. Ronald Hudson, and Adnan Abou-Ayyash, illustrates how the program of Report No. 56-25 can be specifically applied to a typical continuously reinforced pavement with shoulders. May 1972.

Report No. 56-27, "Final Summary of Discrete-Element Methods of Analysis for Pavement Slabs" by W. Ronald Hudson, Harvey J. Treybig, and Adnan Abou-Ayyash, presents a summary of the project developments which can be used for pavement slabs. August 1972.

(P) indicates Preliminary Report.

ABSTRACT

This report summarizes the work in Research Project 3-5-63-56 which has been conducted by the Center for Highway Research. The project staff has produced 27 research reports dealing with the analysis and design of structural members including beam columns, grid-beam systems, slabs, and slabs on foundation.

This report summarizes the portions of the work which are directly related to the analysis and design and of pavement slabs. The details of the discrete-element methods outlined herein may be found in the individual reports which are referenced. A wide variety of variations of the basic slab methods have been developed including (1) variable increment length capability, (2) multiple loading analyses, (3) dynamic loading considerations, and (4) handling of skewed slabs. The methods have been verified by comparison with closed-form solutions and with various small dimension tests. The results have been applied in numerous forms by the Texas Highway Department including the development of design charts, analysis of special problems, and the analysis of field data.

KEY WORDS: slab, discrete element, analysis, rigid pavement design, rigid pavement analysis, computer.

SUMMARY

The development of a complex analysis method for rigid pavements and orthotropic plates has resulted from the need for more sophisticated analysis methods for pavement design. This report describes in detail the evolutionary development of this analytical procedure. The basic SLAB method was developed first as an outgrowth of beam-column solutions. As technology advanced and computer size and capability increased, new developments were continually added to the basic method. These developments included such special capabilities as the handling of variable increments in the coding procedures, multiple loading analysis, dynamic analyses of slabs, and the analysis of skewed slabs. Thus, the basic method has been extended to include many special cases and many economizing features.

The new analytical method has been verified through the use of small-dimension tests on plates and slabs. These tests have mostly been laboratory studies; however, some comparisons and analyses have been made with field pavements.

The new analysis methods have been implemented and applied where possible. For rigid pavements, load placement analyses have been made. Theoretical analysis of continuously reinforced concrete pavements has been conducted and design charts have been developed. Miscellaneous special problems in pavement design have been analyzed using this unique analysis method. This analysis tool can provide basic information which will be most useful in the implementation of more advanced approaches to the design of rigid pavements.

IMPLEMENTATION STATEMENT

This project has resulted in a series of computer programs which can be used to analyze pavement design problems of various kinds. The techniques can be used for routine design, for evaluation of field test results, and for analysis of special problems. In order to be most useful, however, the methods must be available to the design engineers who need them.

The technical advances in this research project can best be implemented by making the computer programs developed on the project operational on the Texas Highway Department computer system. The pavement design personnel of the Texas Highway Department should become familiar with the capabilities of the computer programs which have been developed on this project. The developments from this research can be applied to the analysis and handling of special engineering design problems.

A second important part of the implementation of these research results is the application of the technique as a second-generation subsystem in the RPS (rigid pavement system) design method developed in Project 123, "A System Analysis of Pavement Design and Research Implementation." This program fully implemented in RPS will make the analytical power of the method fully available to pavement design engineers.

TABLE OF CONTENTS

CHAPTER 1. INTRODUCTION

CHAPTER 2. THEORETICAL DEVELOPMENTS

CHAPTER 3. EXPERIMENTAL VERIFICATION

CHAPTER 4. APPLICATION OF PROJECT METHODS

CHAPTER 5. IMPLEMENTATION

CHAPTER 1. INTRODUCTION

Brief History of Need

In 1926, Westergaard completed his treatise on the analysis of stresses in pavement slabs (Ref 1). The equations he developed have become the definitive design equations for pavement slabs in the United States. However, the limitations of conventional mathematics and particularly of hand solutions impose severely limiting assumptions which are not always realistic.

Several large-scale road tests have been conducted in attempts to bridge the gap between theory and reality. These include the Bates Test, 1922, the Maryland Road Test, 1950, and the AASHO Road Test, 1958-61. All three of these full-scale experiments have added to the knowledge of pavement design.

Pavement design involves four general classes of variables: (1) load variables, (2) structural variables, (3) regional variables, and (4) performance variables. Each of these classes is important. Major theoretical efforts have been directed toward evaluating structural variables under a single load. The number of actual variables, and the fact that they interact, prohibits the consideration of all variables in any single road test. Work with AASHO Road Test data has shown that a mechanistic model of structural behavior is essential in the study of load, environment, and performance. Unfortunately, at the time the AASHO Road Test was completed, no satisfactory method of evaluating the behavior of the pavement existed; thus, the attempts to extend the findings of the Road Test have been slow and have involved many assumptions.

Since the initiation of the interstate highway system highway, loads have become more complex in configuration. The geometric design of roadways has brought about a significant change in the lateral distribution of loads in a lane. Likewise slabs are not uniform, homogeneous, and of equal thickness and support. Thus, in 1963, when Project 56 began, the need existed for a slab analysis method which was general, flexible, and capable of handling factors such as slab discontinuities, complex load patterns and their positioning on the slab, and variable foundation conditions.

Project Background

On June 10, 1963, Research Project 56 was initiated. The initial purpose of this project was to adapt methods of numerical analysis in structural mechanics to problems associated with highway bridge design. All of the methods are based on formulating discrete-element models of the corresponding structural problems and solving the resulting equations with a high-speed digital computer.

After an initial year of work on beam-column and grid-beam systems, and a series of conferences, the scope of the program was enlarged to apply these same methods for solution of pavement slabs. Because of fundamental similarities between the bridge and pavement slab problems, the expanded program was maintained as a single project by mutual agreement between the research staff and the sponsors, and the duration of the project was extended.

The discrete-element solution procedures which simulate beam-columns, plates, pavement or bridge slabs, frames, grid-beams, and other layered structural systems can be applied to a wide range of problems. Work on this continuing research project has been concentrated, however, on how these techniques are applied to highway bridges, pavements, and related structures. There is a fundamental efficiency which results from these coordinated efforts which has permitted advances which would have otherwise been slower to develop.

Technical developments on the project in the pavement slab area include the first solution of discontinuous orthotropic plates and pavement slabs, which used an alternating-direction iteration technique (Ref 2). A direct solution for this same analysis was also developed (Ref 3). The slab model was later modified to handle variable increment lengths. The model was revised for a multiple loading analysis of slabs. Dynamic analyses of slabs on nonlinear foundations have been developed. The method has been extended to skewed plates and slabs by the use of triangular elements. The method has been used for the analysis of continuously reinforced concrete pavements. Experimental verification has included small dimension tests in the laboratory on single and multiple layered foundations. Numerous applications of the research results have been made since 1965.

Of the 27 reports prepared within this project, the following reports are those which specifically refer to the technical developments related to slab and pavement analysis.

Report No. 56-6, "Discontinuous Orthotropic Plates and Pavement Slabs" by W. Ronald Hudson and Hudson Matlock, May 1966.

Report No. 56-9, "A Direct Computer Solution for Plates and Pavement Slabs," by C. Fred Stelzer, Jr., and W. Ronald Hudson, October 1967.

Report No. 56-11, "A Discrete-Element Solution of Plates and Pavement Slabs Using a Variable-Increment-Length Model," by Charles M. Pearre, III, and W. Ronald Hudson, April 1969.

Report No. 56-13, "A Discrete-Element Method of Multiple-Loading Analysis for Two-Way Bridge Floor Slabs," by John J. Panak and Hudson Matlock, January 1970.

Report No. 56-17, "Dynamic Analysis of Discrete-Element Plates on Nonlinear Foundations," by Allen E. Kelly and Hudson Matlock, January 1970.

Report No. 56-18, "A Discrete-Element Analysis for Anisotropic Skew Plates and Grids," by Mahendrakumar R. Vora and Hudson Matlock, August 1970.

Report No. 56-15, "Experimental Verification of Discrete-Element Solutions for Plates and Pavement Slabs," by Sohan L. Agarwal and W. Ronald Hudson, April 1970.

Report No. 56-16, "Experimental Evaluation of Subgrade Modulus and Its Application in Small-Dimension Slab Studies," by Qaiser S. Siddiqi and W. Ronald Hudson, April 1970.

Report No. 56-22, "Analysis of Bending Stiffness Variation at Cracks in Continuous Pavements," by Adnan Abou-Ayyash and W. Ronald Hudson, June 1971.

Report No. 56-26, "Application of Slab Analysis Methods to Rigid Pavement Problems," by Harvey J. Treybig, W. Ronald Hudson, and Adnan Abou-Ayyash, June 1972.

The other reports in the sequence are listed on page iv. Some of these give background data which may be helpful as background.

CHAPTER 2. THEORETICAL DEVELOPMENTS

The technical developments of the discrete-element slab analysis method have been evolutionary. The first SLAB computer program was reported by Hudson and Matlock (Ref 2) and numerous developments have been added as technology advanced and computer size and capability increased.

Basic SLAB Method

Hudson and Matlock (Ref 2) developed a method of solving for the deflected shape of orthotropic plates and pavement slabs. From this deflected shape the stresses, deflections, loads, and bending moments are determined. The principal features incorporated in the initial discrete-element method are

(1) representation of structural members by a physical model of bars and springs which are grouped for analyses into two orthogonal systems of beams,

(2) a rapid method for direct solution of individual beams that serve as line elements of a two-dimensional slab, and

(3) an alternating direction iteration technique which coordinates the solutions of individual beams and which, thereby, ties the system together.

The method allows for freely discontinuous variation of input parameters including bending stiffness and load. Combination loading is provided for and includes lateral loads, in-plane forces, and applied couples or moments. Freely variable foundation conditions apply not only to the general slab-on-foundation case, but also to orthotropic plates with various configurations of structural support, and could include uniform isotropic plates with simple supports as a special case.

The discrete-element model (Fig 1) is helpful in visualizing the problem and forming the solution. The model consists of

(1) infinitely stiff and weightless bar elements to connect the joints;

(2) elastic joints where bending occurs, made of an elastic, homogeneous, and orthotropic material which can be described by four independent elastic constants;

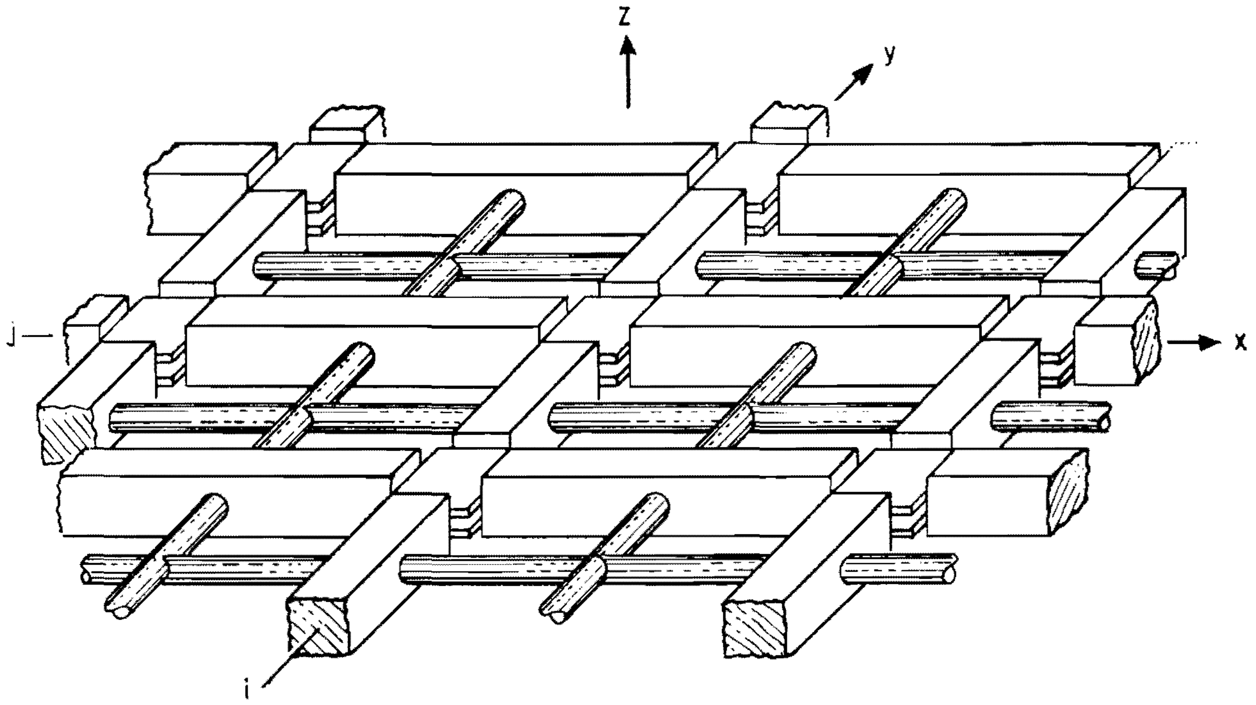


Fig 1. Finite-element model of a plate or slab (after Ref 4).

(3) torsion bars which represent the torsional stiffness of the plate; and

(4) elastic support springs which provide foundation support.

All properties and loads can be freely variable from point to point. Concentrated or distributed loads can be handled, including transverse loads, in-plane forces, and external couples. Elastic restraints are provided by vertical support springs.

The alternating-direction iteration method was originally used to solve the equations describing the behavior of the model because it is well adapted and easy to visualize. The model and method are too complex for hand calculations. For example cases having closed-form solutions, the discrete-element solution with 8 to 10 increments produces results within 2 to 5 percent of the closed-form solution in a few iterations. If the number of increments is increased to 16, the error comparison reduces to 1 to 3 percent (Ref 2).

A computer program, SLAB 17, which solves the equations implicitly for the deflection patterns was developed in 1966. The program is written in FORTRAN-63 for the CDC 1604 computer. Minor changes of input formats are required to convert it for use on an IBM 7090. Compile time is 90 to 100 seconds but binary decks compile in about 15 seconds. Automatic plot routines are available for use with the program.

Direct Solution for Basic Slab Method

In the basic SLAB method (Ref 2) Hudson and Matlock presented a method for the solution of discontinuous orthotropic plates and pavement slabs which utilizes an alternating-direction iterative technique in the computer program. In that method, efficient solutions depend on choosing the proper closure parameters for input values; subsequent investigation has shown that this is sometimes difficult to do, clearly defining a need for a one-pass method of solving discontinuous orthotropic plates and pavement slabs.

A direct solution of the simultaneous equations formed by applying the basic equation at all mesh points was developed by Stelzer and Hudson (Ref 3). The method is not dependent on the selection of closure parameters as is the iterative technique of Hudson and Matlock (Ref 2).

The method is verified by solving problems with available closed-form answers. All problems solved in Ref 2 have been solved by the direct method with consistent results (Ref 3). The method will solve problems which have discontinuities not only in stiffness and support but also in loading and boundary shape. The FORTRAN program is practical for use on computers with a large core storage capacity such as the CDC 6600 Computer.

SLAB Method with Variable Increment Sizes

The basic SLAB method developed by Hudson and Matlock (Ref 2) and later modified by Stelzer and Hudson (Ref 3) used equal lengths of increments in the discrete-element model. The modeling accuracy of both methods of solution increased with the number of increments used in the model. It has been found that the increased number of increments generally affected the solution only near points of abrupt or rapid changes in load, support, or stiffness of the plate and that complex loading patterns were difficult to represent in the fixed length models. These findings clearly defined a need for a method which would permit the use of variable increments in the discrete grid so that small increments could be used near abrupt changes and larger increments in areas where detailed accuracy is less important. Pearre and Hudson have described a method to satisfy these requirements (Ref 4).

The primary difference in the capability of this computer program and the previous two was the variable grid pattern, for which an example is shown in Fig 2. Figures 3 and 4 show a comparison of the constant increment length discrete-element model and the variable increment length discrete-element model.

The FORTRAN computer program has been checked out and example problems have been solved which illustrate that this model improves the accuracy of the direct solution slab programs. The errors in deflection near loads can be cut as much as 80 percent without increasing the solution time. The variable-increment-length solutions obtained are generally equivalent to constant-increment-length solutions using two to three times as many increments in each direction and thus represent a time saving of approximately 95 percent over the previous methods (Ref 4).

The variable increment method has application to a broad variety of problems since complex conditions can be met more exactly than with previous methods. The method is more accurate for the distribution of uniform loads in the near vicinity of supports because it can have a very small increment length near the edge of the slab. Also, the method is more accurate than previous

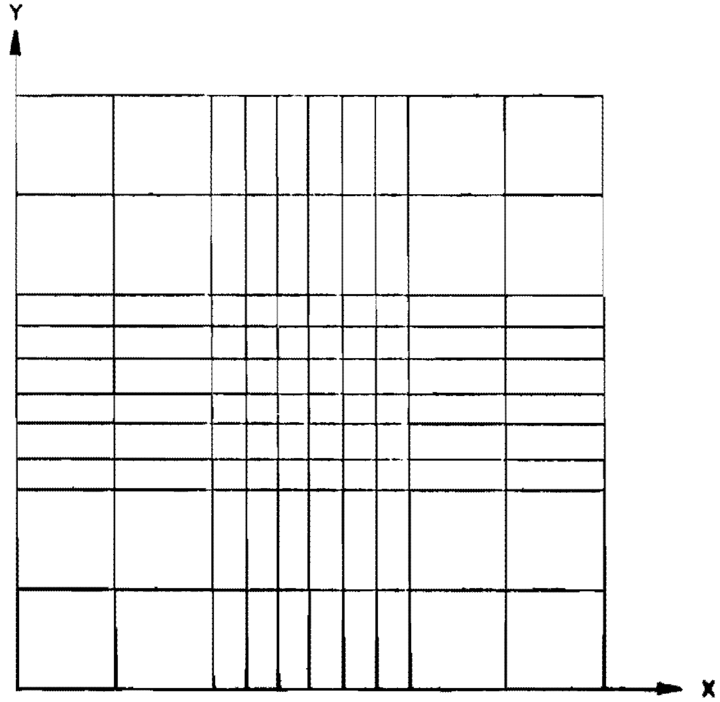


Fig 2. Variable-increment grid pattern (after Ref 4).

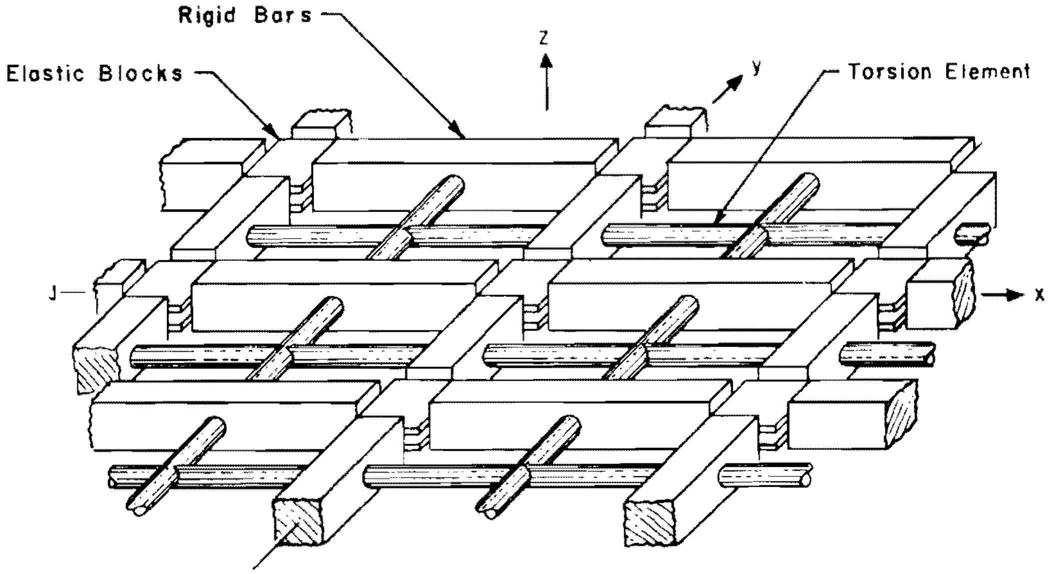


Fig 3. Constant-increment model
(after Ref 2).

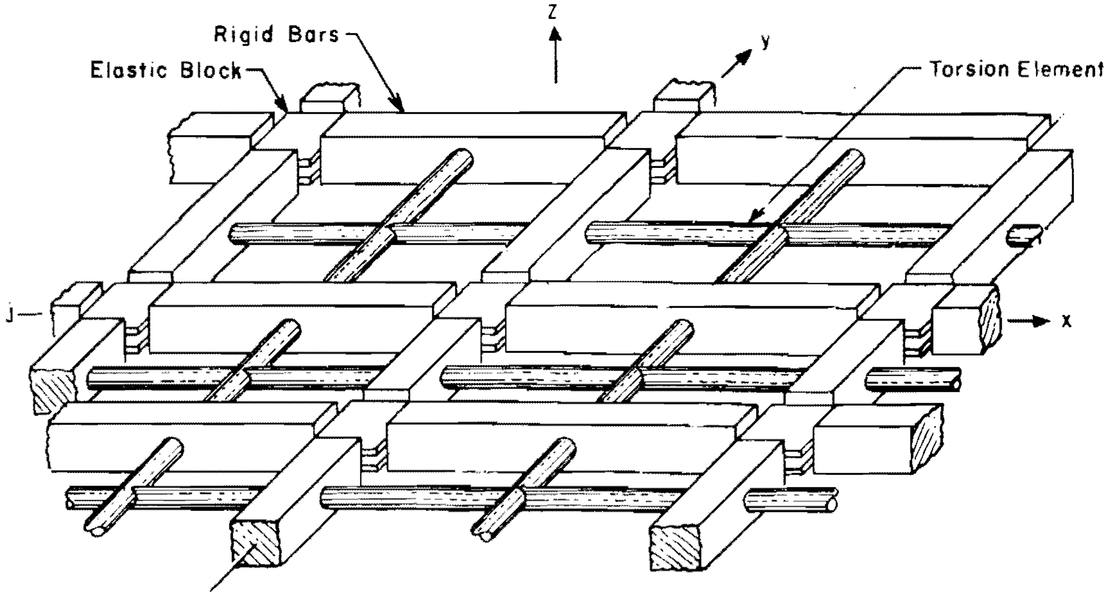


Fig 4. Variable-increment model
(after Ref 4).

methods when the same number of increments are used and is therefore more economical than previous methods.

Multiple-Loading Analysis of Slabs

This slab analysis method was developed primarily for two-way bridge slabs with a multiple-load capability (Ref 5). The procedure, however, is applicable to pavement slabs as well as other types of slab structures.

A set of simultaneous equations is written and solved by a direct solution process (Ref 5). The analogous computer model used for the formulation of the equations can be related to the actual structure and is the same model as that presented in Refs 2 and 3.

The first problem in a multiple-loading series is designated as the "parent" problem and subsequent problems as the "offspring." One property that has been observed for the technique is that as the problem gets larger the percentage of total solution time for the offspring problems becomes smaller. A time as low as 8 percent of the parent problem time has been observed for the largest problems that have been solved.

The analysis method as applicable to pavements is much like previous ones (Ref 23) except that it is much more efficient. The computer program, SLAB 30, is an improved version of the one developed by Stelzer (Ref 3), with the significant addition of the multiple-loading capability. The multiple-loading feature particularly is of interest to the highway bridge designer, who is primarily concerned with moving load patterns on a structure.

The program output consists of deflections, bending moments, twisting moments, maximum principal moments and their directions, and support reactions. In addition, optional concise moment output may be obtained in specific areas. The program has probably been used more than any of the previous ones because of its increased efficiency.

The computer program is written in ASA FORTRAN and operates on either CDC 6600 or IBM 360/50 systems and is compatible with other comparable systems, such as the UNIVAC 1108.

Dynamic Analysis of Discrete-Element Slabs

Salani and Matlock (Ref 6) developed an alternating-direction implicit (ADI) iterative procedure for the solution of plate vibration problems. Kelly and Matlock (Ref 7), on the other hand, utilized an efficient direct solution. Nonlinear analysis is performed by adjusting the load rather than the stiffness coefficients. Since the stiffness matrix of the structure is not changed during the solution procedure, the multiple load method of Panak and Matlock (Ref 5) is applied for the iterative procedure.

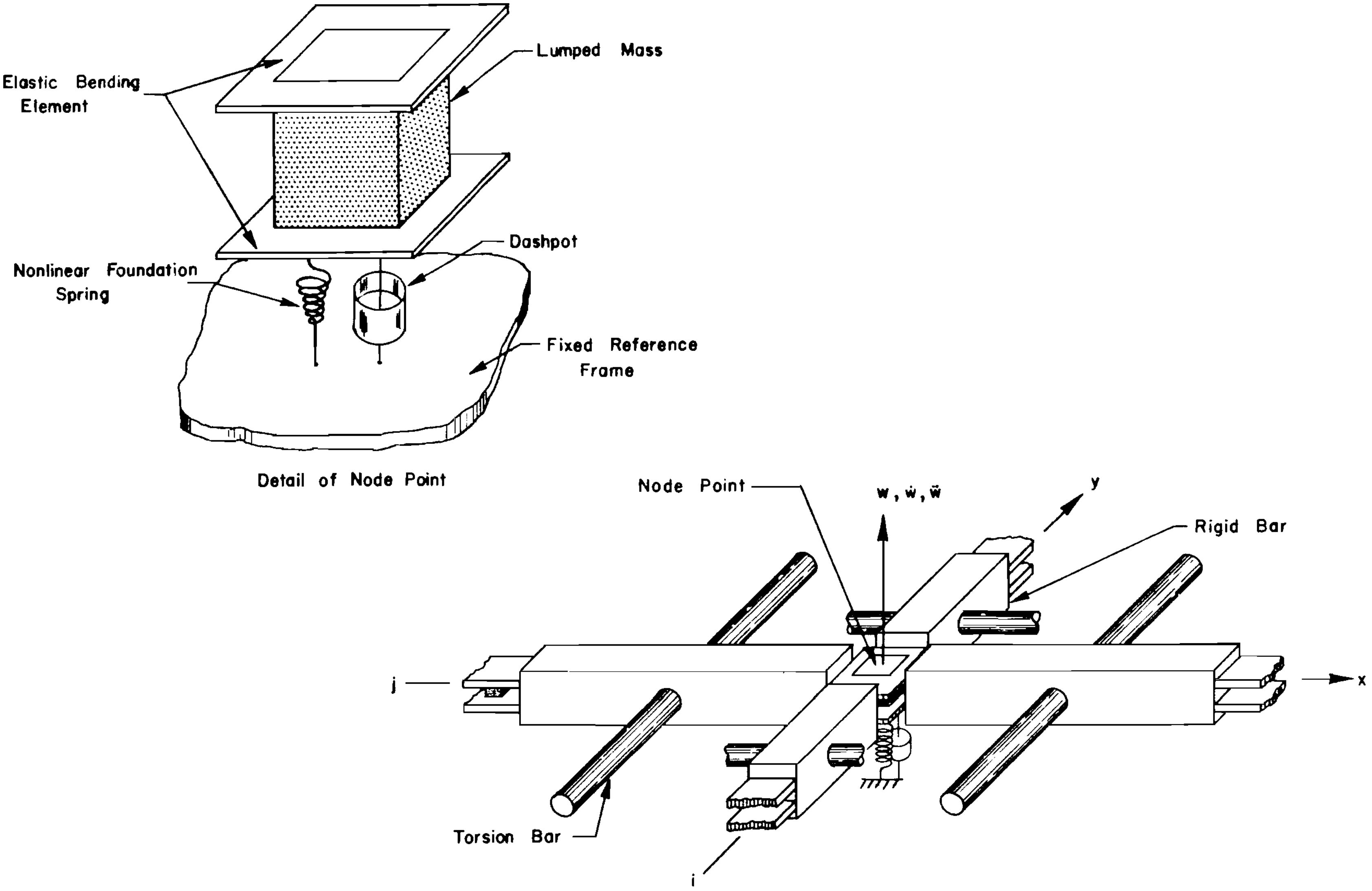
The discrete-element plate model for static analysis is modified to include the mass and damping properties of the system. Both mass and damping are concentrated at the model joints or node points. Figure 5 shows a typical joint from the model.

Equations of motion for the discrete-element model are developed by the addition of the node point inertia and damping forces to the static equilibrium equations. The resulting second-order simultaneous differential equations are solved numerically by a step-by-step method of Wilson and Clough (Ref 27) based on the assumption that the acceleration varies linearly between time stations. The results of this assumption are a second-order variation of velocity and a third-order variation of the deflection (Fig 6).

The stability of the linear acceleration algorithm is investigated and recommendations are made for the selection of a stable time-step increment. The time step required for stability is related to an estimate of the smallest period of free vibration of the structure. It is shown that the time-step increment for numerical analysis should be smaller than one-fourth of the minimum estimated period.

The resistance-deflection characteristics of the nonlinear foundation are described by curves composed of straight line segments (Fig 7). For the formulation of the stiffness matrix of the discrete-element model, a linear approximation of the nonlinear curve is required. Nonlinear analysis is performed by the addition of a load correction to the right-hand side of the equations. The correction load applied at each node point is therefore the difference between the resistance developed by the linear approximation and that specified by the curve (Fig 7). The equations are solved repeatedly until the changes in deflections caused by correction loads are less than a specified closure tolerance.

The computer program was developed for high-speed, fast-access, multiprocessing computer systems with large core storage. It utilizes 16 disk files for peripheral storage of program data.



Detail of Node Point

Fig 5. Joint detail of discrete-element model for dynamic analysis (after Ref 7).

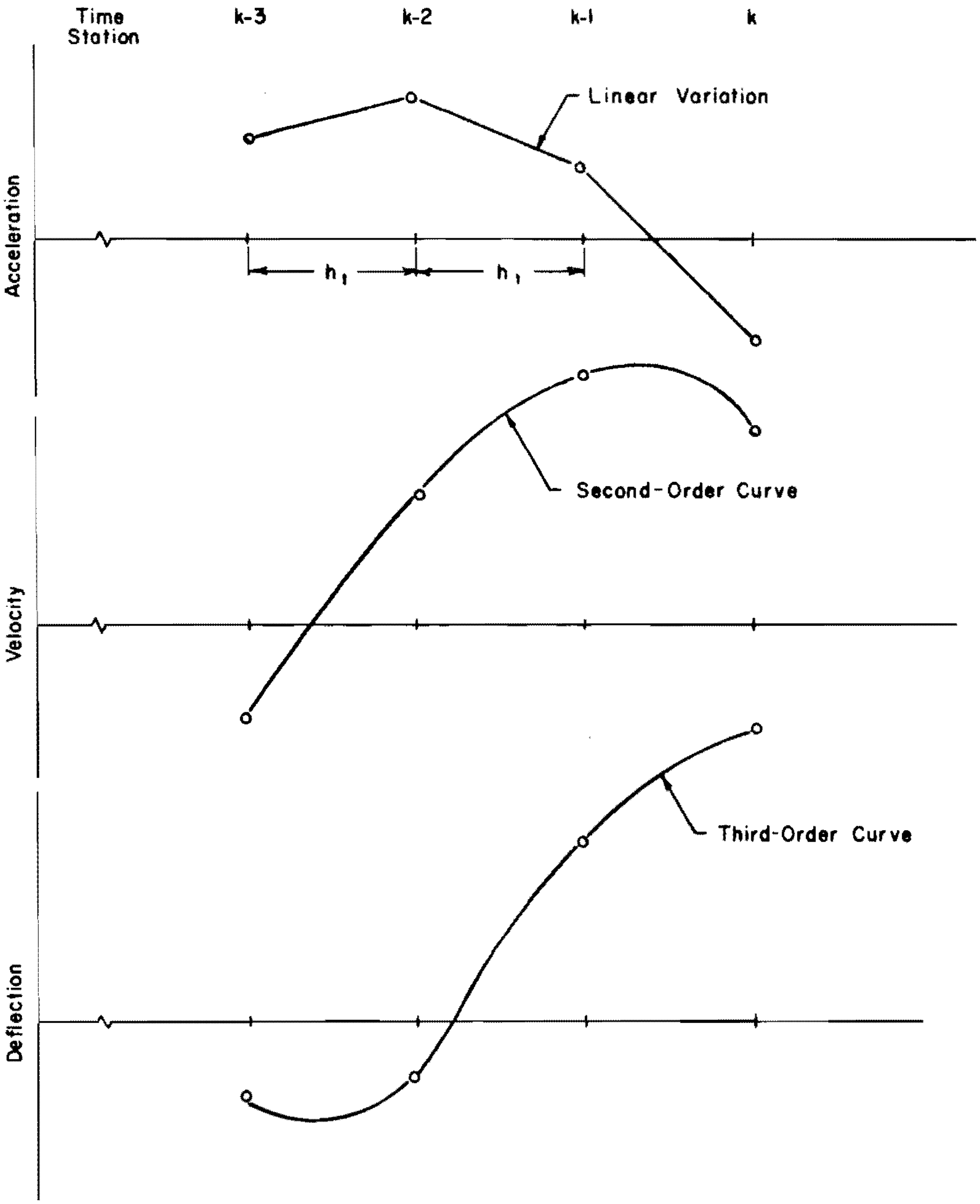


Fig 6. Node-point response for linear acceleration algorithm (after Wilson and Clough, Ref 27).

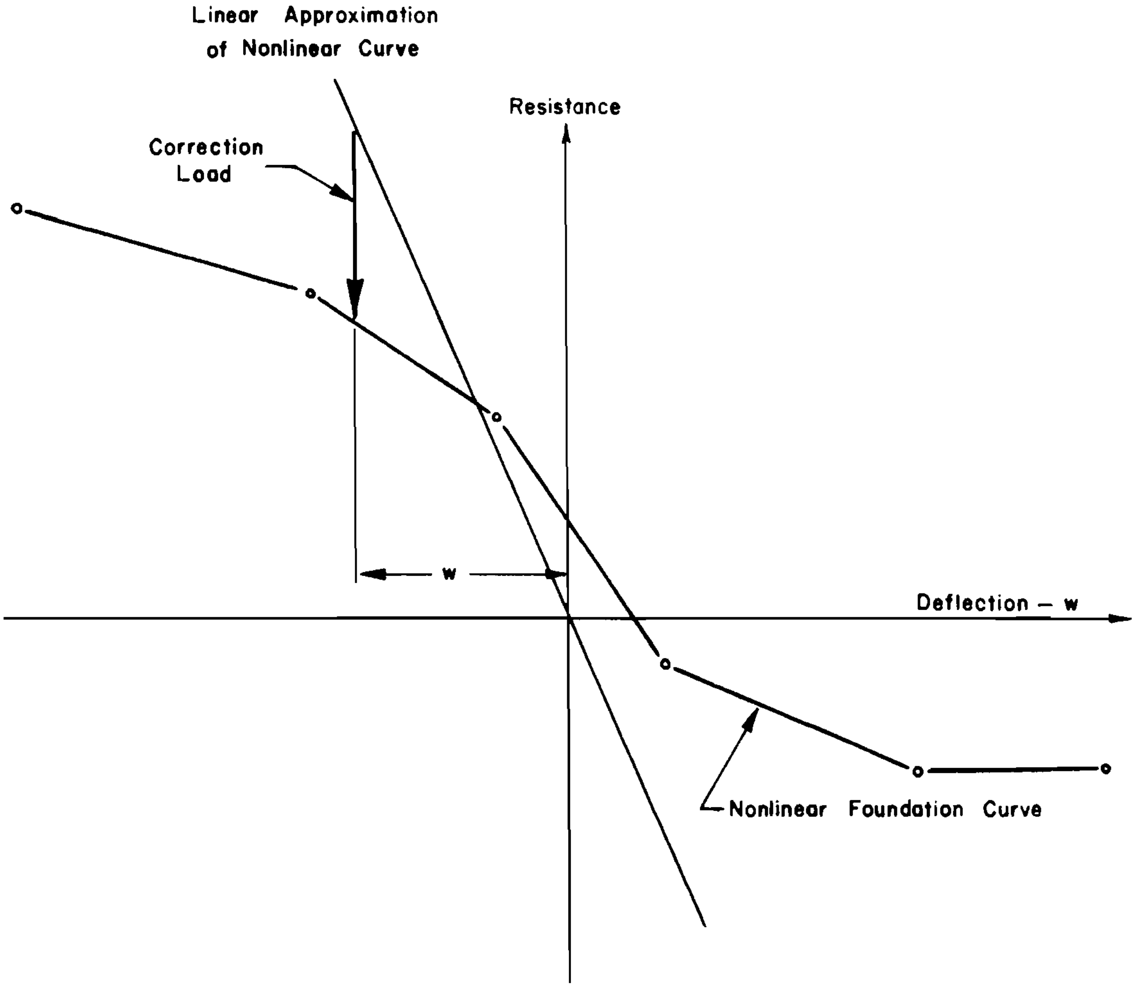


Fig 7. The load iteration method for nonlinear analysis (after Ref 7).

To facilitate the program's extension or modification to include future developments, subroutines are used extensively. The program consists of a main driver and 27 subroutines, fourteen of which are for matrix and vector operations.

The simultaneous equations which result from the linear acceleration algorithm are solved by an efficient recursion-inversion, multiple load technique for large systems of banded equations (Refs 5 and 9).

Although the dynamic load must be specified by the program user, the simplified input form allows great flexibility, and either periodic or nonperiodic loads, as well as stationary loads or loads moving with constant velocity, can be input.

Analysis of Skew Slabs

The discrete-element method of analysis of anisotropic skew plates and grids (Ref 10) was developed primarily for handling complex skew slab bridge structures. The method, however, is flexible and can be applied to skewed pavement slabs such as bridge approach slabs or jointed pavements with skewed joints.

A mechanical model consisting of a tridirectional system of rigid bars and elastic joints is used to simulate anisotropic skew plates plus slab-and-grid systems in which the grid-beams may run in any three directions (Fig 8). The model allows for the free linear elastic variation of stiffnesses and support characteristics. Loads are applied at each joint to represent any degree of concentrated or uniform loading.

By use of concepts of a continuum composed of interconnected fibers, stress-strain relationships for the anisotropic slab model are derived. Each grid-beam may thus be considered the same as a previously developed discrete-element beam-column (Refs 11 and 12).

A computer program, SLAB 44, is written to apply the discrete-element formulation of an anisotropic skew-plate and grid-beam system (Fig 8) in which the grid-beams may run in any three directions.

The computer program SLAB 44 is written in FORTRAN for the CDC 6600 computer. The program is easily made compatible with IBM 360, UNIVAC 1108, and other similar systems.

The input value of slab stiffness may be related either to orthogonal directions or three specific directions. The program output consists of

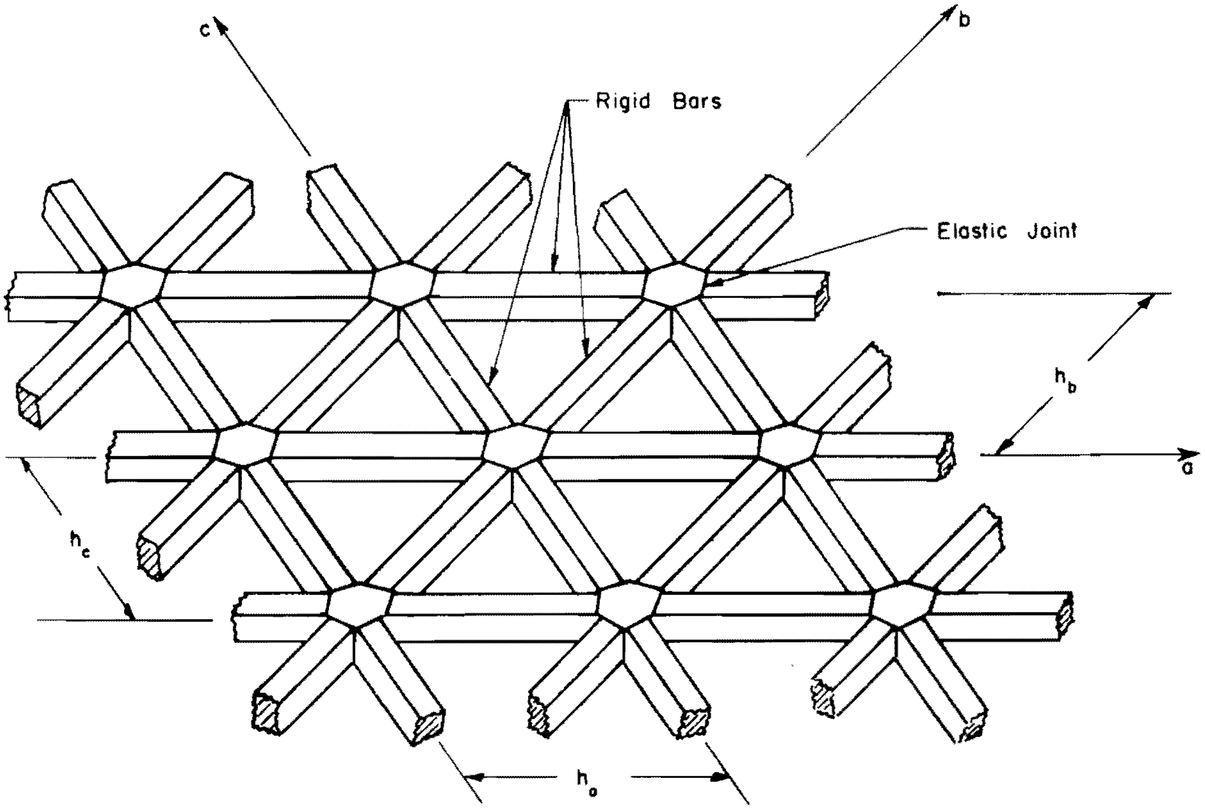


Fig 8. Discrete-element model for anisotropic skew plate showing all components (after Ref 10).

deflections, bending moments and twisting moments (or bending moments in three directions), largest principal moments together with their directions, and support reactions (or a statics check at each joint of the discrete-element model).

CHAPTER 3. EXPERIMENTAL VERIFICATION

Introduction

The validity of any analytical method can best be proven by comparing its solutions with actual test results. To obtain test results for use in verifying discrete-element analytical methods for slabs-on-foundation, Agarwal and Hudson (Ref 13) conducted a study of small-dimension plates and slabs with the basic approach divided into two parts:

(1) a check of the modeling and method of solution for plates on simple supports for a variety of stiffness and load conditions, and

(2) an investigation of the modeling of the soil-structure interaction problem of a slab-on-foundation using linear and nonlinear characteristics according to the Winkler assumption.

Study of Plates

Tests for the first part of the study were conducted on both aluminum and plexiglas plates, 25 inches square and resting on four-point supports. Preliminary tests involving four-edge support were conducted also, but because of shortcomings in the test set-up they were not continued. The point supports were placed 22 inches apart and a 25 by 25-inch plate was set on them. Test loading was done in increments of 10 pounds with a maximum of 40 pounds, plus a seating load of 10 pounds. Deflections were measured by 0.0001-inch dial gages, and strains, where measured, were obtained using rosettes. The following four-point support tests were performed:

(1) deflections and strains on aluminum plate under center loading,

(2) deflections and strains on aluminum plate under off-center loading,

(3) deflections and strains on aluminum plate with a 6 by 1/2-inch slot cut in the plate,

(4) deflections for isotropic plexiglas plate, and

(5) deflections for orthotropic (stiffness one one side) plexiglas plate.

Analytical solutions based on the discrete-element model were found using the independently determined plate properties for each case.

A comparison of measured and computed deflections was made and the percentage error calculated as a function of the maximum measured value.

A summary of percentage error for deflections for all the tests is presented in Table 1 for points "under the load," "near the load," and "near the periphery." The following observations can be made from this table.

For linearly elastic materials such as aluminum, the analytical deflections agreed with the experimental data within 4 percent for continuous plates under center and off-center loadings and for plates with a discontinuity under center loading. For an approximately linearly elastic material like plexiglas the discrepancy for deflections was within about 2 percent for isotropic plates and 5 percent for orthotropic plates.

From the strains of the rosettes, principal stresses were determined and compared with the computed principal stresses. The percentage errors were calculated using the individual strain values of the strain gages fixed at the top and bottom of the plate, with the averages of these strain values thus canceling any possible effects of in-plane forces. A summary of percentage error for all tests is given in Table 2. It can be observed from this table that for linearly elastic materials such as aluminum, the agreement between computed and measured principal stresses for all three cases is within 6 percent.

Based on the above tests, the validity of the analytical solution for plates of linearly elastic materials on rigid supports under a variety of load and stiffness conditions is proven.

Slab-on-Foundation Study

After verifying the method of solution for linear materials on rigid supports, tests for the second part of the study were conducted on a 9 by 9 by 1/8-inch aluminum slab resting on clay soil. The soil was prepared by extrusion and placed in a supporting box at an average density of 116 lb/cu ft and a moisture content of 38 percent. Load was applied by a mechanical screw jack, measured by a load cell, and recorded on a digital voltmeter. Deflections of the slab were measured by linear variable differential transformers and dial gages, and strains were measured by electrical resistance strain rosettes. Tests were conducted for center.load and two-point corner loads. Load was applied continuously up to 255 pounds for center load and 208 pounds for corner

TABLE 1. PERCENTAGE ERROR FOR DEFLECTIONS, FOUR-POINT SUPPORT (REF 13)

Plate Material and Load Condition	Percentage Error for Points		
	Under the Load	Near the Load	Near the Periphery
Aluminum, center	0.8 to 1.1	-0.78 to 1.2	0.2 to 0.4
Aluminum, off-center	-0.02	0.2 to 1.0	0.1 to 2.8
Aluminum, with cut, center	-2.4 to -2.8	-1.1 to -3.8	-0.2 to 3.1
Plexiglas, center	0.5 to 1.1	0.6 to 2.0	0.4 to -2.4
Plexiglas, orthotropic, center	4.4 to 5.1	zero to 3.6	-0.2 to 0.2

TABLE 2. PERCENTAGE ERROR FOR PRINCIPAL STRESS (REF 13)

	Percentage Error			
	Rosette 1		Rosette 2	
	Stress 1	Stress 3	Stress 1	Stress 3
Aluminum plate, center load	-1.8 to 2.5	3.8 to 2.9	-1.4 to 1.5	3 to 4.3
Aluminum plate, off-center load	5.2 to 6.8	-5.0 to 6.5	5.8	-4.8
Aluminum plate with cut	0.5 to 0.8	0.1 to 3.2	6.1	2.8

loads, and records for loads, strains, and LVDT's were obtained at regular intervals on a 40-channel voltmeter with a digital scanning system.

In order to characterize the soil subgrade as a Winkler foundation, linear and nonlinear support characteristics were obtained from rigid plate load tests. These characteristics were also determined from stress-strain relationships of the soil obtained from unconfined compression tests. Discrete-element solutions were obtained using these linear and nonlinear soil springs.

The measured and computed deflections for a particular load on the slab were compared and the error was calculated as percentage of maximum measured deflection.

Figure 9 shows that the measured and computed deflections used different linear and nonlinear springs along the center line for a center load of 100 pounds (categorized as the low load). It is observed from this figure that good correlation exists between the experimental and analytical solutions using linear springs (secant modulus corresponding to maximum deflection) or nonlinear springs. However, for a load of 200 pounds (categorized as high load), good correlation exists only with nonlinear springs, as shown in Fig 10. Similar observations obtaining good agreement between experimental and computed deflections using nonlinear springs only for loads producing large deflections were also made for corner load tests.

Calculations were made for principal stresses from the strain readings of the rosettes for both tests, and for stresses along the edge from the readings of the strain gages fixed along the edge in the corner load test. From the comparison of measured and computed principal stresses and stresses along the edge, it was observed that nonlinear soil springs should be used to get a good agreement for loads producing large deflections.

Slab Tests Under Cyclic Loading

Pavements are subjected to numerous repetitive loads, which vary in frequency, rate, duration, and stress level. The problem of the soil-pavement system under repetitive loads is quite complex. However, some idea of the phenomenon may be obtained by testing slab sections under a constant rate of loading and cycling either the load or deflection for a few cycles. With this purpose in mind cyclic test loading was conducted for both center and corner loading. The load was cycled for the maximum deflection in the corner load

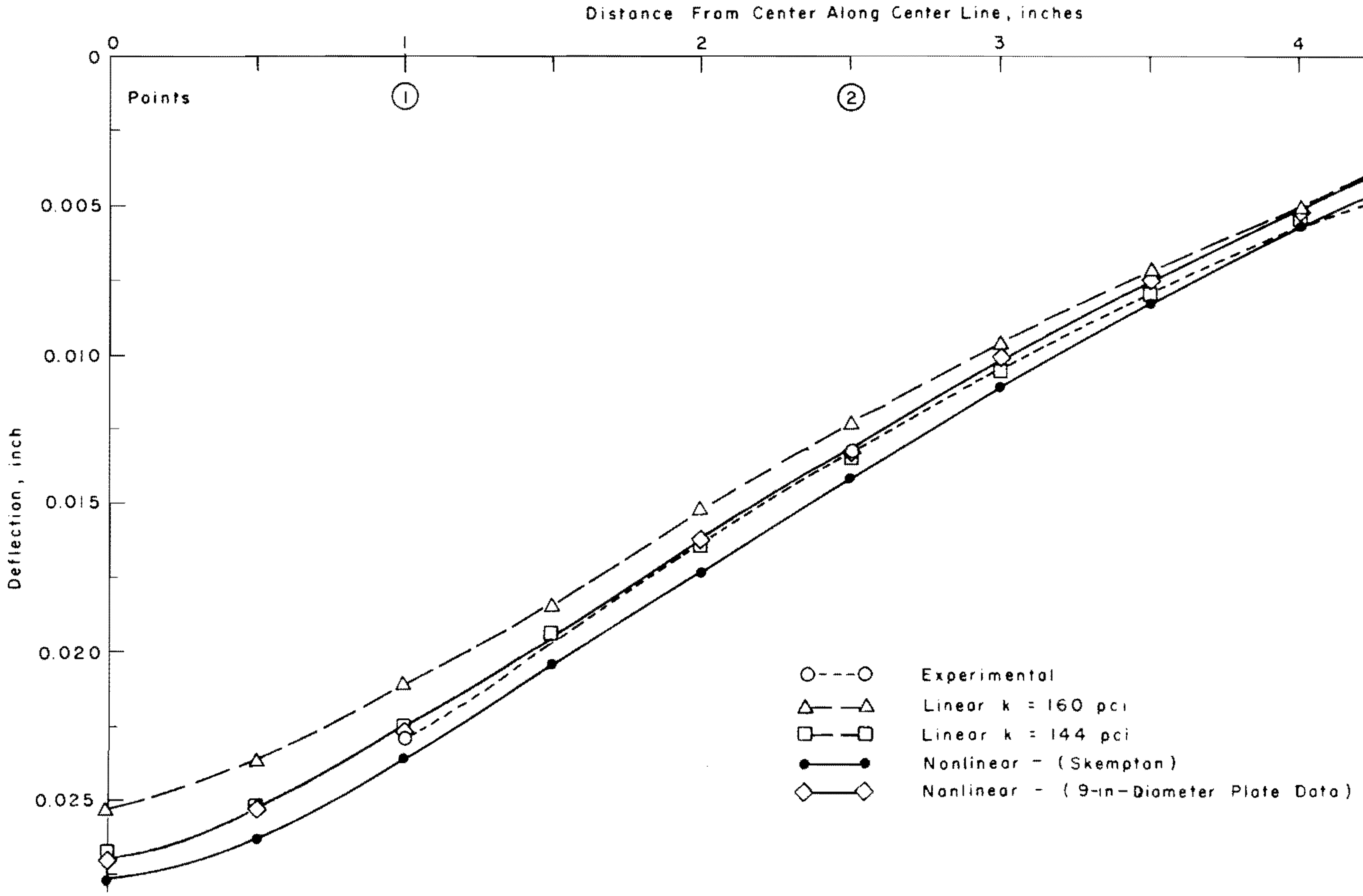


Fig 9. Experimental and analytical deflections on center line under center load of 100 pounds (series 330) (after Ref 13).

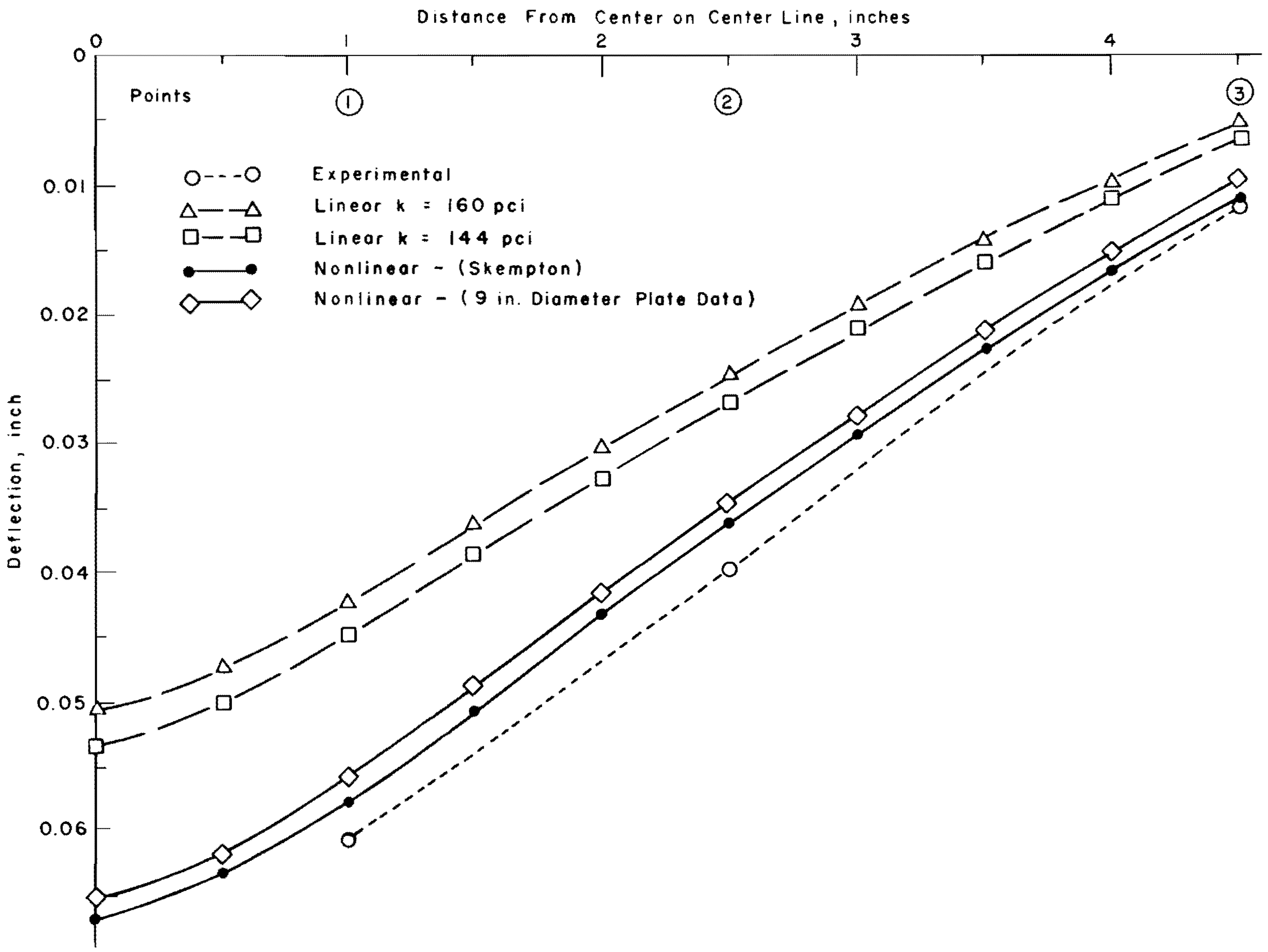


Fig 10. Experimental and analytical deflections on center line under center load of 200 pounds (series 330) (after Ref 13).

test for 10 cycles and for a constant deflection in the center load test for 7 cycles. The effect of such cycling was studied at the points controlling the deflections and on the rest of the slab. It was observed from these tests that, first, a stage of load stabilization seems to begin after a few cycles (Fig 11); such stabilization takes place earlier if the load is cycled for a deflection less than the maximum obtained during the first cycle. Second, the slopes and shapes of curves for deflections and stresses for cycle 1 and for those obtained in the stabilized stage after cycling are approximately the same. Third, the net deflections and stresses after cycling are, in general, greater than those for the first cycle. Fourth, by modifying the nonlinear support representation, based on the experimental evidence, it is possible to obtain computed solutions for cyclic loading comparable to the experimental results.

Modulus of Subgrade Reaction

The foregoing experimental evaluation of plates and slabs included an experimental evaluation of the modulus of subgrade reaction of the clay subgrade. Plate load tests were taken with circular rigid plates whose diameters ranged from 2 to 9 inches. The small-dimension slab tested on clay (Ref 13) was subsequently tested on the layered system also.

The load-deflection data of the plate tests on the clay subgrade are shown in Fig 11. The plate diameter influences the load-deformation characteristics of the clay soil and the pressure required to produce a given plate deflection increases as the plate diameter decreases. Similar observations were made from load-deflection data for a layered system (Ref 14). Using the data shown in Fig 12, the k-value for the clay can be calculated using tangent and/or secant moduli approaches. The initial straight line portion of the load-deflection curve gives the initial tangent modulus K (Fig 13). The secant modulus, however, is obtained by selecting points on the load deflection curve, depending on the deflection criteria considered. The ratio of load and deflection at each point gives an estimate of k .

From the load-deflection data for the plate tests on the layered system, an increase in k-value of 40 percent was obtained for the 1-1/2-inch thick layer of asphaltic material over the clay subgrade.

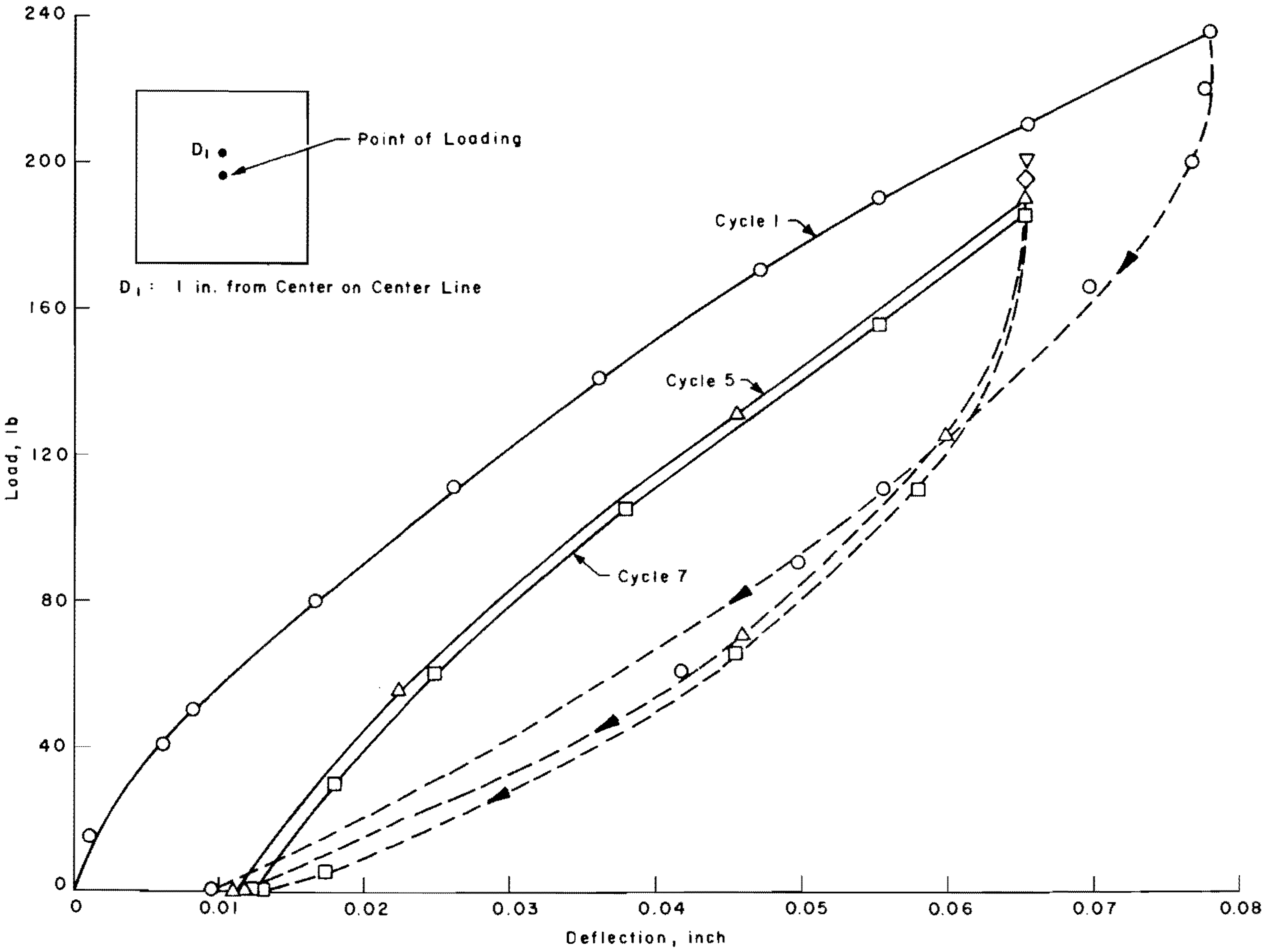


Fig 11. Load versus deflection for dial 1 under center cyclic load slab test (after Ref 13).

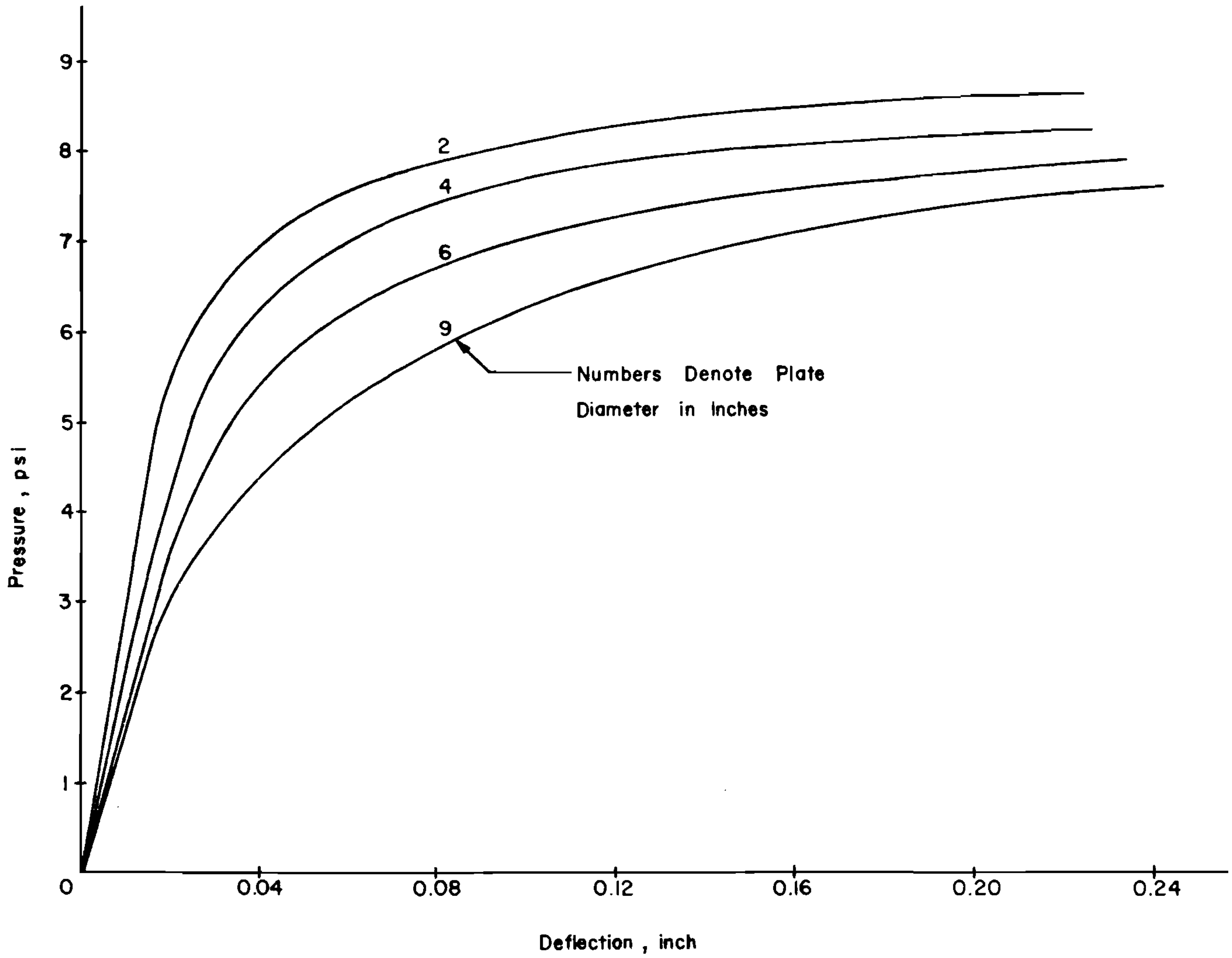


Fig 12. Average pressure versus deflection curves for plate load tests on clay, Series A, B, and C (Ref 14).

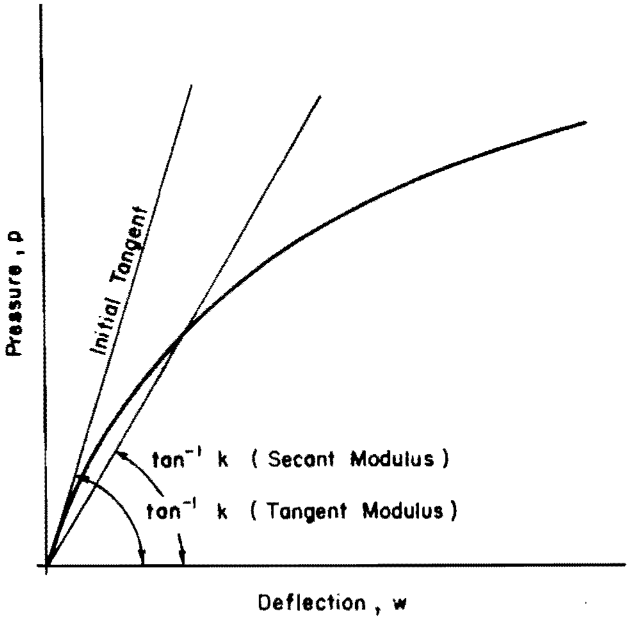


Fig 13. Tangent and secant moduli approaches for finding k-value (after Ref 14).

CHAPTER 4. APPLICATION OF PROJECT METHODS

The SLAB analysis method has been applied to various civil engineering problems in many ways. The two major applications of the SLAB method have been to highway pavement design problems by the Highway Department and project staff. Others have used the methods successfully for the analysis of airfield pavements. These studies were not made on this project, but serve to illustrate the broad applicability of the method. A few examples of these applications which are most significant and which stand alone are summarized in the following paragraphs.

Load Placement Analysis

Historically, different types of rigid pavements have been characterized by the load position selected for evaluating the design stress conditions. The SLAB method, with its inherent capabilities, has been used by other persons to conduct an extensive study on the effect of load placement on rigid pavement behavior (Ref 15). These effects of load placement have been determined for typical highway loading as well as airfield loadings. In this application, the researchers have gone one step further than determining the stress or behavior of the pavement with respect to load position and have related this to expected pavement life. Figures 14 and 15 show the relationship of pavement thickness, load position, and stress conditions for a typical highway load and for a typical jumbo jet aircraft loading respectively. For the design conditions which are shown in Figs 14 and 15, these pavement stresses have been related to pavement life through the use of the pavement life relationships developed at the Road Test (Ref 26). In Figs 16 and 17 the relative performance effects of load position can be estimated for the highway and the airfield loading conditions respectively.

This foregoing application of the SLAB method to the study of load positions with respect to rigid pavement design analysis should be carried one step further, that is, to evaluate the true effects of load transfer in these pavements, because no single pavement consists of a single pavement slab. This

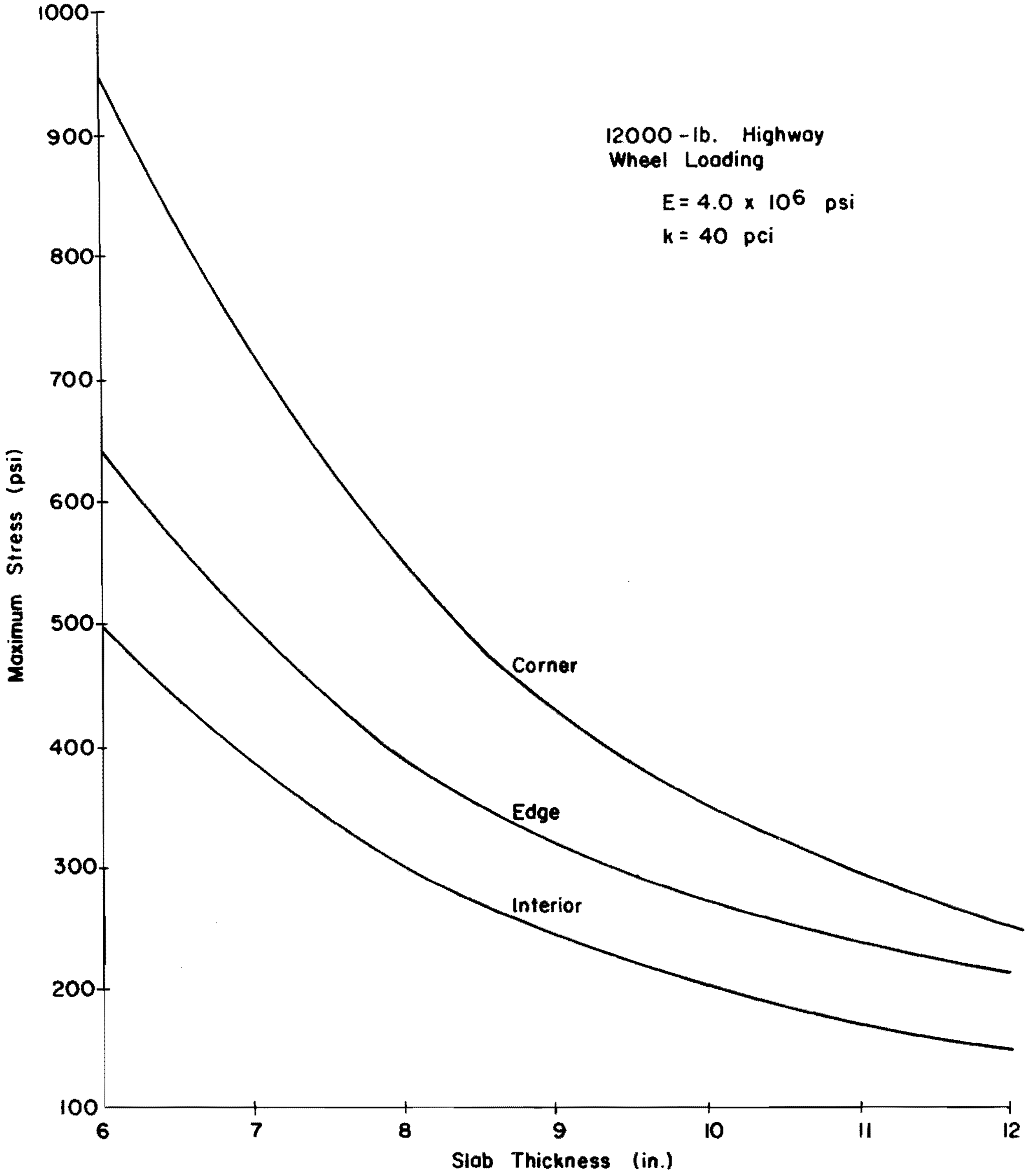


Fig 14. Influence of load placement and slab thickness on stress in highway pavement with k-value of 40 pci (Ref 15).

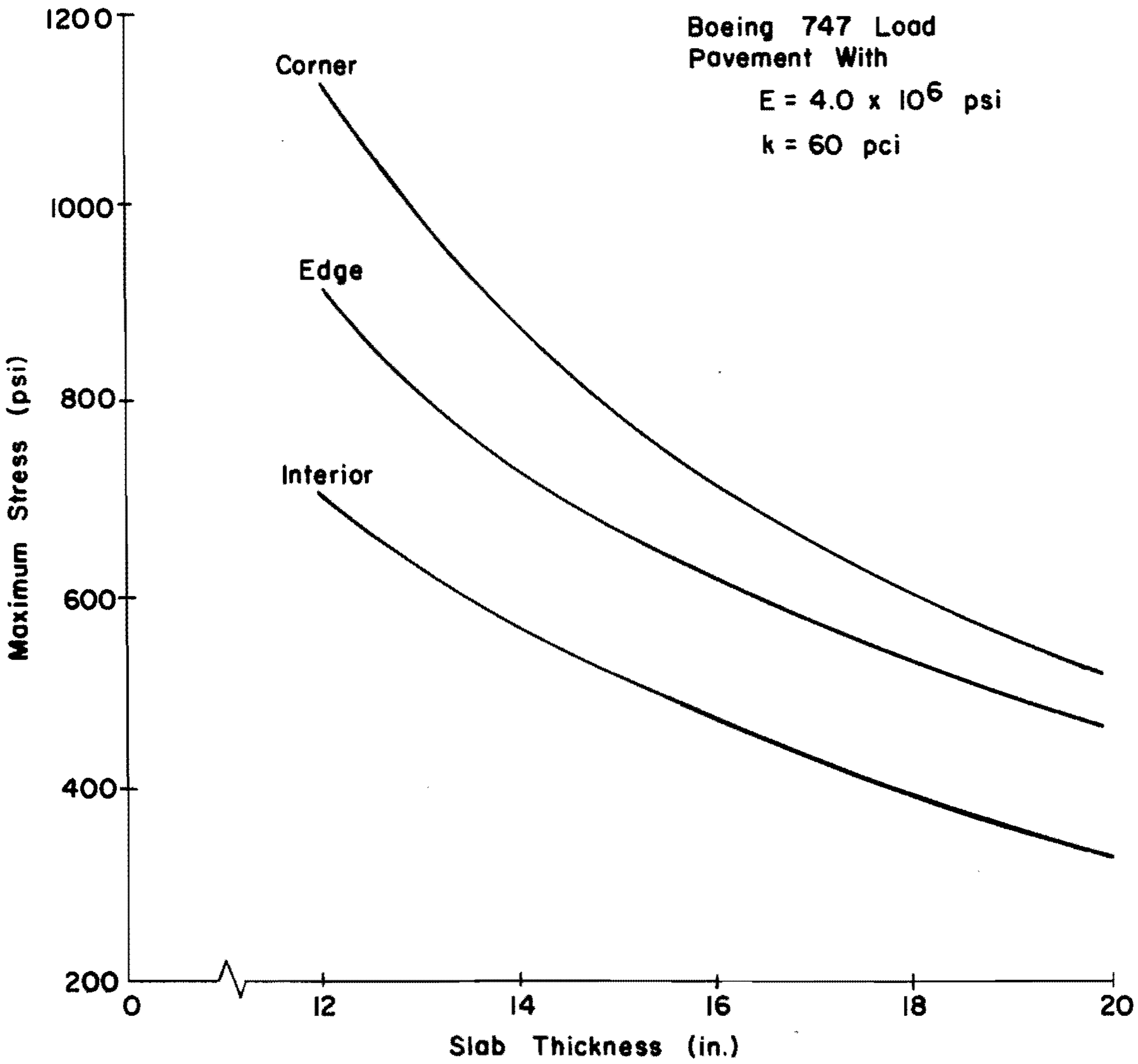


Fig 15. Effect of B-747 load placement and slab thickness on stress in airfield pavement with k-value of 60 pci (after Ref 15).

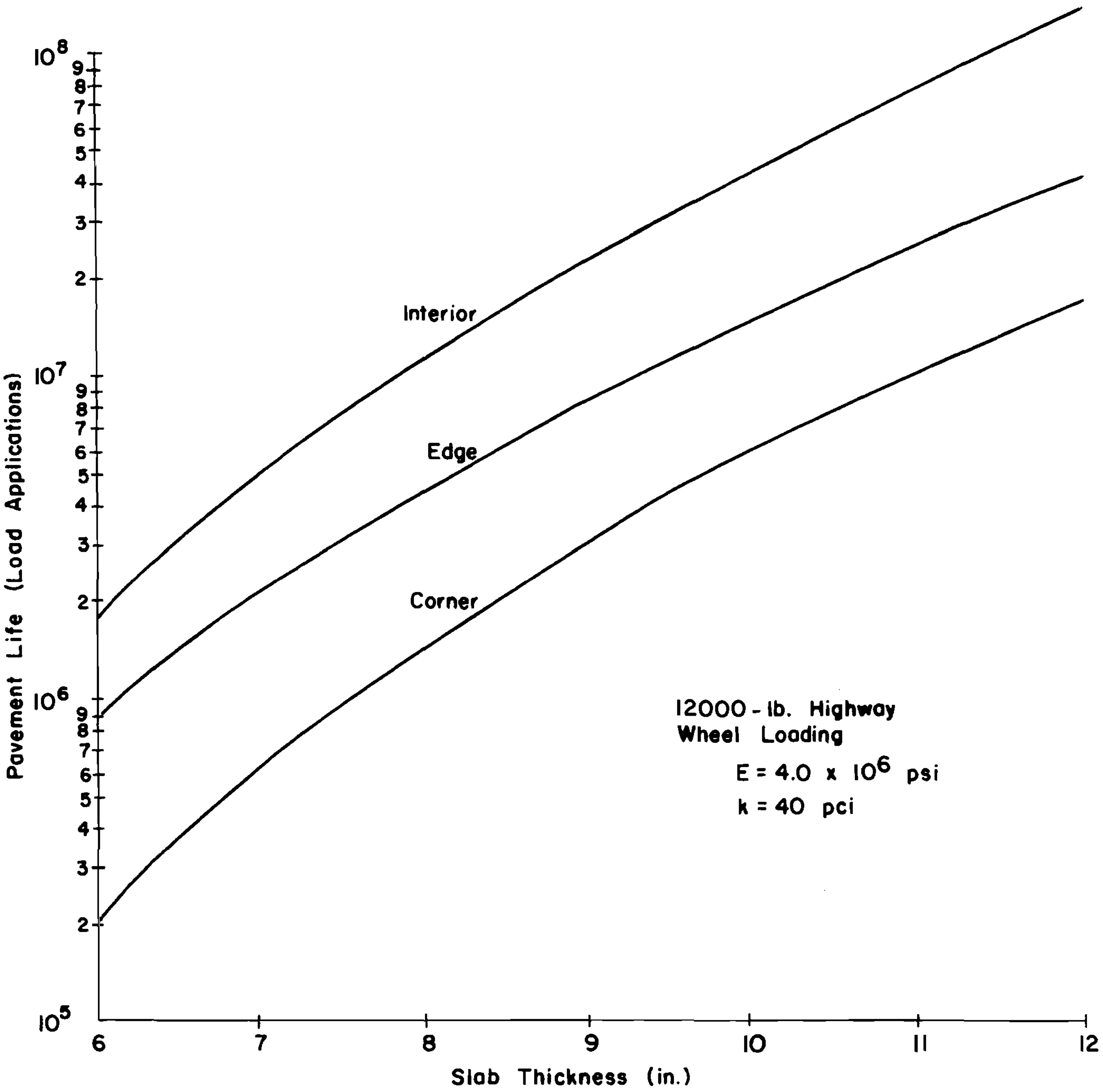


Fig 16. Influence of load placement and slab thickness on life of highway pavement with k-value of 40 pci (after Ref 15).

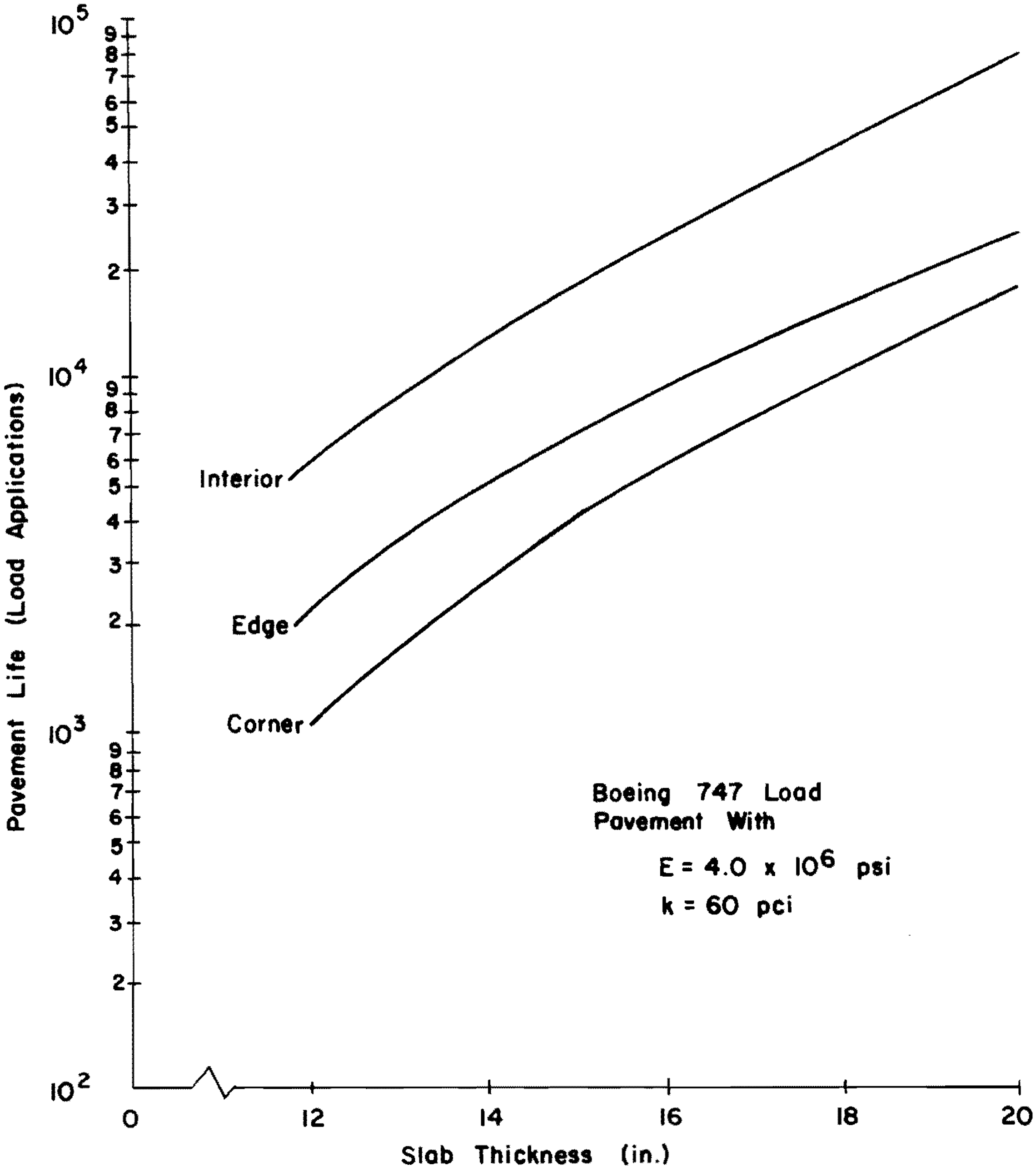


Fig 17. Influence of B-747 load placement and slab thickness on life of airfield pavement with k-value of 60 pci (after Ref 15).

is no doubt an area of research where an existing analysis tool could be used. There have been some significant applications of this method in design where the design problem was solved with discrete-element SLAB methods used to simulate load transfer and jointing conditions. Additional work is needed in this important area.

Analysis of Field Deflection Data

SLAB method has been used to analyze numerous continuously reinforced concrete pavements in the State of Texas which were used in a performance study by the Texas Highway Department (Refs 18 and 19). To analyze the effect of various design variables on pavement behavior, the SLAB programs were used. Pavement responses such as deflections and stresses were computed for a wide variety of input variables. Figure 18 shows a comparison of the field deflections and the computed deflections using the discrete-element slab computer program for analyzing these respective pavements (Ref 18). This application of the discrete-element SLAB analysis method was made prior to the detailed analysis of continuous reinforced concrete pavements using the discrete-element method, which is discussed in the next paragraph.

Theoretical Analysis of CRCP

The bending rigidity of structural members is affected significantly by discontinuities and such is the case in continuously reinforced concrete pavement which contains a series of tiny volume change cracks. These discontinuities have been analyzed theoretically by Abou-Ayyash and Hudson by application of the discrete-element method of slab analysis (Ref 19). It has been shown that the bending stiffness at cracked locations in continuously reinforced concrete pavement is reduced between 80 and 90 percent from the uncracked stiffness value. This is illustrated in Fig 19, which relates the longitudinal reinforcement percentage to the percentage reduction in bending stiffness for several different concrete strengths.

After theoretically characterizing the volume change cracks in continuously reinforced pavement, a sensitivity analysis of continuously reinforced pavement was made using the SLAB analysis method. For the range in the variables studied, it was found that slab bending stiffness and subgrade modulus explained most of the variations in deflections and principal moments. Crack spacing showed minor influence on slab behavior.

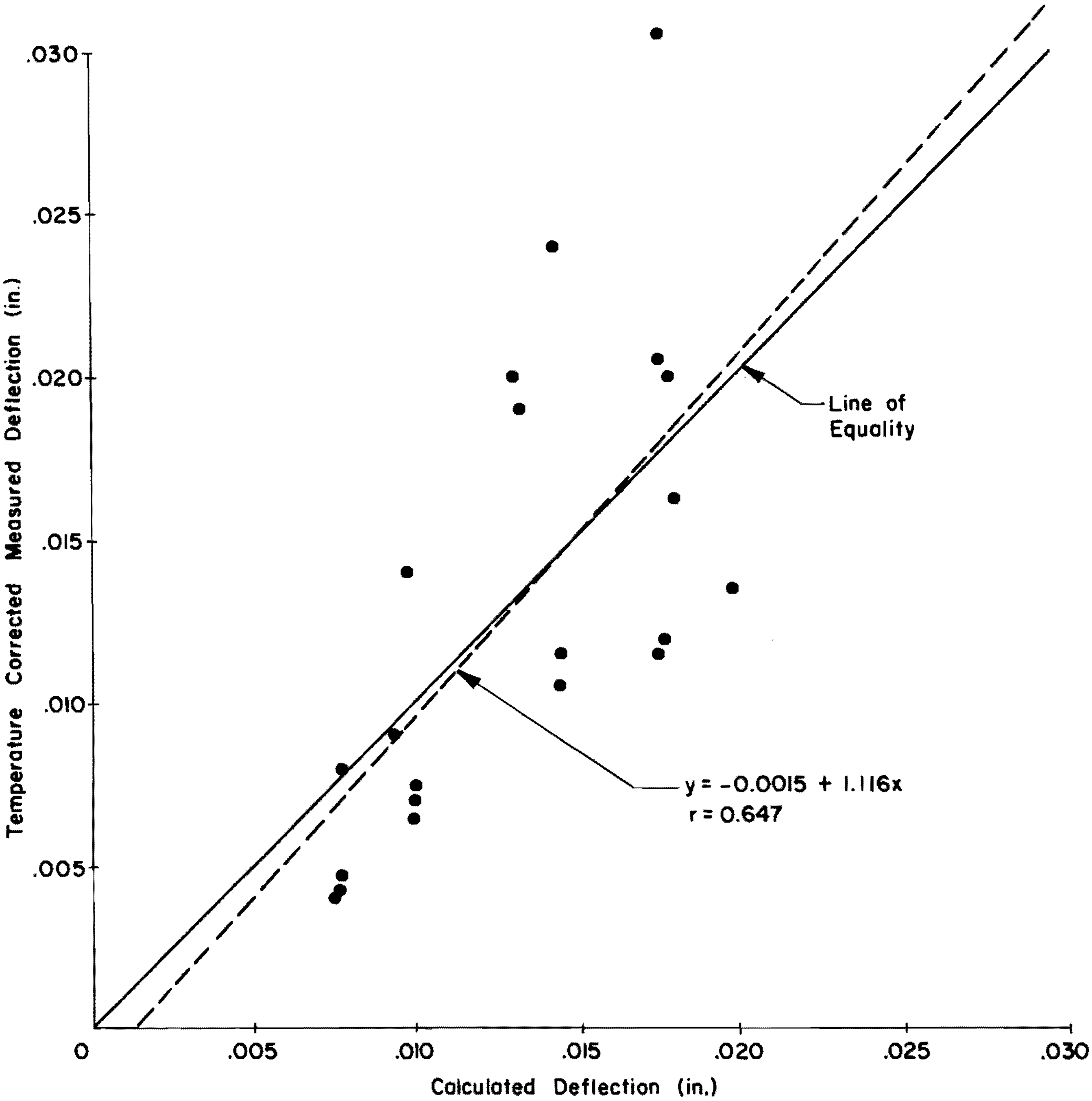
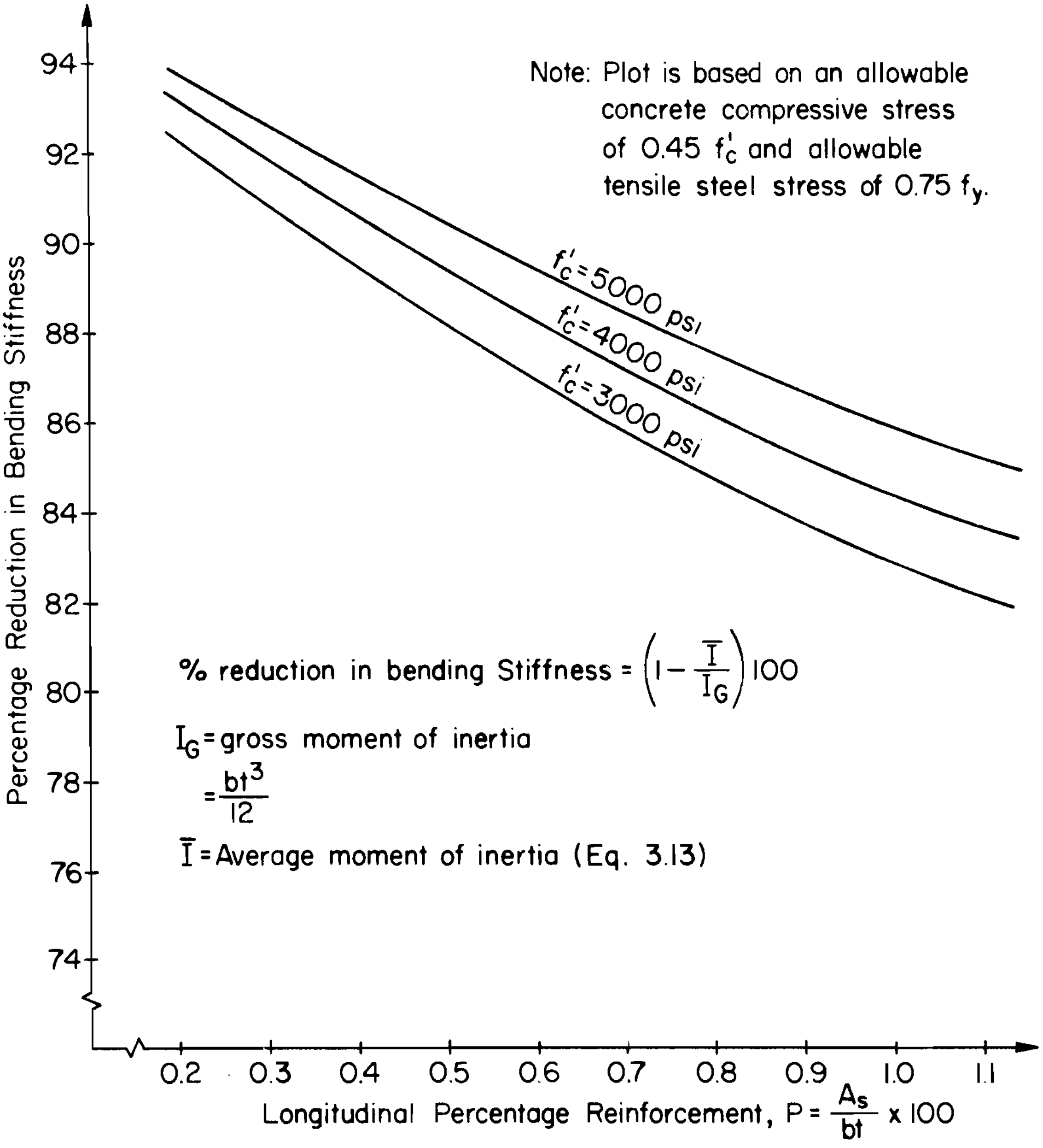


Fig 18. Comparison of measured and calculated deflections (after Ref 18).



Note: Plot is based on an allowable concrete compressive stress of 0.45 f'_c and allowable tensile steel stress of 0.75 f_y.

% reduction in bending Stiffness = $\left(1 - \frac{\bar{I}}{I_G}\right) 100$

I_G = gross moment of inertia

$= \frac{bt^3}{12}$

$\bar{I}$ = Average moment of inertia (Eq. 3.13)

Fig 19. Variation of the percentage reduction in bending stiffness at crack location with longitudinal percentage reinforcement (after Ref 19).

Design Charts

The discrete-element slab analysis method has been used to develop several different kinds of rigid pavement design charts. A design chart has been developed which allows the designer to take into account loss of support in the selection of a subgrade k value for use in rigid pavement design. This chart is reproduced here from Ref 20 as Fig 20. Still another design chart which has been developed using the SLAB method relates to a new deflection based pavement design criteria where a composite k value is selected for a maximum permissible design deflection for both jointed and continuously reinforced types of rigid pavements. This chart is shown as Fig 21 and has been extracted from Ref 20.

Design chart applications have also been made for a tracked air-cushioned vehicle. Design analyses have been made for the various loading conditions which are related to this tracked air-cushioned vehicle, namely, the air-cushion itself, the emergency ski load, and the lateral load due to wind loading, which is applied to the pavement slab through an anchored rail. A sample of these design charts, Fig 22 is included herein for illustrative purposes.

Analysis of Concrete Shoulder Pavements

The SLAB method has been applied to the analysis of continuously reinforced concrete pavement shoulders. This problem is essentially a load placement problem; however, it has several other characteristics which make it worth mentioning separately from the load placement study. This application included the careful modeling of transverse volume change cracks in the continuously reinforced pavements. Joints were modeled as cracked sections as well. The analysis showed the value of concrete shoulder pavements in terms of pavement life. Thus through the use of concrete shoulders, either insurance of pavement life would be realized or for a given pavement life a thickness reduction in the pavement could be anticipated, and possibly a savings realized on a large project (Ref 22). A typical profile of stress across a pavement with and without a concrete shoulder has been included here to illustrate the value of a concrete shoulder and is shown as Fig 23.

Other applications of the SLAB method to a lesser degree have included studies by the Corps of Engineers of the C-5A test pavements located at Vicksburg at the Waterways Experiment Station, an analysis of an experimental

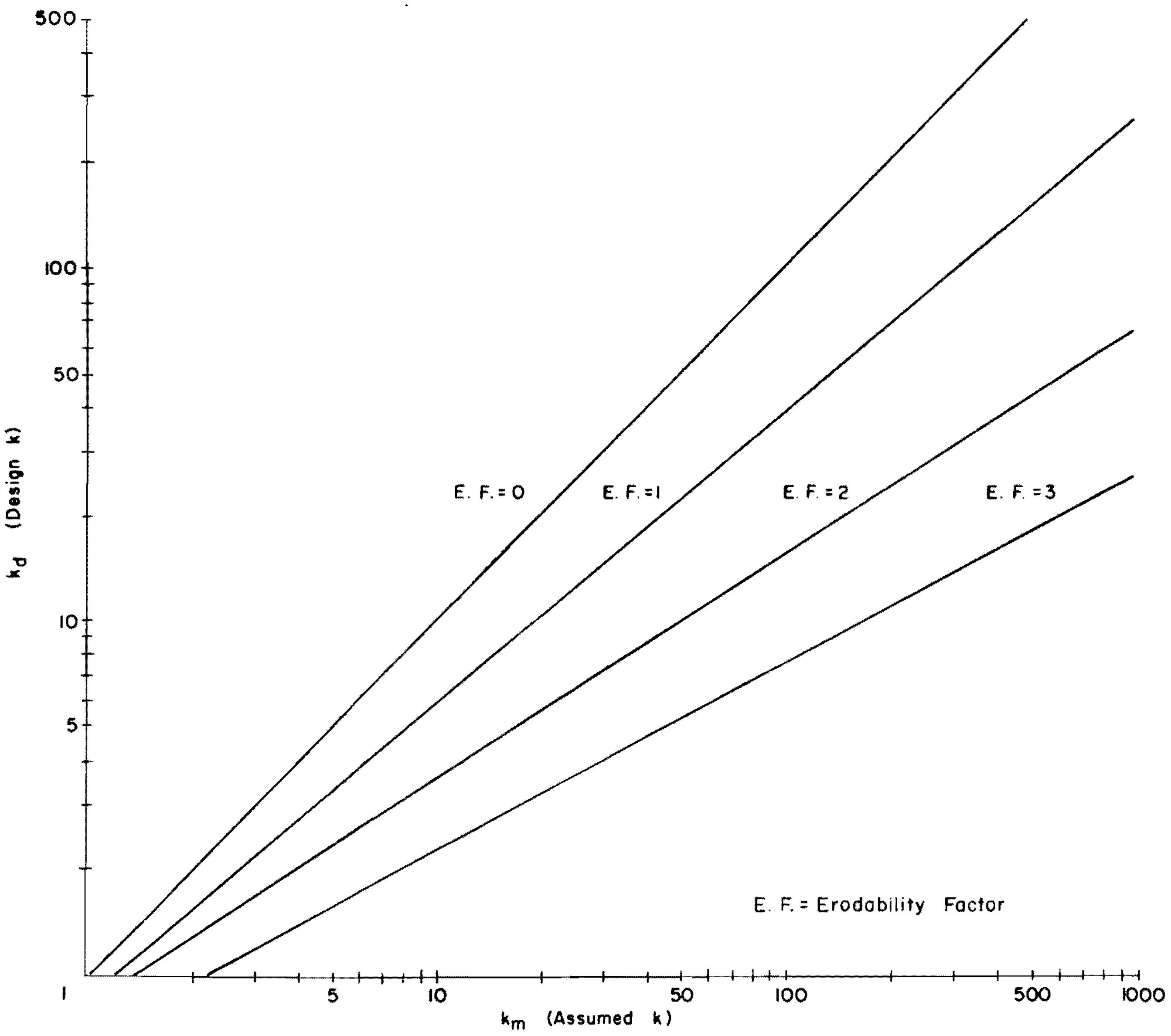


Fig 20. Correction of k-value for effect of erosive nature of subbase (after Ref 20).

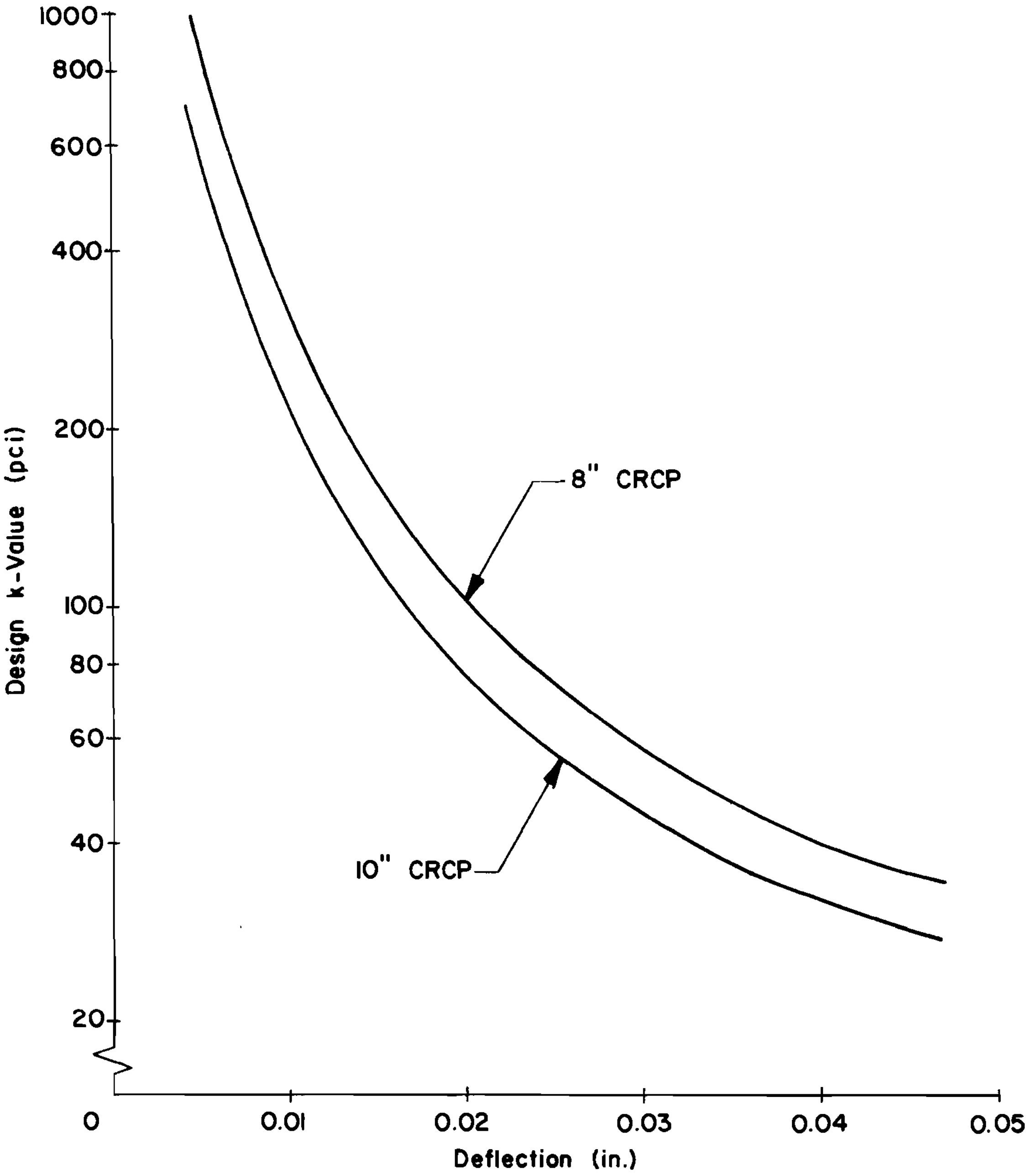


Fig 21. Variation of maximum permissable design deflection with composite subgrade support and pavement thickness (after Ref 20).

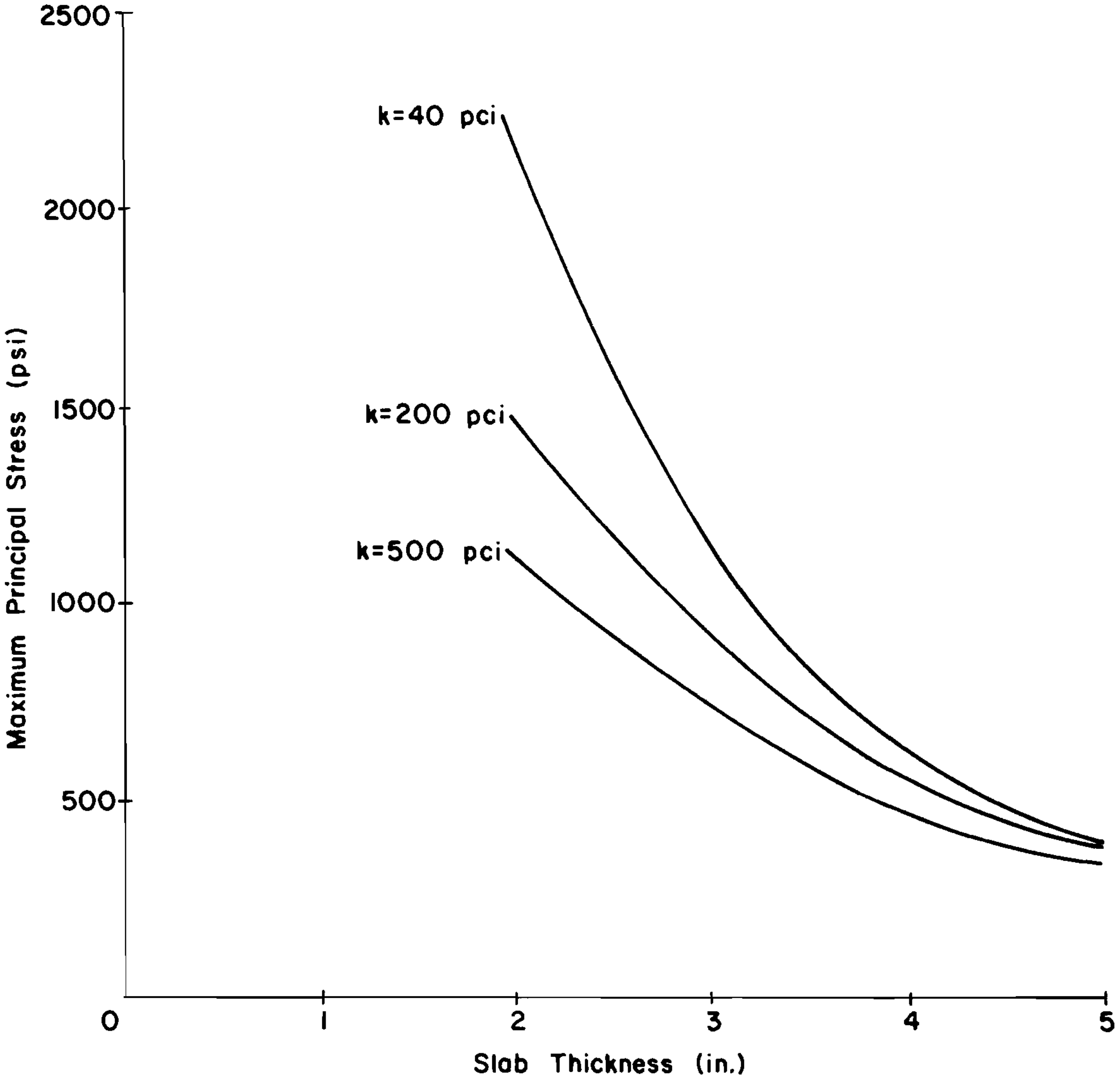


Fig 22. Variation of maximum principal stress with slab thickness for different subgrade modulus values.

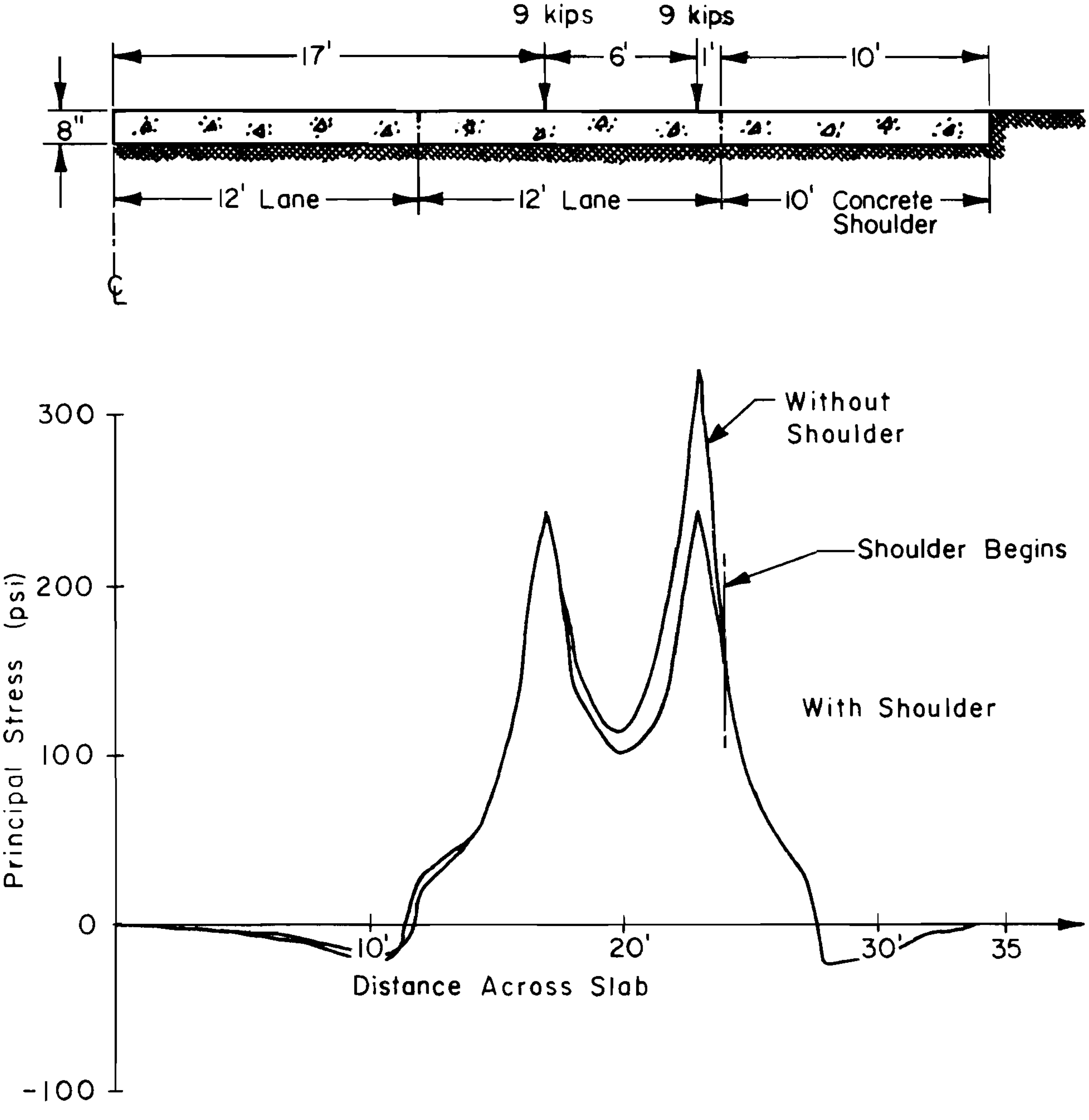


Fig 23. Transverse section and stress profile for 8-inch CRCP, with and without concrete shoulders (Ref 22).

pavement studies, and an analysis of experimental studies which have been used to verify the SLAB method itself (Ref 13).

Other Proposed Applications

There are many technical problems which the SLAB method could be applied to with worthwhile effort. The SLAB method could be used to analyze proposed overloads for which permits are requested for carrying these loads over state highway pavements and bridges. This analysis would not only give insight into what stress might occur, but also what deflection patterns would be encountered. A second application might be to special unique design problems such as a hydraulic inlet in CRCP which is in essence somewhat of an anchor in the continuously reinforced pavement and can cause cracking distress. Other design applications include skewed bridge approach slabs, ramp connections of freeways, and bridge approaches which are partially structurally supported and partially supported by foundations.

The third proposed application is in the rigid pavement design system being developed at The University of Texas (Ref 23). This proposed application has a wide variety of inputs that can improve the existing structural models as well as allow the addition of other necessary ones. Some of these applications would be as follows:

(1) Load transfer across rigid pavement joints is an important problem that has not received enough past attention. The SLAB method offers a unique tool to analyze the problem. Appropriate reductions in bending and twisting stiffnesses can be applied at the joints to simulate an existing load transfer condition.

(2) The effect of nonlinear material properties and nonlinear subgrade support on slab behavior can be investigated. This analysis offers a better characterization of actual conditions.

(3) Crack width in continuously reinforced concrete has a significant influence on slab response and hence pavement performance. To simulate this effect, different nonlinear moment curvature relationships for different crack widths can be used. Correlating the resulting deflections and/or stresses with field observations, a crack width design criterion can be established.

(4) Variations in deflections and stresses due to various tire pressures and corresponding areas of contact can be studied by using the SLAB method. This is highly significant since the rate of change of these responses is very high in the vicinity of the loaded area.

(5) Another proposed application is the study of the influence of skew joints on pavement behavior. Only a slight amount of work has been done in this area.

CHAPTER 5. IMPLEMENTATION

Implementation of technical developments in this research project has been somewhat slower than desired as a result of the complexities involved. Computer programs which replace hand computation methods of analysis are not readily implementable in a large-production organization without direct access to computer facilities and trained personnel. The computer programs for slab analysis which have been developed on this research project have been written in what is termed ASA standard FORTRAN so that the programs will be compatible with computers other than those for which they were developed with a minimum of FORTRAN conversion. The programs have been developed for several different computers during the past five years. The first working SLAB program, SLAB 17 (Ref 2), was developed on The University of Texas CDC 1604 computer. At that time the Texas Highway Department operated a similar computer, the CDC 1604A. This program was implemented on the Highway Department computer with reasonable ease.

As time and technology progressed, The University of Texas acquired the CDC 6600 computer facility. With this computing facility, SLAB programs became larger and more complex. The next SLAB program developed which was implemented on the Texas Highway Department computer facility was SLAB 30 (Ref 5). By this time, the Highway Department no longer used the 1604A, but had obtained an IBM system 360 computer facility. The SLAB 30 computer program was implemented on that computer and has been used for some time.

The most recent computer development which has been implemented on the Highway Department IBM 360 computer is SLAB 49. This computer program is very similar to SLAB 30, but has in it many economizing features with regard to computer time as well as other general output capabilities or user oriented capabilities (Ref 24). This program, SLAB 49, represents the most up-to-date discrete-element method of SLAB analysis. The program has such features as printer plots, microfilm plots, perspective view plots, and print-out of only a selected portion of the complete set of computed output.

User's Guides

Two user's guides or instruction manuals have been prepared for the use of the SLAB programs. The first such guide was developed by the Texas Highway Department for program SLAB 30 as it was operational on the Texas Highway Department IBM 360 computer system. This user's guide provided information for the design engineer which was required to solve design analysis problems using the computer method (Ref 25).

The second user's guide was made in the form of an applications report, (Ref 22) wherein a typical pavement design problem was selected and analyzed using the computer method SLAB 49. All of the computations, decisions, coding, etc. are presented in detail so that the design engineer may follow this example in solving similar problems with the same computer method. This applications report includes all of the required inputs which the design engineer must provide, the decisions he must make with regard to materials properties, design loads, etc. Also included is the computed output for further analysis and selection of design values.

With such user guides, the comprehensive, complex slab analysis method can be simplified for the practicing engineer to use in solution of design problems. The one thing required to get implementation going and keep it going is to have experienced people with some computer capabilities using the method, to understand the values which may be realized through the use of the computer aided design analysis as well as other engineering functions.

Implementation of SLAB in Project 123

In furthering the developments in the rigid pavement design systems developed through Research Project 123 at The University of Texas at Austin, the use of the discrete-element slab method is being considered in several capacities for inclusion in the design system. The first thought would be to include the SLAB program as a separate structural analysis routine in the RPS program. This, however, may not be economically feasible, due to the time requirements for solution of the SLAB program. Therefore, other techniques are being considered. One technique which is currently under consideration is the development of a regression equation for solving for stress and deflection in the pavement slab. This regression equation would have to be developed from a large number of slab solutions which would encompass the entire range of load configurations, load ranges, pavement material property ranges, and pavement types. In general, it would be a very comprehensive sensitivity analysis of the SLAB program and would be a costly undertaking from the standpoint of computer time and manpower required. Some development work in this area has been done (Ref 19) for continuously reinforced concrete pavements for a single loading condition where some of the paving materials parameters were studied over reasonable ranges. Other attempts are also being made to include the SLAB method in the rigid pavement design system.

CHAPTER 6. SUMMARY

Twenty-seven reports on this research project have been prepared and forwarded to the sponsor. Of these ten are directly applicable to SLAB and pavement analysis and design. These reports document the technical developments and verifications of these analytical techniques and illustrate how the method can be applied to the engineering design of rigid pavements. This report has summarized some of the important developments on this research project which are pertinent to pavement slab analysis and design.

The analytical method has been developed, verified, and refined as well as implemented to some degree. Continued use in engineering practice and in educational functions will help to acquaint more engineers with the method and help to make it a routine capability in pavement design offices.

The following is a list of brief recommendations which are offered at the conclusion of this project:

(1) The SLAB method should continue to be used for special problem analysis by the Texas Highway Department.

(2) The analytical capability should be incorporated in the rigid pavement design system as early as possible.

(3) The method should be extended to be more generally applicable to composite rigid pavements and to efficiently handle nonlinear materials properties as well as nonlinear subgrade support values.

REFERENCES

1. Love, A. E. H., A Treatise on the Mathematical Theory of Elasticity, Fourth Edition, Dover Publications, New York, 1944.

2. Hudson, W. Ronald, and Hudson Matlock, "Discontinuous Orthotropic Plates and Slabs," Research Report No. 56-6, Center for Highway Research, The University of Texas at Austin, May 1966.

3. Stelzer, C. Fred, Jr., and W. Ronald Hudson, "A Direct Computer Solution for Plates and Pavement Slabs," (DSLAB 5), Research Report No. 56-9, Center for Highway Research, The University of Texas at Austin, October 1967.

4. Pearre, Charles M., III, and W. Ronald Hudson, "A Discrete-Element Solution of Plates and Pavement Slabs Using a Variable-Increment-Length Model," Research Report No. 56-11, Center for Highway Research, The University of Texas at Austin, April 1969.

5. Panak, John J., and Hudson Matlock, "A Discrete-Element Method of Multiple-Loading Analysis for Two-Way Bridge Floor Slabs," Research Report No. 56-13, Center for Highway Research, The University of Texas at Austin, January 1970.

6. Salani, Harold J., and Hudson Matlock, "A Finite-Element Method for Vibrating Beams and Plates," Research Report No. 56-8, Center for Highway Research, The University of Texas at Austin, June 1967.

7. Kelley, Allen E., and Hudson Matlock, "Dynamic Analysis of Discrete-Element Plates on Nonlinear Foundations," Research Report No. 56-17, Center for Highway Research, The University of Texas at Austin, January 1970.

8. Wilson, Edward L., and Ray W. Clough, "Dynamic Response by Step-by-Step Matrix Analysis," Symposium on the Use of Computers in Civil Engineering, Lisbon, Portugal, October 1962.

9. Endres, Frank L., and Hudson Matlock, "An Algebraic Equation Solution Process Formulated in Anticipation of Banded Linear Equations," Research Report No. 56-19, Center for Highway Research, The University of Texas at Austin, January 1971.

10. Vora, Mahendrakumar R., and Hudson Matlock, "A Discrete-Element Analysis for Anisotropic Skew Plates and Grids," Research Report No. 56-18, Center for Highway Research, The University of Texas at Austin, August 1970.

11. Matlock, Hudson, and T. A. Haliburton, "A Finite-Element Method of Solution for Linearly Elastic Beam-Columns," Research Report No. 56-1, Center for Highway Research, The University of Texas at Austin, September 1966.

12. Matlock, Hudson, and T. P. Taylor, "A Computer Program to Analyze Beam-Columns Under Movable Loads," Research Report No. 56-4, Center for Highway Research, The University of Texas at Austin, June 1968.

13. Agarwal, Sohan L., and W. Ronald Hudson, "Experimental Verification of Discrete-Element Solutions for Plates and Pavement Slabs," Research Report No. 56-15, Center for Highway Research, The University of Texas at Austin, April 1970.

14. Siddiqi, Qaiser S., and W. Ronald Hudson, "Experimental Evaluation of Subgrade Modulus and Its Application in Model Slab Studies," Research Report No. 56-16, Center for Highway Research, The University of Texas at Austin, January 1970.

15. Treybig, Harvey J., W. Ronald Hudson, and B. F. McCullough, "Effect of Load Placement on Rigid Pavement Behavior," Paper No. 8543, Transportation Engineering Journal, Proceedings of ASCE, November 1971.

16. McCullough, B. F., and Harvey J. Treybig, "A Statewide Deflection Study of Continuously Reinforced Concrete Pavement in Texas," Highway Research Record No. 239, 1968, pp 150-174.

17. Treybig, Harvey, "Performance of Continuously Reinforced Concrete Pavement in Texas," Highway Research Record No. 291, 1969, pp 32-47.

18. Treybig, Harvey J., "Observation and Analysis of Continuously Reinforced Concrete Pavement," Research Report No. 46-7, Texas Highway Department, April 1968.

19. Abou-Ayyash, Adnan, and W. Ronald Hudson, "Analysis of Bending Stiffness Variation at Cracks in Continuous Pavements," Research Report No. 56-22, Center for Highway Research, The University of Texas at Austin, August 1971.

20. McCullough, B. F., and W. R. Hudson, "Subbase Design Manual for Portland Cement Concrete Pavements," A Report submitted to United States Steel Corporation, Pittsburgh, Pennsylvania, Materials Research and Development Corporation, November 1970.

21. Treybig, Harvey J., and B. F. McCullough, "A Recommended Procedure for the Design of Rigid Pavements," Research Report No. 502-1F, Center for Highway Research, The University of Texas at Austin, December 1971.

22. Treybig, Harvey J., W. Ronald Hudson, and Adnan Abou-Ayyash, "Application of Slab Analysis Methods to Rigid Pavement Problems," Research Report No. 56-26, Center for Highway Research, The University of Texas at Austin, May 1972.

23. Kher, Ramesh K., W. Ronald Hudson, and B. Frank McCullough, "A Systems Analysis of Rigid Pavement Design," Research Report No. 123-5, published jointly by Texas Highway Department; Center for Highway Research, The University of Texas at Austin; Texas Transportation Institute, Texas A&M University, November 1970.

24. Panak, John J., and Hudson Matlock, "A Discrete-Element Method of Analysis for Orthogonal Slab and Grid Bridge Floor Systems," Research Report No. 56-25, Center for Highway Research, The University of Texas at Austin, August 1971.

25. "Users Manual-Computer Program SLAB 30," Texas Highway Department, Bridge Division, January 1970.

26. Langsner, George, T. S. Huff, and Wallace J. Liddle, "Use of Road Test Findings by AASHO Design Committee," Special Report 73, Highway Research Board, 1962.

27. Wilson, Edward L., and Ray W. Clough, "Dynamic Response by Step-by-Step Matrix Analysis," Symposium on the Use of Computers in Civil Engineering, Lisbon, Portugal, October 1962.

THE AUTHORS

W. Ronald Hudson is an Associate Professor of Civil Engineering at The University of Texas at Austin. He has had a wide variety of experience as a research engineer with the Texas Highway Department and the Center for Highway Research at The University of Texas at Austin and was Assistant Chief of the Rigid Pavement Research Branch of the AASHO Road Test. He is the author of numerous publications and was the recipient of the 1967 ASCE J. James R. Croes Medal. He is presently concerned with research in the areas of (1) analysis and design of pavement management systems, (2) measurement of pavement roughness performance, (3) slab analysis and design, and (4) tensile strength of stabilized subbase materials.

Harvey J. Treybig is currently associated with Austin Research Engineers, Inc. While at the Center for Highway Research at The University of Texas at Austin he was involved in the application and implementation of rigid pavement analysis methods and systems analysis methods for pavement design. Prior to this, he was involved in continuously reinforced concrete pavement design and research at the Texas Highway Department.

Adnan Abou-Ayyash is a graduate research assistant with the Center for Highway Research at The University of Texas at Austin. A native of Lebanon, his previous experience in that country includes design and construction of various pavement structures. He is currently involved with the application of research on computer simulation of slabs to real, practical problems. His research interest is primarily in (1) slab analysis and design, and (2) pavement system.