AN ALGEBRAIC EQUATION SOLUTION PROCESS FORMULATED IN ANTICIPATION OF BANDED LINEAR EQUATIONS

by

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Development of Methods for Computer Simulation of Beam-Columns and Grid-Beam and Slab Systems

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PREFACE

Computer storage and time requirements for structural problems will often determine whether or not a particular program is feasible to use on an extensive basis. For multi-dimensioned problems requiring fine mesh spacing and involving nonlinear or time-dependent behavior, careful attention must be given to the efficiency of the solution process, even with the largest and fastest computers in use today.

This report describes a system of equation solving routines that may be applied to a wide variety of problems by utilizing them within appropriate programs. The routines will not be directly apparent to the structural or pavement design engineer in routine work; instead, it is the one who develops the program or the one who is concerned with fitting a program to run on a particular computer who will be directly involved with the material in this report.

The routines have been incorporated in many of the programs described in other reports of the current project. Because they are potentially very useful in optimizing solution processes in future developments, it was decided to document the routines separately by means of this report.

LIST OF REPORTS

Report No. 56-1, "A Finite-Element Method of Solution for Linearly Elastic Beam-Columns" by Hudson Matlock and T. Allan Haliburton, presents a finite-element solution for beam-columns that is a basic tool in subsequent reports.

Report No. 56-2, "A Computer Program to Analyze Bending of Bent Caps" by Hudson Matlock and Wayne B. Ingram, describes the application of the beam-column solution to the particular problem of bent caps.

Report No. 56-3, "A Finite-Element Method of Solution for Structural Frames" by Hudson Matlock and Berry Ray Grubbs, describes a solution for frames with no sway.

Report No. 56-4, "A Computer Program to Analyze Beam-Columns under Movable Loads" by Hudson Matlock and Thomas P. Taylor, describes the application of the beam-column solution to problems with any configuration of movable non-dynamic loads.

Report No. 56-5, "A Finite-Element Method for Bending Analysis of Layered Structural Systems" by Wayne B. Ingram and Hudson Matlock, describes an alternating-direction iteration method for solving two-dimensional systems of layered grids-over-beams and plates-over-beams.

Report No. 56-6, "Discontinuous Orthotropic Plates and Pavement Slabs" by W. Ronald Hudson and Hudson Matlock, describes an alternating-direction iteration method for solving complex two-dimensional plate and slab problems with emphasis on pavement slabs.

Report No. 56-7, "A Finite-Element Analysis of Structural Frames" by T. Allan Haliburton and Hudson Matlock, describes a method of analysis for rectangular plane frames with three degrees of freedom at each joint.

Report No. 56-8, "A Finite-Element Method for Transverse Vibrations of Beams and Plates" by Harold Salani and Hudson Matlock, describes an implicit procedure for determining the transient and steady-state vibrations of beams and plates, including pavement slabs.

Report No. 56-9, "A Direct Computer Solution for Plates and Pavement Slabs" by C. Fred Stelzer, Jr., and W. Ronald Hudson, describes a direct method for solving complex two-dimensional plate and slab problems.

Report No. 56-10, "A Finite-Element Method of Analysis for Composite Beams" by Thomas P. Taylor and Hudson Matlock, describes a method of analysis for composite beams with any degree of horizontal shear interaction.

Report No. 56-11, "A Discrete-Element Solution of Plates and Pavement Slabs Using a Variable-Increment-Length Model" by Charles M. Pearre, III, and W. Ronald Hudson, presents a method of solving for the deflected shape of freely discontinuous plates and pavement slabs subjected to a variety of loads.

Report No. 56-12, "A Discrete-Element Method of Analysis for Combined Bending and Shear Deformations of a Beam" by David F. Tankersley and William P. Dawkins, presents a method of analysis for the combined effects of bending and shear deformations.

Report No. 56-13, "A Discrete-Element Method of Multiple-Loading Analysis for Two-Way Bridge Floor Slabs" by John J. Panak and Hudson Matlock, includes a procedure for analysis of two-way bridge floor slabs continuous over many supports.

Report No. 56-14, "A Direct Computer Solution for Plane Frames" by William P. Dawkins and John R. Ruser, Jr., presents a direct method of solution for the computer analysis of plane frame structures.

Report No. 56-15, "Experimental Verification of Discrete-Element Solutions for Plates and Slabs" by Sohan L. Agarwal and W. Ronald Hudson, presents a comparison of discrete-element solutions with the small-dimension test results for plates and slabs, along with some cyclic data on the slab.

Report No. 56-16, "Experimental Evaluation of Subgrade Modulus and Its Application in Model Slab Studies" by Qaiser S. Siddiqi and W. Ronald Hudson, describes an experimental program developed in the laboratory for the evaluation of the coefficient of subgrade reaction for use in the solution of small dimension slabs on layered foundations based on the discrete-element method.

Report No. 56-17, "Dynamic Analysis of Discrete-Element Plates on Nonlinear Foundations" by Allen E. Kelly and Hudson Matlock, presents a numerical method for the dynamic analysis of plates on nonlinear foundations.

Report No. 56-18, "Discrete-Element Analysis for Anisotropic Skew Plates and Grids" by Mahendrakumar R. Vora and Hudson Matlock, describes a tridirectional model and a computer program for the analysis of anisotropic skew plates or slabs with grid-beams.

Report No. 56-19, "An Algebraic Equation Solution Process Formulated in Anticipation of Banded Linear Equations" by Frank L. Endres and Hudson Matlock, describes a system of equation-solving routines that may be applied to a wide variety of problems by utilizing them within appropriate programs.

Report No. 56-20, "Finite-Element Method of Analysis for Plane Curved Girders" by William P. Dawkins, presents a method of analysis that may be applied to plane-curved highway bridge girders and other structural members composed of straight and curved sections.

Report No. 56-21, "Linearly Elastic Analysis of Plane Frames Subjected to Complex Loading Condition" by Clifford O. Hays and Hudson Matlock, presents a design-oriented computer solution of plane frame structures that has the capability to economically analyze skewed frames and trusses with variable cross-section members randomly loaded and supported for a large number of loading conditions.

ABSTRACT

A general method for the solution of large, sparsely banded, positive-definite, coefficient matrices is presented. The goal in developing the method was to produce an efficient and reliable solution process and to provide the user-programmer with a package which is problem-independent, efficient, and easy to use, so that program development time can be spent in problem analysis rather than on solution technique.

The procedures have been developed specifically to deal with matrices generated by three and five-wide difference operators, whether symmetrical or unsymmetrical.

KEY WORDS: structural analysis, numerical analysis, computers, mathematics, banded equations, finite differences.

SUMMARY

We see that for systems having numerous constant vectors or coefficient matrices which are large, sparse, or possess a moderate degree of bandedness direct methods are generally preferable to iterative methods.

Since we have considered both symmetric and nonsymmetric problems as well as three and five-wide difference operators, four separate routines have been programmed. For the nonsymmetric case having a three-wide difference operator TRIP 3 (three-wide recursion inversion procedure) was developed, TRIP 4 is for the symmetric case. For the five-wide nonsymmetric case there is FRIP 3, and for the symmetric case FRIP 4. The above four routines are flow charted and listed in Appendix A.

Each of these routine drives (or calls upon) a group of secondary matrix manipulation routines referred to as the SUMP pack (Submatrix Manipulation Package). The particular group used by these four routines is SUMP 6. For a more complete description of these secondary routines see Appendix B.

The user of the solution procedure must in some way transmit his stiffness matrix to it, and since for the solution procedure only one partitioned level is needed at each recurrence of the algorithm, a shuttle routine which is called at each step must be provided by the user. It is here that he computes or fetches the appropriate level or partition of his stiffness matrix. A complete description of this routine is given in Appendix C while an example of the use of the entire package is presented in Appendix D.

IMPLEMENTATION STATEMENT

The routines developed and explained herein are of immediate benefit to the engineer who is developing a structural analysis program. The main body of the report is concerned with the theoretical development of the procedures and the appendices pertain to their implementation. These routines provide the engineer-programmer not only with a ready-to-use, efficient solution package, but with one that requires a minimum of input and reorganization of the natural form of the stiffness matrix.

TABLE OF CONTENTS

NOMENCLATURE

Symbol	Definition
a_i	Submatrix element of coefficient matrix
A	Coefficient matrix
A_i	Preliminary unknown vector
b_i	Submatrix element of coefficient matrix
B_i	Recursion coefficient
c_i	Submatrix element of coefficient matrix
C_i	Recursion coefficient
d_i	Submatrix element of coefficient matrix
D_i	Recursion multiplier
e_i	Submatrix element of coefficient matrix
E_i	Recursion multiplier
f_i	Subvector element of constant vector
F , f_ℓ	Family of constant vectors
K	Order of partitioned submatrix
L	N/K
N	Order of coefficient matrix
N_1 , N_2 , ... , N_5	Band widths, respectively, of a_i , b_i , ... , e_i
w_i	Subvector element of unknown vector

Symbol	Definition
W , w_ℓ	Family of unknown vectors
X	Coordinate in short direction of grid
Y	Coordinate in long direction of grid
Z^{-1}	Inverse of the matrix Z (any matrix)
Z^T	Transpose of the matrix Z (any matrix)
ϕ	A zero vector, matrix, or region

CHAPTER 1. INTRODUCTION

In physical problems, closed-form solutions are not always available; therefore, discrete methods must be employed to approximate the true solution. One can think of representing a continuous problem by a satisfactory discrete-element model and solve directly for the solution of the model and thereby obtain an approximate solution to the real problem.

In either case, the solution technique boils down to the solving of a system of linear algebraic equations that exhibit a high degree of sparseness and whose non-zero elements tend to cluster about the main diagonal of the coefficient matrix. Most physical systems may be ordered so that this banding phenomena will be observed, but, when care is taken, the width of the band may be reduced, thereby compacting the band and allowing for a more efficient solution. Furthermore, if we enforce a preset method of ordering the grid points, then the form of our finite-difference operator will produce banding within the main band.

It is the purpose of this report to present an efficient, general method for the solution of this type of problem.

Recursion Technique

The general problem can be subdivided in several ways. It can be symmetric or non-symmetric, or it can have three subbands or five. These different considerations are taken up in Chapter 2, along with the associated proofs that verify the method.

Requirements

In Chapter 3 the amount of work (computations) along with the amount of storage required is taken into consideration. Graphs showing actual times involved for a wide range of problems are presented.

Use

The total package can be visualized as three main parts. The first part contains the recursion algorithm and is listed and flow charted in Appendix A. The next part drives a secondary group of matrix manipulation routines, more commonly referred to as the SUMP package. These are located and flow charted in Appendix B. The third and most crucial part to the user is the FSUB routine, where the coefficient matrix is defined. A description of what is necessary for this routine is included in Appendix C, and the relationship of the complete group to the user's main program is discussed in Appendix D.

CHAPTER 2. RECURSION INVERSION

The General Problem

While the type of coefficient matrix we will concern ourselves with generally arises in physical problems where a discrete-element model of the real problem is assumed and a finite difference operator is applied, the solution technique is in no way bound to problems of this type. Any coefficient matrix which is positive definite and has the form of Fig 2.1 can be solved using this method. Although this method can be extended, in this discussion we will consider only three and five-wide type problems, respectively resulting from three and five-wide difference operators. We will also consider both the symmetrical and nonsymmetrical cases. These limits were selected since the one-wide case is trivial and the five-wide is extensive enough for most two-dimensional problems.

There is some disagreement as to whether one should consider direct or iterative solution techniques to solve this problem. The main advantages of the iterative processes are that they require less storage and are not as susceptible to round-off error. The main advantages of the direct methods are that for problems that give rise to coefficient matrices with "small" band widths they require less work. Also when it is known that in general the problems have the same coefficient matrix A and differ only in the constant vector f ($Aw_\ell = f_\ell$ or $AW = F$), which is quite often the case, the direct methods are capable of solving the succeeding problems for as little as 5 percent of the effort required to solve the initial problem.

In addition, if the problems are of the type that produces systems of heterogeneous coefficients, there may be complications in the iterative methods, whereas the direct methods are not as sensitive.

The power of this method is its ability to handle efficiently matrices exhibiting second level banding which lends itself to partitioning into submatrices which in turn are also banded.

Therefore, in the methods to be presented, we are assuming the widths of the second level bands (N_1, N_2, ...) are small relative to K, the order of the submatrices, and the overall band width is small and problems of the type $AW = F$ are expected.

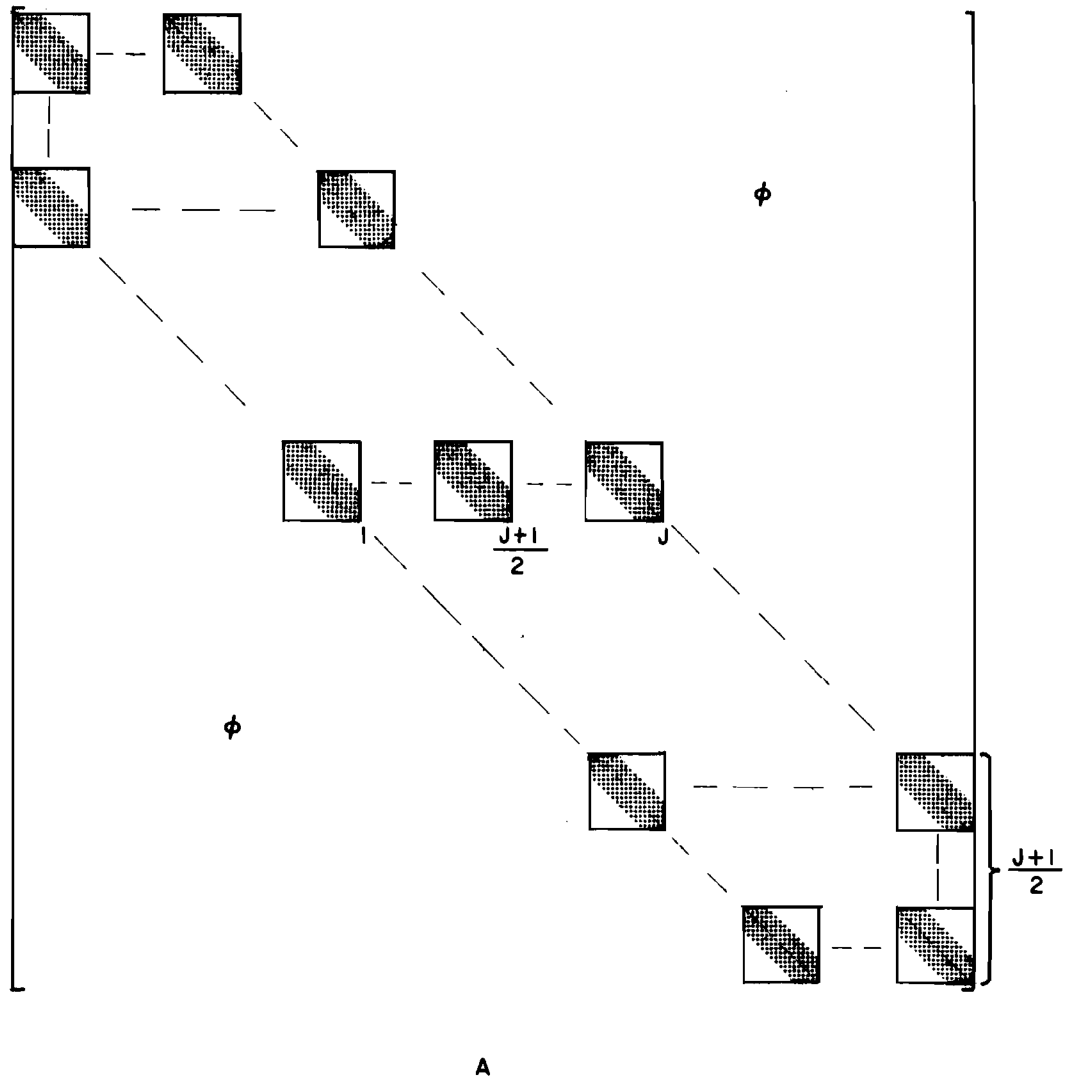


Fig 2.1. Coefficient matrix for a general J-wide operator.

Figure 2.1 shows a general J-wide partitioned matrix. Figures 2.2, 2.3, and 2.4 show a specific instance of a coefficient matrix resulting from the application of a five-wide difference operator to a real grid.

In most instances, we also know that the coefficient matrix is symmetric in addition to being positive definite. Since we can take advantage of this additional knowledge, we will develop two procedures: one for the symmetric case, one for the nonsymmetric case.

We have stated that we will limit ourselves to coefficient matrices which can be partitioned into three and five-wide bands. In terms of our difference operator, this restricts its vertical width but leaves the horizontal dimensions completely general. If we think of our coefficient matrix without reference to a difference operator, this merely means the respective widths of the bands are arbitrary up to the width of the submatrix itself. Of course, for the symmetric case, the bands must be symmetric about the main diagonal, e.g., $N_1 = N_5$ and $N_2 = N_4$ (refer to Figs 2.5 through 2.10).

Nonsymmetric Case

In the case where the operator is applied once and only once at each grid point, the respective widths of the subbands will be N_1 , N_2 , and N_3 for the three-wide case and N_1 , N_2 , N_3 , N_4 , and N_5 for the five-wide.

To insure this form and the narrowest band width, we must enforce the ordering of the mesh points as shown in Fig 2.11.

If we think of applying our operator, this tells us we apply it first to the bottom row of points from left to right. Then we go to the second row and so on, moving upward. If we now look at the partitioned matrix of Fig 2.4, we see that each submatrix has order K ($K \times K$ elements) and the stiffness matrix is composed of $L \times L$ submatrices. If we look at that portion of the coefficient matrix formed by applying the operator to the i^{th} row and for convenience assume we have a five-wide operator, we have the nonsymmetric case shown in Fig 2.12. Referring to the figure, we have

$$a_i w_{i-2} + b_i w_{i-1} + c_i w_i + d_i w_{i+1} + e_i w_{i+2} = f_i \qquad (1)$$

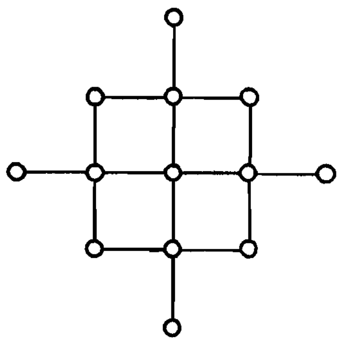


Fig 2.2. Typical five-wide difference operator.

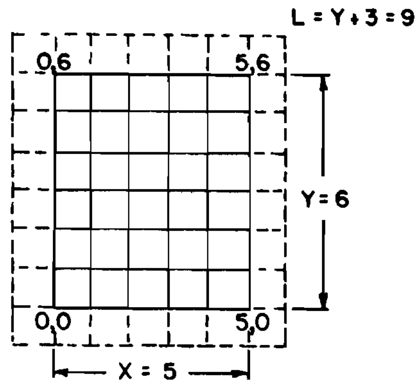


Fig 2.3. A specific discrete model.

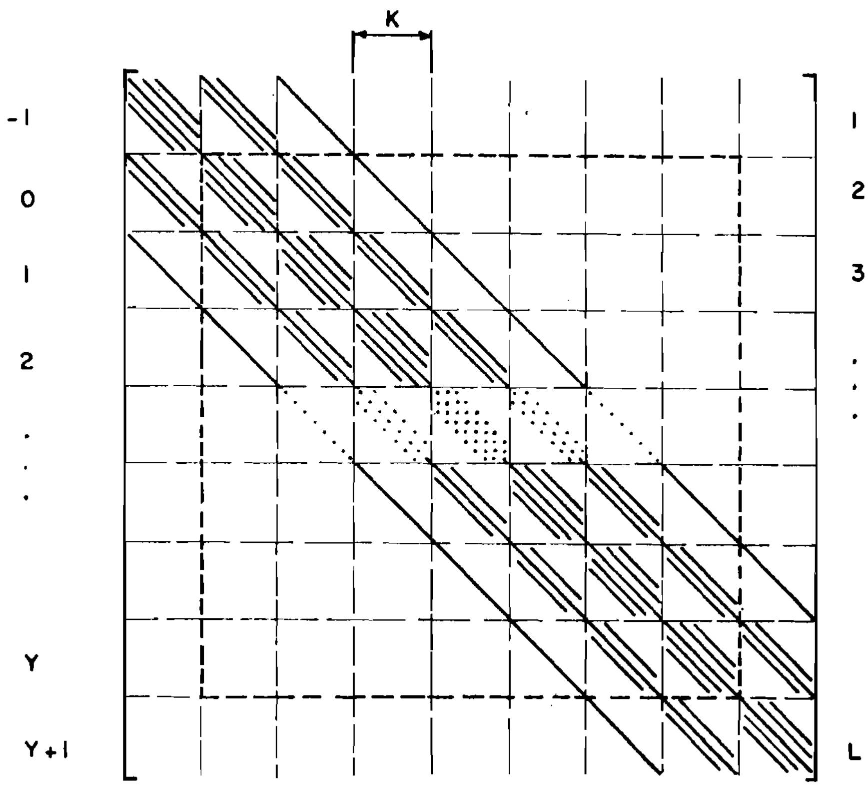


Fig 2.4. Coefficient matrix resulting from the application of the difference operator of Fig 2.2 to the discrete model of Fig 2.3.

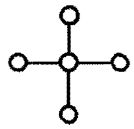


Fig 2.5. A typical three-wide difference operator.

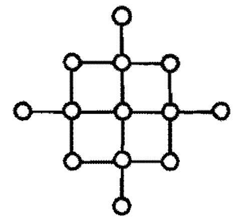


Fig 2.6. A typical five-wide difference operator.

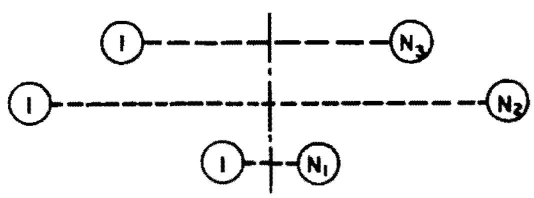


Fig 2.7. General three-wide operator.

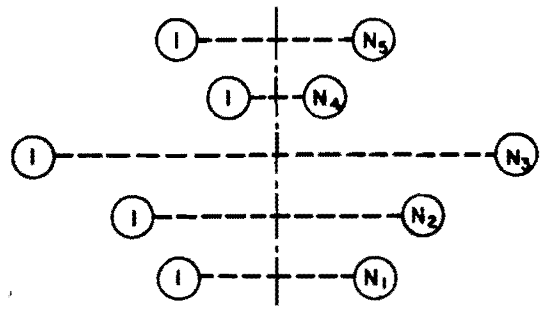


Fig 2.8. General five-wide operator.

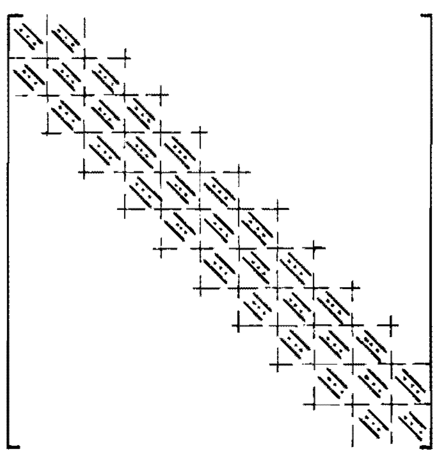


Fig 2.9. Coefficient matrix resulting from a general three-wide difference operator.

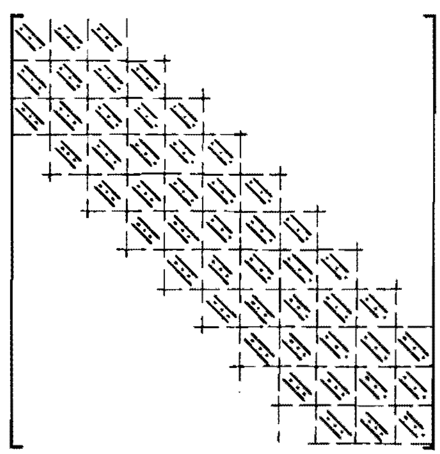


Fig 2.10. Coefficient matrix resulting from a general five-wide difference operator.

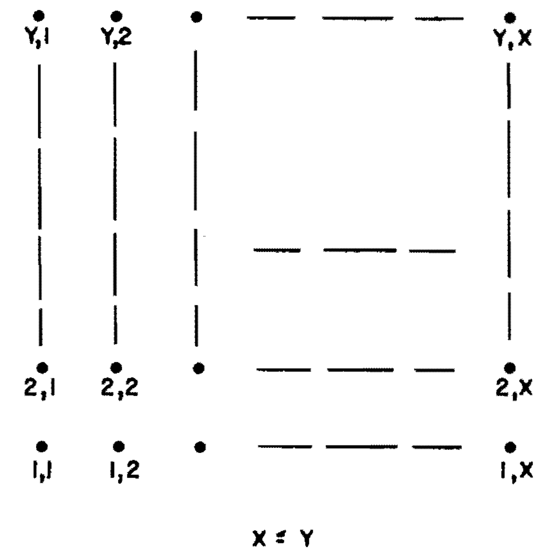


Fig 2.11. Imposed ordering of grid points.

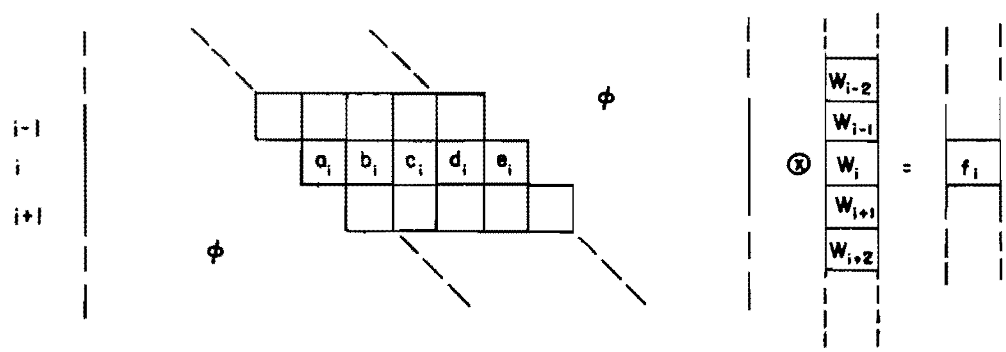


Fig 2.12. i^{th} partition of five-wide stiffness matrix.

Assume

$$w_i = A_i + B_i w_{i+1} + C_i w_{i+2}$$

Then

$$w_{i-1} = A_{i-1} + B_{i-1} w_i + C_{i-1} w_{i+1}$$

and

$$w_{i-2} = A_{i-2} + B_{i-2} w_{i-1} + C_{i-2} w_i \tag{2}$$

Substitution into Eq 1 to eliminate the variables w_{i-2} and w_{i-1} gives rise to an equation with the form of Eq 2 where

$$A_i = D_i (E_i A_{i-1} + a_i A_{i-2} - f_i)$$

$$B_i = D_i (E_i C_{i-1} + d_i)$$

$$C_i = D_i e_i$$

and where

$$D_i = -(a_i C_{i-2} + E_i B_{i-1} + c_i)^{-1}$$

$$E_i = a_i B_{i-2} + b_i$$

A , B , and C are referred to as recursion coefficients and D and E as recursion multipliers for the five-wide case.

Now with the appropriate starting values, we have an efficient two-pass, matrix elimination procedure.

For the three-wide band, we have a similar situation where by referring to Fig 2.13 we see that

$$b_i w_{i-1} + c_i w_i + d_i w_{i+1} = f_i \tag{3}$$

Assume

$$w_i = A_i + C_i w_{i+1}$$

Then

$$w_{i-1} = A_{i-1} + C_{i-1} w_i \tag{4}$$

Substitution into Eq 3 to eliminate w_{i-1} gives rise to an equation with the form of Eq 4 where

$$A_i = D_i (b_i A_{i-1} - f_i)$$

$$C_i = D_i d_i$$

and

$$D_i = (b_i C_{i-1} + c_i)^{-1}$$

In the preceding derivations, the sign (-) represents matrix negation, the sign (+) represents matrix addition, ($^{-1}$) represents matrix inversion, and concatination indicates matrix multiplication.

Symmetric Case

For the symmetric five-wide case, if we look at the i^{th} partition we have what is shown in Fig 2.14. Applying the same type of analysis as done for the nonsymmetric case gives rise to the identical recursion equations with a_i

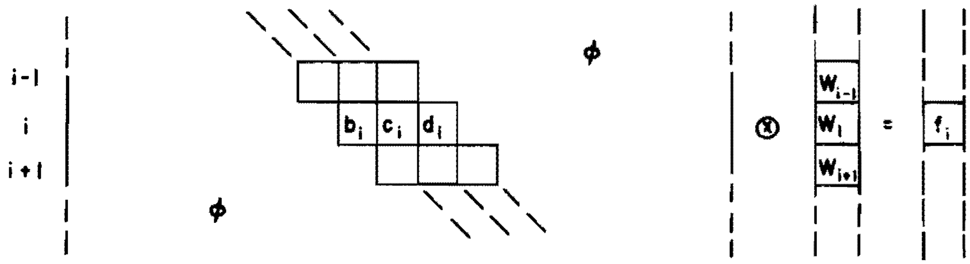


Fig 2.13. i^{th} partition of three-wide stiffness matrix.

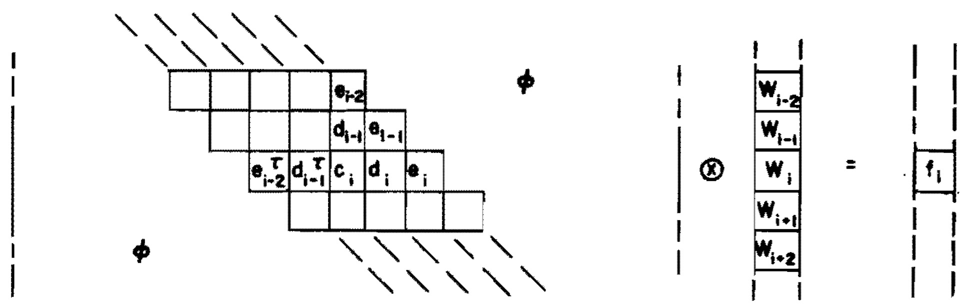


Fig 2.14. i^{th} partition of five-wide symmetric stiffness matrix.

replaced by e^T_{i-2} and b_i by d^T_{i-1} . This yields several important results for the five-wide case, namely, that $E_i B_{i-1}$, and therefore D_i , are symmetric and $B_i = D_i E^T_{i+1}$. We shall later see that this leads to approximately a 50 percent reduction in time for both the three and five-wide cases.

Making the substitutions in Eq 3 gives us

$$A_i = D_i(E_i A_{i-1} + e^T_{i-2} A_{i-2} - f_i)$$

$$B_i = D_i(E_i C_{i-1} + d_i)$$

$$C_i = D_i e_i$$

and

$$D_i = -(e^T_{i-2} C_{i-2} + E_i B_{i-1} + c_i)^{-1}$$

$$E_i = e^T_{i-2} B_{i-2} + d^T_{i-1} \tag{5}$$

In the following definitions and theorems A and B are square matrices and x is a vector.

Definitions

I. The statement that A is symmetric means that for all a , $a_{ij} = a_{ji}$.

II. The statement that A is positive definite means that for any non-zero vector x , $x^T A x > 0$.

Theorems

I. If A and B are symmetric, then A + B is symmetric.

II. If A is symmetric, then for all scalers c , cA is symmetric.

III. $A + A^T$ is symmetric.

IV. If A is symmetric, then BAB^T is symmetric.

V. $(A_1A_2 \ldots A_N)^T = A_N^T \ldots A_2^TA_1$.

VI. If A is symmetric, then A^{-1} is symmetric.

VII. $(A_1 + A_2 + \ldots + A_N)^T = A_1^T + A_2^T + \ldots A_N^T$.

Proof of the symmetry of D for the five-wide case:

Given Eq 5 and the boundary conditions

$$e_{-1}^T = d_0^T = e_0^T = B_{-1} = C_{-1} = B_0 = C_0 = E_1 = \emptyset$$

and that the coefficient matrix S and its diagonal partition c_i are symmetric and positive definite.

Let $\bar{D}_i = -D_i^{-1}$

For $i = 1$ and $i = 2$

$$\bar{D}_1 = E_1B_0 + e_{-1}^TC_{-1} + c_1$$

$$= c_1 \text{ which is symmetric}$$

Therefore D_1 is symmetric from VI and II

$$\bar{D}_2 = E_2B_1 + e_0^TC_0 + c_2$$

$$= d_1^TD_1d_1 + c_2$$

$d_1^TD_1d_1$ is symmetric because of IV and finally $\bar{D}_2$ is symmetric because of I. Therefore D_2 is symmetric from VI and II.

Assume D_i is symmetric for $i = k-2$ and $i = k-1$

We now need to show that $B_k = D_k E^{\top}_{k+1}$.
To do this we will use an inductive proof within our main induction.

For $j = 1$

$$B_1 = D_1(E_1C_0 + d_1) = D_1d_1$$

$$E_2 = e^{\top}_0 B_0 + d^{\top}_1 = d^{\top}_1$$

$$E^{\top}_2 = d_1$$

Therefore

$$B_1 = D_1E^{\top}_2$$

Assume for $j = k-1$

$$B_{k-1} = D_{k-1}E^{\top}_k$$

$$B^{\top}_{k-1} = E_k D^{\top}_{k-1}$$

Therefore

$$D_k(B^{\top}_{k-1}e_{k-1} + d_k) = D_k(E_k D^{\top}_{k-1}e_{k-1} + d_k) \tag{6}$$

From Eq 5

$$E_{k+1} = e^{\top}_{k-1}B_{k-1} + d^{\top}_k$$

Therefore

$$E^{\intercal}_{k+1} = B^{\intercal}_{k-1}e_{k-1} + d_k \tag{7}$$

From Eq 5

$$B_k = D_k(E_kC_{k-1} + d_k) \tag{8}$$

and

$$C_{k-1} = D_{k-1}e_{k-1} = D^{\intercal}_{k-1}e_{k-1} \tag{9}$$

Substituting Eq 9 into Eq 8 gives

$$B_k = D_k(E_kD^{\intercal}_{k-1}e_{k-1} + d_k) \tag{10}$$

and substituting Eq 7 and Eq 10 into Eq 6 gives

$$B_k = D_kE^{\intercal}_{k+1}$$

Therefore

$$B_j = D_jE^{\intercal}_{j+1}$$

$$j = 1 , 2 , \ldots , k$$

and

$$\bar{D}_k = E_kB_{k-1} + e^{\intercal}_{k-2}C_{k-2} + c_k$$

$$= E_kD_{k-1}E^{\intercal}_k + e^{\intercal}_{k-2}D_{k-2}e_{k-2} + c_k \tag{11}$$

Since D_{k-1} and D_{k-2} have been assumed to be symmetric we have that $E_k D_{k-1} E_k^T$ and $e_{k-2}^T D_{k-2}^T e_{k-2}$ are both symmetric because of IV and finally $\overline{D}_k$ is symmetric from the successive application of I. Therefore D_k is symmetric from VI and II and by induction we conclude that D_i is symmetric for all $i = 1, 2, \ldots$. Given this we can now conclude that

$$B_i = D_i E_{i+1}^T \quad \text{for all} \quad i = 1, 2, \ldots \tag{12}$$

and therefore $E_i B_{i-1}$ is symmetric for all $i = 1, 2, \ldots$.
The proof is similar but less involved for the three-wide case.
Making these substitutions into Eq 5 gives us our final set of equations for the symmetric case.

Five-wide case

$$A_i = D_i(E_i A_{i-1} + e_{i-2}^T A_{i-2} - f_k)$$

$$B_i = D_i E_{i+1}^T$$

$$C_i = D_i e_i$$

$$D_i = -(e_{i-2}^T C_{i-2} + E_i B_{i-1} + c_i)^{-1}$$

$$E_{i+1} = e_{i-1}^T B_{i-1} + d_i^T \tag{13}$$

Three-wide case

$$A_i = D_i(d_{i-1}^T A_{i-1} - f_i)$$

$$C_i = D_i d_i$$

$$D_i = -(d_{i-1}^T C_{i-1} + c_i)^{-1} \tag{14}$$

If we compare the recursion inversion algorithm with the elimination (a la Gauss) methods, we see analogous patterns. The first phase, in which the recursion coefficients and multipliers are calculated, is analogous to the triangularization phase of the elimination methods. The second phase, in which the recursion equation is solved for the unknown vectors, is analogous to the back-substitution phase.

In the following section we will see that there is also a similar pattern in the way multiple constant vectors can be handled.

Multiple Constant Vectors

A very important part of any solution scheme is its ability to handle problems where for the same coefficient matrix there are numerous constant vectors. We represent this by $AW = F$ where $F = (f_1, f_2, \ldots, f_\ell)$. This actually represents ℓ individual problems,

$$Aw_1 = f_1 \ , \quad Aw_2 = f_2 \ , \text{ and so forth.}$$

One way to handle this problem, which seems appealing at first glance, is the classical solution $w_i = A^{-1}f_i$, $i = 1, 2, \ldots, \ell$. This requires N^2 multiplications for each vector, which is approximately the same amount required for back substitution, given an upper triangularized matrix or some analogous factorization. But since it requires approximately N^3 multiplications to compute A^{-1} and only $N^3/3$ for the elimination or decomposition methods, we see that the latter are preferable. The above analysis was done with the assumption that A had no special properties. In the case where A is banded, the comparison between the two alternatives becomes even more striking, because the decomposition methods can readily take advantage of the banding whereas the inverse of a banded matrix is not necessarily banded and therefore in general cannot take advantage of this additional information.

If we now consider the recursion inversion approach, we see that the introduction of multiple f's requires only that for each vector we need to recalculate recursion coefficient A and circumvent the calculations of B , C , D , and E, all of which is approximately equal to the work required for banded decomposition. In addition, we see that the work required to calculate A is approximately equal to that needed for the transformation of the constant vector F and banded back substitution.

For the sake of discussion and for descriptive purposes in the flow charts, we categorize problems in the following manner:

$$\text{Given} \quad AW = F \text{ , where } \quad F = f_1$$

We refer to this as a standard problem

$$\text{when} \quad AW = F \text{ , where } \quad F = (f_1, f_2, \ldots, f_\ell)$$

f_1 is referred to as the parent problem and f_1, $i = 2, 3, \ldots, \ell$ are referred to as offspring problems.

While it is true that f_1 through f_ℓ are all of equal importance from the solution standpoint, the distinction between f_1 and the other f's was made because in the recursion inversion procedure, the original vector is operated on concurrently with the calculation of the recursion coefficients.

CHAPTER 3. EFFICIENCY, SPEED, AND SIZE

To program the recursion inversion procedure in the most efficient manner, it is necessary that the implied matrix operations not be done explicitly, but by banded matrix operation routines which do only the necessary operations. Also, to save storage as well as time, we pack these submatrices, thereby eliminating the zero elements which lie between the bands. Also, to minimize internal core memory requirements, only one level (horizontal partition) will actually be generated at any given time. The packing procedures necessary are explained fully in Appendix C.

Computations

To compute the amount of work necessary, let us first consider in particular the five-wide nonsymmetric case. Let us also assume N_1 and N_5 are small relative to K (the submatrix order), where N_1 and N_5 are the respective band widths of the outside bands. Therefore, the overall band width is $\geq 4K + 1$, and for simplicity and because of the above assumption, let us use $4K$.

If we look back at the recursion coefficients, we see that the dominant operations are the two full matrix multiplications in B and the one multiplication and inversion to get D. Referring to Fig 2.4 for the definition of K and L, we define N as K times L. Each of these requires K^3 multiplications. These must be repeated L times giving us an approximate estimate of

$$
\begin{aligned}
& 4K^3L \\
= \ & 4K^2N \\
= \ & (2K)^2N
\end{aligned}
$$

which is the amount needed for a comparable form of banded Gaussian elimination.

If we consider back substitution or the necessary work required for additional constant vectors, we see that this involves only the computation of the recursion coefficient A and then the application of the recursion equation (Eq 2, p 4). The former requires roughly $(2K + 1)N$ multiplications while the latter takes $2KN$ making the total $(4K + 1)N$, which also compares almost identically with banded Gaussian back-substitution.

We have used banded Gaussian elimination as a yardstick because, as was mentioned in Chapter 1, it is much more expedient than taking the actual inverse and in fact it has been proven that no direct method can take less computation than Gaussian elimination for a nonsymmetric matrix.

We find that for the three-wide nonsymmetric case, the recursion inversion method also is approximately equivalent to banded Gaussian elimination from the standpoint of computations involved. As was mentioned earlier, the additional information of symmetry reduces both solutions by approximately 50 percent.

Accuracy

Error analysis indicates that for well-conditioned problems at least four significant digits will remain for up to 100,000 equations, where a 60-bit word length is used. In both the nonsymmetric and symmetric case, the routines are such that one can take advantage of the accurate accumulation of inner-products, which can be very helpful for problems that are ill-conditioned. Greater advantage of this can be taken in the symmetric case because a compacted inversion routine is employed which also takes advantage of the accurate accumulation of inner-products.

Storage

For all practical purposes the storage requirements are consumed by the recursion coefficients (Table 3.1). Although it is not necessary, the entire solution vector has been retained in the core for convenience.

For a single vector problem, the auxiliary storage is reduced to $N(2K + 1)$ as one would expect. For the symmetric case we no longer need B_{i-2} but its space is replaced by E_{i+1} . Also, only C_{i-2} actually occurs in the revised formulas, but C_{i-1} is needed as temporary storage. However, by judicious arrangement, we have been able to use B_{i-1} as a temporary and

TABLE 3.1. STORAGE FOR FIVE-WIDE CASE (NONSYMMETRIC)

	Total Storage	Core	Auxiliary
A_i	$K \cdot L$	K	$K \cdot L$
A_{i-1}	K	K	0
A_{i-2}	K	K	0
B_i	$K^2L = K{\cdot}N$	K^2	$K^2 \cdot L$
B_{i-1}	K^2	K^2	0
B_{i-2}	K^2	K^2	0
C_i	$K^2L = K{\cdot}N$	K^2	$K^2 \cdot L$
C_{i-1}	K^2	K^2	0
C_{i-2}	K^2	K^2	0
D	$K^2 \cdot L$	K^2	$K^2 \cdot L$*
E	$K^2 \cdot L$	K^2	$K^2 \cdot L$*
W_i	$K \cdot L$	$K \cdot L$	0

*(This storage is necessary for multiple vector problems only.)

Total core storage $= 8K^2 + 3K + K \cdot L$

Total auxiliary storage $= 4K^2L + K \cdot L = N(4K + 1)$

delete the need for a storage slot for C_{i-2} . Therefore, the core storage for the symmetric five-wide case is reduced by K^2 .

Table 3.2 describes an analogous situation.

For a single vector problem, the auxiliary storage is reduced to $N(K + 1)$. It should now be obvious why K is chosen to be the smaller of the pair K , L . Although the order of the stiffness matrix $(N = K \cdot L)$ is not changed, the band width is directly proportional to K ; therefore it should be the smaller to reduce both the core storage and computational requirements.

TABLE 3.2. STORAGE FOR THREE-WIDE CASE

	Total Storage	Core	Auxiliary
A_i	$K \cdot L$	K	$K \cdot L$
A_{i-1}	K	K	0
C_i	K^2L	K^2	K^2L
D_i	K^2L	K^2	K^2L*
W_i	$K \cdot L$	$K \cdot L$	0

*(This storage is necessary only for multiple vector problems.)

Total core storage $(T_c) = 2K^2 + 2K + K \cdot L$

Total auxiliary storage $= 2K^2L + KL = N(2K + 1)$

Let us assume we had a problem that yielded a five-wide problem in which $K = 10$ and $L = 20$:

$$T_c = 8 \cdot 10^2 + 3 \cdot 10 + 10 \cdot 20 = 1030$$

In these computations we have been neglecting the storage necessary for the coefficient matrix itself, but, since we only need one partition at a time, this storage is negligible. If, in our example, we assume a typical 1-3-5-3-1 operator (refer to Fig 2.2), the additional storage required would be $(1 + 3 + 5 + 3 + 1)10 = 130$, which is not only small compared to 1030 but, as L increases, will not change since it is dependent only on K . Also as K increases, this number increases linearly, whereas our total core storage requirement (T_c) increases quadratically.

Timing

Figures 3.1 and 3.2 show the time in seconds for the solutions to problems having square grids $(L = K)$. For rectangular grids $(L > K)$ the time increases in direct proportion to L and may be estimated by multiplying the time from Fig 3.1 or Fig 3.2 by $\frac{L}{K}$.*

---

*All runs were made on a CDC 6600, with the SCOPE 2.0 RUN compiler.
Basic execution time in minor cycles:

division	-	29 m.c.
multiplication	-	10 m.c.
addition	-	4 m.c.

1 m.c. = 250 nanoseconds.

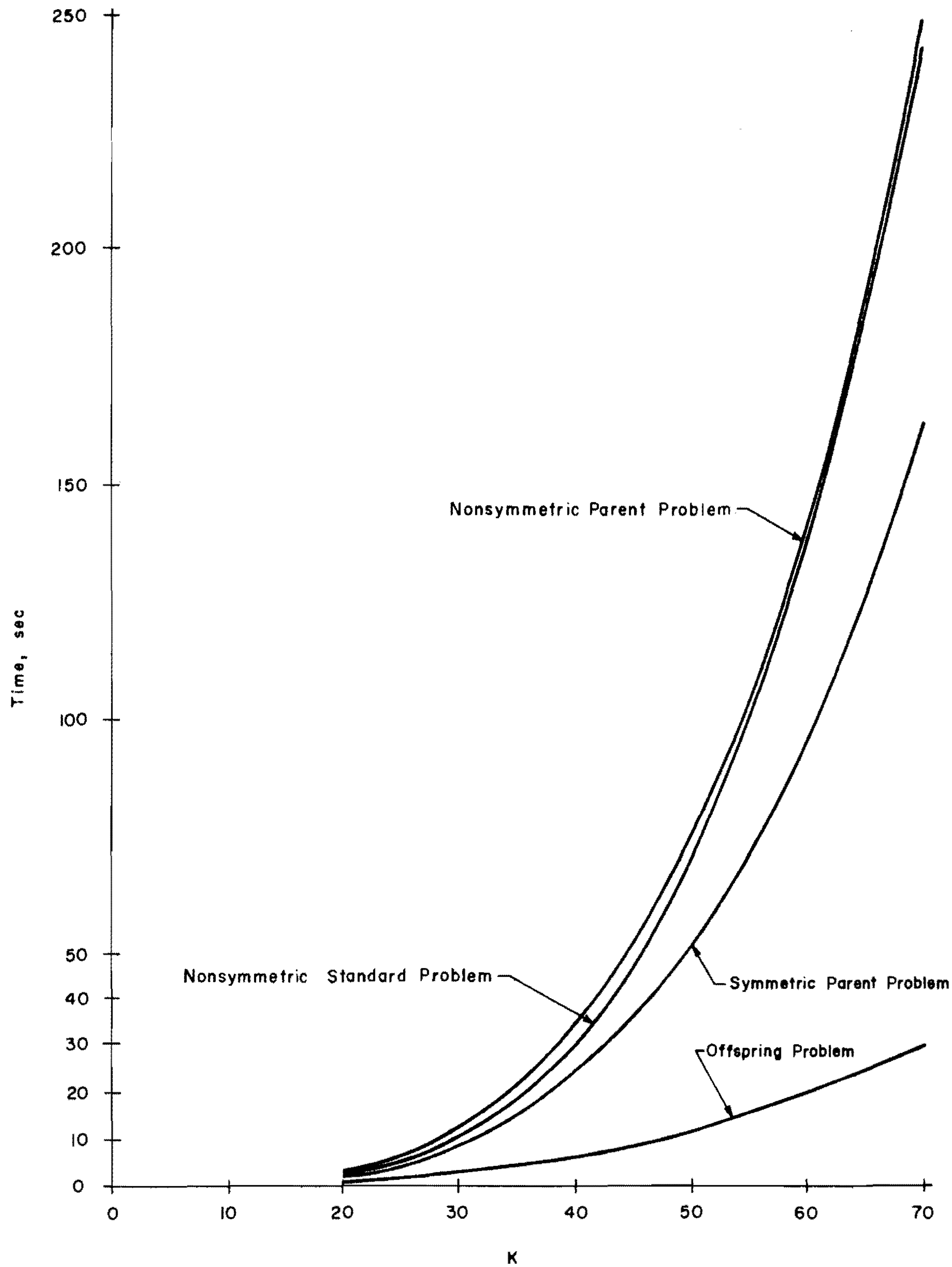


Fig 3.1. Time graph for three-wide solver.

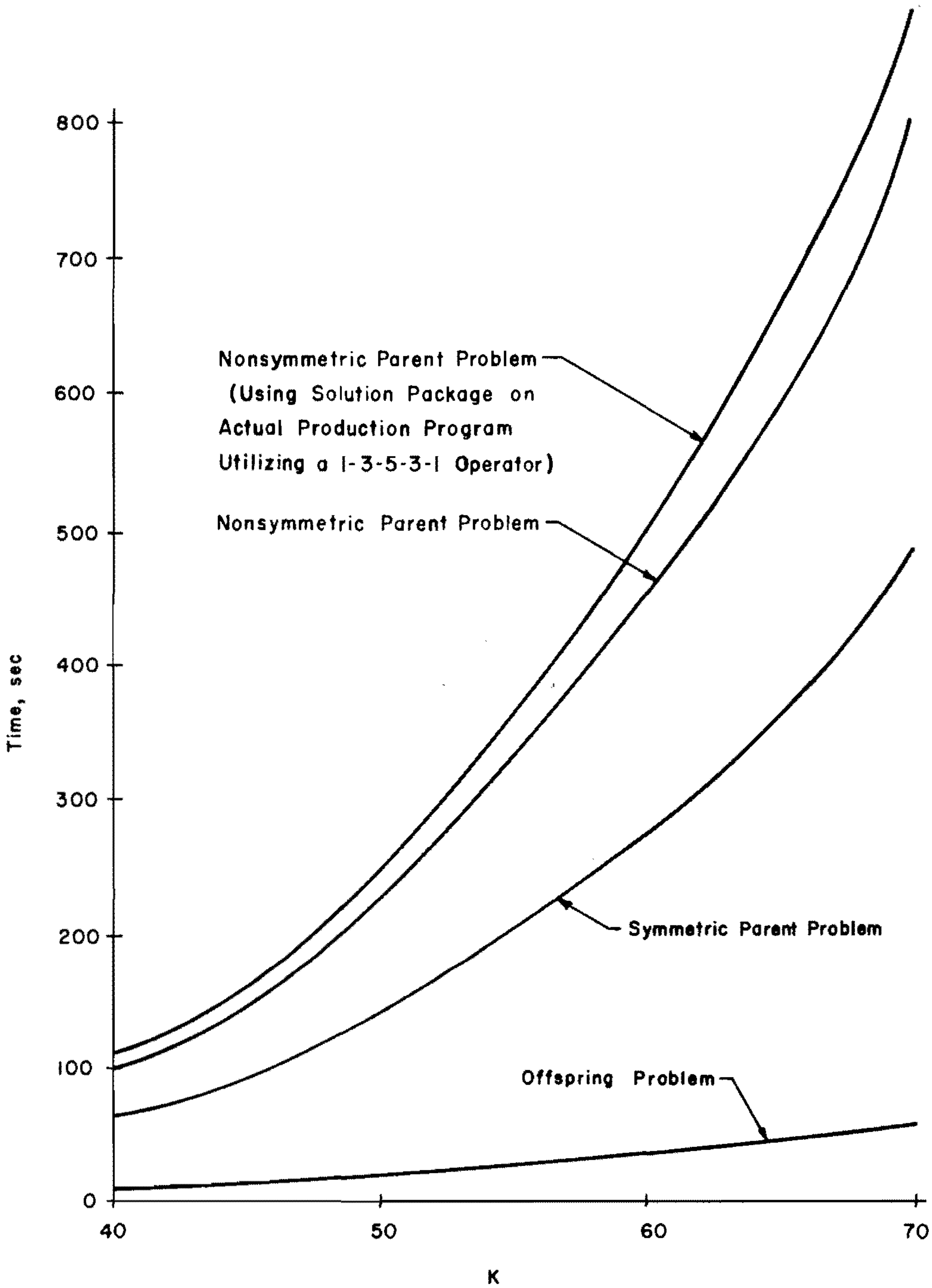


Fig 3.2. Time graph for five-wide solver.

REFERENCES

1. The American Standards Association, "American Standard FORTRAN," New York, 1966.

2. Panak, John J., and Hudson Matlock, "A Discrete-Element Method of Multiple-Loading Analysis for Two-Way Bridge Floor Slabs," Research Report No. 56-13, Center for Highway Research, The University of Texas at Austin, January 1970.

3. Stelzer, C. Fred, Jr., and W. Ronald Hudson, "A Direct Computer Solution for Plates and Pavement Slabs," Research Report No. 56-9, Center for Highway Research, The University of Texas, Austin, October 1967.

APPENDIX A

FLOW CHARTS AND LISTINGS OF MAIN ROUTINES

APPENDIX A. FLOW CHARTS AND LISTINGS OF MAIN ROUTINES

TRIP 3	-	Listing
TRIP 4	-	Listing
TRIP 3 & 4	-	Flow chart
FRIP 3	-	Listing
FRIP 4	-	Listing
FRIP 3 & 4	-	Flow chart

TRIP: Three-wide Recursion Inversion Procedure

FRIP: Five-wide Recursion Inversion Procedure

```
C******* NOTATION FOR TRIP 3
C         A      -     RECURSION COEFFICIENT ( A(I) )
C         AM1    -     RECURSION COEFFICIENT ( A(I-1) )
C         ATM    -     TEMPORARY VECTOR
C         C      -     RECURSION COEFFICIENT ( C(I) )
C         CM1    -     RECURSION COEFFICIENT ( C(I-1) )
C         D      -     RECURSION MULTIPLIER  ( D(I) )
C         BB     -     SUB-MATRIX ( LITTLE B(I) )
C         CC     -     SUB-MATRIX ( LITTLE C(I) )
C         DD     -     SUB-MATRIX ( LITTLE D(I) )
C         FF     -     SUB-MATRIX VECTOR ( LITTLE F(I) )
C         W      -     SOLUTION VECTOR ( STORED AS TWO-DIMENSIONAL )
C         N1     -     BAND WIDTH OF BB
C         N2     -     BAND WIDTH OF CC
C         N3     -     BAND WIDTH OF DD
C         ML     -     PROBLEM TYPE SWITCH     NEGATIVE FOR OFFSPRING
C                                              ZERO FOR STANDARD
C                                              POSITIVE FOR PARENT
C         NK     -     ORDER OF SUBMATRICES
C         NL     -     MATRIX ORDER OF OVERALL COEFFICIENT MATRIX
C         NF     -     STARTING VALUE FOR MAIN DO LOOP
C         L1     -     VARIABLE DIMENSION PARAMETER ( REQUIRED )
C         L2     -     VARIABLE DIMENSION PARAMETER ( OPTIONAL )
C         L3     -     VARIABLE DIMENSION PARAMETER ( OPTIONAL )
```

```
      SUBROUTINE TRIP3  ( L1,L2,L3,ML,A,AM1,ATM,C,D,                    20MY8
     1                   BB,CC,DD,FF,W,N1,N2,N3 )                       20MY8
C*******            THE LATEST REVISION DATE FOR THIS PROGRAM IS - - - - 01AG8
C******* THIS GROUP OF 10 SUBROUTINES PROVIDES THE USER WITH AN         23MR8
C        EFFICIENT GENERAL SPARSELY BANDED EQUATION SOLVER              11JA8
C        (THE MATRIX IS ASSUMED TO BE POSITIVE DEFINITE)                12MR8
C        WHICH CAN HANDLE UP TO 3 GROUPS OF BANDS , EACH                11JA8
C        OF ARBITRARY WIDTH                                             11JA8
C******* THIS ROUTINE SUPERVISES 9 SUBROUTINES , 8 OF WHICH             23MR8
C        ARE SELF-SUFFICIENT AND COME AS A PACKAGE( SUMP 6 ), THE       01AG8
C        REMAINING ONE GENERATES AND PACKS THE STIFFNESS                11JA8
C        MATRIX AS OUTLINED IN THE APPENDIX OF THE RELATED REPORT       23MR8
C        THIS ROUTINE MUST BE SUPPLIED BY THE USER SINCE                11JA8
C        IT DESCRIBES HIS PARTICULAR PROBLEM                            11JA8
C******* IN THE MAIN PROGRAM THE FOLLOWING CAN BE EQUIVALENCED          20MY8
C        ( ATM , CC )                                                   20MY8
C******* SCRATCH TAPES SHOULD BE REQUESTED FOR TAPES 1 AND 2            05MR8
C        TAPE 3 WORKS APPROPRIATELY AS A DISK FILE , BUT A SCRATCH      20MY8
C        TAPE CAN BE USED IF NECCESARY OR DESIRED.                      20MY8
      DIMENSION  A(L1    ) , AM1(L1    ) , ATM(L1    ) ,                20MY8
     1           C(L1,L1)  ,   D(L1,L1)  ,   W(L2,L3)  ,                20MY8
     2          BB(L1,N1)  ,  CC(L1,N2)  ,  DD(N3,L1)  ,  FF( L1  )     08AP8
      COMMON /RI/ NK , NL , NF                                          01FE8
      REWIND 1                                                          11JA8
      REWIND 2                                                          11JA8
      REWIND 3                                                          18JA8
          IF( ML ) 140, 100, 100                                        11JA8
C         SET INITIAL CONDITIONS                                        11JA8
  100     DO 135  I = 1 , NK                                            01FE8
          DO 130  J = 1 , NK                                            01FE8
               C(I,J) = 0.0                                             12MR8
  130     CONTINUE                                                      11JA8
  135     CONTINUE                                                      11JA8
  140     DO 150  I = 1 , NK                                            01FE8
               A(I) = 0.0                                               20MY8
  150     CONTINUE                                                      11JA8
```

```
C*****************************************************************************
C        BEGIN FORWARD PASS  --  SOLVE FOR RECURSION COEFFICIENTS
C*****************************************************************************
C                       . . . . . . . . . . . . . . . . . . . . . . . . . . ..
          DO 1000  J = NF , NL                                        01FE8  .
               JJ = J                                                 11JA8  .
C         FORM SUB-MATRICES                                           11JA8  .
      CALL FSUB31 ( L1,L2,L3,BB,CC,DD,FF,ML,JJ,N1,N2,N3 )             12MR8  .
      CALL RFV    ( AM1, A   , L1 , 1   , NK )                        20MY8  .
          IF( ML ) 210, 220, 220                                      08AP8  .
C         READ D MULTIPLIER FROM TAPE 3                               12MR8  .
  210 READ (3) (( D(I,K) , I = 1,NK) , K = 1,NK )                     12MR8  .
          GO TO 280                                                   11JA8  .
C         CALCULATE RECURSION MULTIPLIER D                            12MR8  .
  220 CALL MBFV   ( BB , C   , D   , L1 , L1 , NK , N1 )              20MY8  .
      CALL ABF    ( CC , D   , D   , L1 , NK , N2 )                   12MR8  .
      CALL INVR5 ( D   , L1 , NK )                                    15MR8  .
      CALL CFV    ( D   , L1 , L1 , NK , -1.)                         20MY8  .
C         CALCULATE RECURSION COEFFICIENT C                           11JA8  .
      CALL MFB    ( D   , DD , C   , L1 , NK , N3 )                   12MR8  .
C         CALCULATE RECURSION COEFFICIENT A                           11JA8  .
  280 CALL MBFV   ( BB , AM1, ATM, L1 , 1   , NK , N1 )               20MY8  .
      CALL ASFV   ( ATM, FF , ATM, L1 , 1   , NK , -1 )               20MY8  .
      CALL MFFV   ( D   , ATM, A   , L1 , 1   , NK )                  20MY8  .
C         SAVE A COEFFICIENT ON TAPE 1                                11JA8  .
      WRITE (1)   ( A(I), I = 1,NK )                                  01FE8  .
          IF( ML ) 400, 600, 500                                      11JA8  .
  400 READ (2)                                                        18JA8  .
          GO TO 1000                                                  11JA8  .
C         SAVE D MULTIPLIER ON TAPE 3                                 12MR8  .
  500 WRITE (3)   (( D(I,K) , I = 1,NK) , K = 1,NK )                  12MR8  .
C         SAVE C COEFFICIENT ON TAPE 2                                12MR8  .
  600 WRITE (2)   (( C(I,K) , I = 1,NK) , K = 1,NK )                  12MR8  .
 1000     CONTINUE                                                    11JA8  .
C                 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
```

```
C*****************************************************************************
C        BEGIN BACKWARD PASS  --  COMPUTE RECURSION EQUATION
C*****************************************************************************
      BACKSPACE 1                                                    20MY8
      BACKSPACE 2                                                    20MY8
      CALL RFV    ( W(NF,NL), A   , L1 , 1   , NK )                  01AG8
              NLM1 = NL - 1                                          20MY8
C                       . . . . . . . . . . . . . . . . . . . . . . . . . . ..
         DO 2000  L = NF , NLM1                                      20MY8   .
              J = NLM1 + NF - L                                      20MY8   .
      BACKSPACE 1                                                    11JA8   .
      BACKSPACE 2                                                    18JA8   .
C        READ A COEFFICIENT FROM TAPE 1                              11JA8   .
      READ  (1)   ( A(I), I = 1,NK )                                 01FE8   .
C        READ C COEFFICIENT FROM TAPE 2                              12MR8   .
      READ (2)   (( C(I,K) , I = 1,NK) , K = 1,NK )                  12MR8   .
      BACKSPACE 1                                                    11JA8   .
      BACKSPACE 2                                                    18JA8   .
      CALL MFFV   ( C   , W(NF,J+1),AM1, L1 , 1   , NK )             01AG8   .
      CALL ASFV   ( A   , AM1, W(NF,J) , L1 , 1   , NK , +1 )        01AG8   .
 2000    CONTINUE                                                    11JA8   .
C                 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
      RETURN                                                         11JA8
      END                                                            11JA8
```

```
C******* NOTATION FOR TRIP 4
C        A      -    RECURSION COEFFICIENT ( A(I) )
C        AM1    -    RECURSION COEFFICIENT ( A(I-1) )
C        ATM    -    TEMPORARY VECTOR
C        C      -    RECURSION COEFFICIENT ( C(I) )
C        CM1    -    RECURSION COEFFICIENT ( C(I-1) )
C        D      -    RECURSION MULTIPLIER  ( D(I) )
C        DT1    -    SUB-MATRIX ( LITTLE D(I-1) TRANSPOSE )
C        CC     -    SUB-MATRIX ( LITTLE C(I) )
C        DD     -    SUB-MATRIX ( LITTLE D(I) )
C        FF     -    SUB-MATRIX VECTOR ( LITTLE F(I) )
C        W      -    SOLUTION VECTOR ( STORED AS TWO-DIMENSIONAL )
C        N1     -    BAND WIDTH OF DD
C        N2     -    BAND WIDTH OF CC
C        ML     -    PROBLEM TYPE SWITCH     NEGATIVE FOR OFFSPRING
C                                            ZERO FOR STANDARD
C                                            POSITIVE FOR PARENT
C        NK     -    ORDER OF SUBMATRICES
C        NL     -    MATRIX ORDER OF OVERALL COEFFICIENT MATRIX
C        NF     -    STARTING VALUE FOR MAIN DO LOOP
C        L1     -    VARIABLE DIMENSION PARAMETER ( REQUIRED )
C        L2     -    VARIABLE DIMENSION PARAMETER ( OPTIONAL )
C        L3     -    VARIABLE DIMENSION PARAMETER ( OPTIONAL )
```

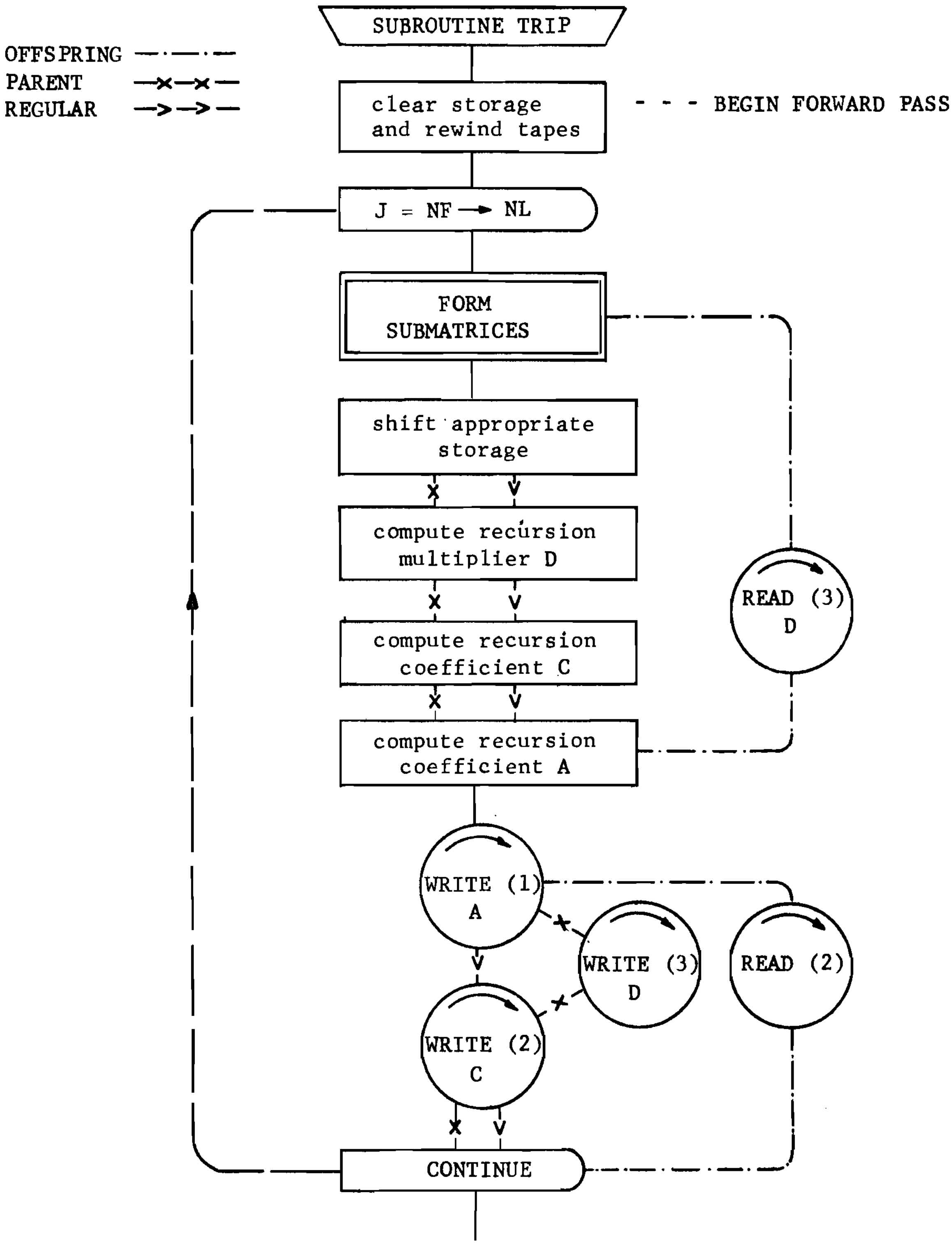
```
      SUBROUTINE TRIP4  ( L1,L2,L3,ML,A,AM1,ATM,C,D,                    20MY8
     1                 DT1,CC,DD,FF,W,N1,N2 )                           20MY8
C*******             THE LATEST REVISION DATE FOR THIS PROGRAM IS - - - - 01AG8
C******* THIS GROUP OF 13 SUBROUTINES PROVIDES THE USER WITH AN         23MR8
C        EFFICIENT GENERAL SPARSELY BANDED EQUATION SOLVER              11JA8
C        (THE MATRIX IS ASSUMED TO BE SYMMETRIC AND POSITIVE DEFINITE)  12MR8
C        WHICH CAN HANDLE UP TO 3 GROUPS OF BANDS , EACH                11JA8
C        OF ARBITRARY WIDTH                                             11JA8
C******* THIS ROUTINE SUPERVISES 12 SUBROUTINES , 11 OF WHICH           23MR8
C        ARE SELF-SUFFICIENT AND COME AS A PACKAGE( SUMP 6 ), THE       01AG8
C        REMAINING ONE GENERATES AND PACKS THE STIFFNESS                11JA8
C        MATRIX AS OUTLINED IN THE APPENDIX OF THE RELATED REPORT       23MR8
C        THIS ROUTINE MUST BE SUPPLIED BY THE USER SINCE                11JA8
C        IT DESCRIBES HIS PARTICULAR PROBLEM                            11JA8
C******* IN THE MAIN PROGRAM THE FOLLOWING CAN BE EQUIVALENCED          20MY8
C        ( ATM , CC )                                                   20MY8
C******* SCRATCH TAPES SHOULD BE REQUESTED FOR TAPES 1 AND 2            05MR8
C        TAPE 3 WORKS APPROPRIATELY AS A DISK FILE , BUT A SCRATCH      20MY8
C        TAPE CAN BE USED IF NECCESARY OR DESIRED.                      20MY8
      DIMENSION  A(L1    )  , AM1(L1    )  , ATM(L1    )  ,             20MY8
     1           C(L1,L1)  ,   D(L1,L1)  ,   W(L2,L3)  ,                20MY8
     2         DT1(L1,N1)  ,  CC(L1,N2)  ,  DD(N1,L1)  ,   FF( L1  )    09AP8
      COMMON /RI/ NK , NL , NF                                          01FE8
      REWIND 1                                                          11JA8
      REWIND 2                                                          11JA8
      REWIND 3                                                          18JA8
         IF( ML ) 140, 100, 100                                         11JA8
C        SET INITIAL CONDITIONS                                         11JA8
  100    DO 135  I = 1 , NK                                             01FE8
         DO 130  J = 1 , NK                                             01FE8
            C(I,J) = 0.0                                                12MR8
  130    CONTINUE                                                       11JA8
  135    CONTINUE                                                       11JA8
  140    DO 150  I = 1 , NK                                             01FE8
            A(I) = 0.0                                                  20MY8
  150    CONTINUE                                                       11JA8
```

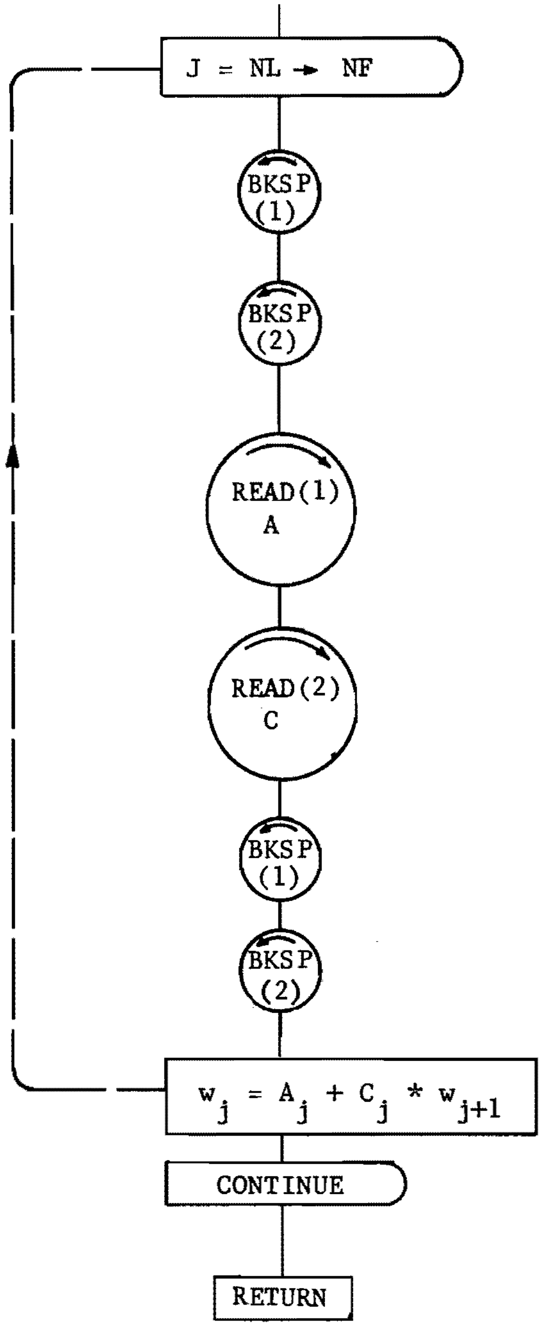
```
C***************************************************************************
C         BEGIN FORWARD PASS  --  SOLVE FOR RECURSION COEFFICIENTS
C***************************************************************************
C                        . . . . . . . . . . . . . . . . . . . . . . . . . . ..
          DO 1000  J = NF , NL                                          01FE8  .
               JJ = J                                                   11JA8  .
C         FORM SUB-MATRICES                                             11JA8  .
      CALL FSUB32 ( L1,L2,L3,DT1,CC,DD,FF,ML,JJ,N1,N2 )                 12MR8  .
      CALL RFV    ( AM1, A   , L1 , 1   , NK )                          20MY8  .
          IF( ML ) 210, 220, 220                                        09AP8  .
C         READ D MULTIPLIER FROM TAPE 3                                 12MR8  .
  210 READ (3) (( D(I,K) , I = 1,NK) , K = 1,NK )                       12MR8  .
          GO TO 280                                                     11JA8  .
C         CALCULATE RECURSION MULTIPLIER D                              12MR8  .
  220 CALL MBFV   ( DT1, C   , D   , L1 , L1 , NK , N1 )                20MY8  .
      CALL ABF    ( CC , D   , D   , L1 , NK , N2 )                     12MR8  .
      CALL INVR6  ( D   , L1 , NK )                                     15MR8  .
      CALL CFV    ( D   , L1 , L1 , NK , -1.)                           20MY8  .
C         CALCULATE RECURSION COEFFICIENT C                             12MR8  .
      CALL MFB    ( D   , DD , C   , L1 , NK , N1 )                     12MR8  .
C         CALCULATE RECURSION COEFFICIENT A                             11JA8  .
  280 CALL MBFV   ( DT1, AM1, ATM, L1 , 1   , NK , N1 )                 20MY8  .
      CALL ASFV   ( ATM, FF , ATM, L1 , 1   , NK , -1 )                 20MY8  .
      CALL MFFV   ( D   , ATM, A   , L1 , 1   , NK )                    20MY8  .
C         SAVE A COEFFICIENT ON TAPE 1                                  11JA8  .
      WRITE (1)  ( A(I), I = 1,NK )                                     01FE8  .
          IF( ML ) 400, 600, 500                                        11JA8  .
  400 READ (2)                                                          18JA8  .
          GO TO 1000                                                    11JA8  .
C         SAVE D MULTIPLIER ON TAPE 3                                   12MR8  .
  500 WRITE (3)  (( D(I,K) , I = 1,NK) , K = 1,NK )                     12MR8  .
C         SAVE C COEFFICIENT ON TAPE 2                                  12MR8  .
  600 WRITE (2)  (( C(I,K) , I = 1,NK) , K = 1,NK )                     12MR8  .
 1000     CONTINUE                                                      11JA8  .
C                 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
```

```
C*****************************************************************************
C        BEGIN BACKWARD PASS  --  COMPUTE RECURSION EQUATION
C*****************************************************************************
      BACKSPACE 1                                                        20MY8
      BACKSPACE 2                                                        20MY8
      CALL RFV    ( W(NF,NL), A   , L1 , 1   , NK )                      01AG8
               NLM1 = NL - 1                                             20MY8
C                          . . . . . . . . . . . . . . . . . . . . . . . . . ..
         DO 2000  L = NF , NLM1                                          20MY8  .
               J = NLM1 + NF - L                                         20MY8  .
      BACKSPACE 1                                                        11JA8  .
      BACKSPACE 2                                                        18JA8  .
C        READ A COEFFICIENT FROM TAPE 1                                  11JA8  .
      READ  (1)   ( A(I), I = 1,NK )                                     01FE8  .
C        READ C COEFFICIENT FROM TAPE 2                                  12MR8  .
      READ (2)  (( C(I,K) , I = 1,NK) , K = 1,NK )                       12MR8  .
      BACKSPACE 1                                                        11JA8  .
      BACKSPACE 2                                                        18JA8  .
      CALL MFFV  ( C   , W(NF,J+1),AM1, L1 , 1   , NK )                  01AG8  .
      CALL ASFV  ( A   , AM1, W(NF,J) , L1 , 1   , NK , +1 )             01AG8  .
 2000    CONTINUE                                                        11JA8  .
C                 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
      RETURN                                                             11JA8
      END                                                                11JA8
```

Flow Chart for Three-Wide Recursion Inversion Process

TRIP 3 , TRIP 4




J = NL → NF
BKSP (1)
BKSP (2)
READ(1) A
READ(2) C
BKSP (1)
BKSP (2)
$w_j = A_j + C_j * w_{j+1}$
CONTINUE
RETURN

```
C******* NOTATION FOR FRIP 3
C        A      -    RECURSION COEFFICIENT ( A(I) )
C        AM1    -    RECURSION COEFFICIENT ( A(I-1) )
C        AM2    -    RECURSION COEFFICIENT ( A(I-2) )
C        ATM    -    TEMPORARY VECTOR
C        B      -    RECURSION COEFFICIENT ( B(I) )
C        BM1    -    RECURSION COEFFICIENT ( B(I-1) )
C        BM2    -    RECURSION COEFFICIENT ( B(I-2) )
C        C      -    RECURSION COEFFICIENT ( C(I) )
C        CM1    -    RECURSION COEFFICIENT ( C(I-1) )
C        CM2    -    RECURSION COEFFICIENT ( C(I-2) )
C        D      -    RECURSION MULTIPLIER  ( D(I) )
C        E      -    RECURSION MULTIPLIER  ( E(I) )
C        AA     -    SUB-MATRIX ( LITTLE A(I) )
C        BB     -    SUB-MATRIX ( LITTLE B(I) )
C        CC     -    SUB-MATRIX ( LITTLE C(I) )
C        DD     -    SUB-MATRIX ( LITTLE D(I) )
C        EE     -    SUB-MATRIX ( LITTLE E(I) )
C        FF     -    SUB-MATRIX VECTOR ( LITTLE F(I) )
C        W      -    SOLUTION VECTOR ( STORED AS TWO-DIMENSIONAL )
C        N1     -    BAND WIDTH OF AA
C        N2     -    BAND WIDTH OF BB
C        N3     -    BAND WIDTH OF CC
C        N4     -    BAND WIDTH OF DD
C        N5     -    BAND WIDTH OF EE
C        ML     -    PROBLEM TYPE SWITCH     NEGATIVE FOR OFFSPRING
C                                            ZERO FOR STANDARD
C                                            POSITIVE FOR PARENT
C        NK     -    ORDER OF SUBMATRICES
C        NL     -    MATRIX ORDER OF OVERALL COEFFICIENT MATRIX
C        NF     -    STARTING VALUE FOR MAIN DO LOOP
C        L1     -    VARIABLE DIMENSION PARAMETER ( REQUIRED )
C        L2     -    VARIABLE DIMENSION PARAMETER ( OPTIONAL )
C        L3     -    VARIABLE DIMENSION PARAMETER ( OPTIONAL )
```

```
      SUBROUTINE FRIP3  ( L1,L2,L3,ML,A,AM1,AM2,ATM,B,BM1,BM2,C,            20MY8
     1                 CM1,CM2,D,E,AA,BB,CC,DD,EE,FF,W,N1,N2,N3,N4,N5 )   20MY8
C*******           THE LATEST REVISION DATE FOR THIS PROGRAM IS - - - -  01AG8
C******* THIS GROUP OF 10 SUBROUTINES PROVIDES THE USER WITH AN            20MY8
C        EFFICIENT GENERAL SPARSELY BANDED EQUATION SOLVER                 04JA8
C        (THE MATRIX IS ASSUMED TO BE POSITIVE DEFINITE)                   12MR8
C        WHICH CAN HANDLE UP TO 5 GROUPS OF BANDS , EACH                   04JA8
C        OF ARBITRARY WIDTH                                                04JA8
C******* THIS ROUTINE SUPERVISES  9 SUBROUTINES , 8 OF WHICH               20MY8
C        ARE SELF-SUFFICIENT AND COME AS A PACKAGE( SUMP 6 ), THE          01AG8
C        REMAINING ONE GENERATES AND PACKS THE STIFFNESS                   04JA8
C        MATRIX AS OUTLINED IN THE APPENDIX OF THE RELATED REPORT          23MR8
C        THIS ROUTINE MUST BE SUPPLIED BY THE USER SINCE                   04JA8
C        IT DESCRIBES HIS PARTICULAR PROBLEM                               04JA8
C******* IN THE MAIN PROGRAM THE FOLLOWING CAN BE EQUIVALENCED             20MY8
C        ( ATM , BB )                                                      20MY8
C******* SCRATCH TAPES SHOULD BE REQUESTED FOR TAPES 1 AND 2               05MR8
C        TAPE 3 WORKS APPROPRIATELY AS A DISK FILE , BUT A SCRATCH         20MY8
C        TAPE CAN BE USED IF NECCESARY OR DESIRED.                         20MY8
      DIMENSION  A(L1    )  , AM1(L1    )  , AM2(L1    )  ,                20MY8
     1          B(L1,L1)  , BM1(L1,L1)  , BM2(L1,L1)  , ATM(L1    )  ,     20MY8
     2          C(L1,L1)  , CM1(L1,L1)  , CM2(L1,L1)  ,                    20MY8
     3          D(L1,L1)  ,    E(L1,L1)  ,    W(L2,L3)  ,                  20MY8
     4         AA(L1,N1)  ,   BB(L1,N2)  ,   CC(L1,N3)  ,   DD(L1,N4)  ,   04JA8
     5                        EE(N5,L1)  ,   FF(L1)                        04JA8
      COMMON /RI/ NK , NL , NF                                             02FE8
      REWIND 1                                                             04JA8
      REWIND 2                                                             04JA8
      REWIND 3                                                             17JA8
         IF( ML ) 140, 100, 100                                            04JA8
C        SET INITIAL CONDITIONS                                            04JA8
  100    DO 135  J = 1 , NK                                                01FE8
         DO 130  I = 1 , NK                                                01FE8
              BM1(I,J) = 0.0                                               04JA8
              CM1(I,J) = 0.0                                               04JA8
              B(I,J)  = 0.0                                                04JA8
              C(I,J)  = 0.0                                                04JA8
  130    CONTINUE                                                          04JA8
  135    CONTINUE                                                          04JA8
  140    DO 150  I = 1 , NK                                                01FE8
              A(I) = 0.0                                                   20MY8
              AM1(I) = 0.0                                                 20MY8
  150    CONTINUE                                                          04JA8
```

```
C***********************************************************************
C        BEGIN FORWARD PASS  --  SOLVE FOR RECURSION COEFFICIENTS      04JA8
C***********************************************************************
C                                  . . . . . . . . . . . . . . . . . . . . . . . . . . ..
         DO 1000  J = NF , NL                                          01FE8  .
              JJ = J                                                   04JA8  .
C        FORM SUB-MATRICES                                             04JA8  .
     CALL FSUB51 (L1,L2,L3,AA,BB,CC,DD,EE,FF,ML,JJ,N1,N2,N3,N4,N5 )    12MR8  .
     CALL RFV    ( AM2, AM1, L1 , 1  , NK )                            20MY8  .
     CALL RFV    ( AM1, A  , L1 , 1  , NK )                            20MY8  .
          IF( ML ) 210, 180, 180                                       04JA8  .
 180 CALL RFV    ( BM2, BM1, L1 , L1 , NK )                            20MY8  .
     CALL RFV    ( BM1, B  , L1 , L1 , NK )                            20MY8  .
     CALL RFV    ( CM2, CM1, L1 , L1 , NK )                            20MY8  .
     CALL RFV    ( CM1, C  , L1 , L1 , NK )                            20MY8  .
          GO TO 220                                                    04JA8  .
C        READ D AND E MULTIPLIERS FROM TAPE 3                          17JA8  .
 210 READ (3)   (( D(I,K) , E(I,K) , I = 1,NK) , K = 1,NK )            01FE8  .
          GO TO 280                                                    04JA8  .
C        CALCULATE RECURSION MULTIPLIER E                              04JA8  .
 220 CALL MBFV   ( AA , BM2, E   , L1 , L1 , NK , N1 )                 20MY8  .
     CALL ABF    ( BB , E   , E   , L1 , NK , N2 )                     01FE8  .
C        CALCULATE RECURSION MULTIPLIER D                              04JA8  .
     CALL MFFV   ( E  , BM1, D   , L1 , L1 , NK )                      20MY8  .
     CALL MBFV   ( AA , CM2, C   , L1 , L1 , NK , N1 )                 20MY8  .
     CALL ASFV   ( D  , C  , D   , L1 , L1 , NK , +1 )                 01AG8
     CALL ABF    ( CC , D  , D   , L1 , NK , N3 )                      01FE8  .
     CALL INVR5  ( D  , L1 , NK )                                      01FE8  .
     CALL CFV    ( D  , L1 , L1 , NK , -1.)                            20MY8  .
C        CALCULATE RECURSION COEFFIECENT C                             04JA8  .
     CALL MFB    ( D  , EE , C   , L1 , NK , N5 )                      01FE8  .
C        CALCULATE RECURSION COEFFIECENT B                             04JA8  .
     CALL MFFV   ( E  , CM1, B   , L1 , L1 , NK )                      20MY8  .
     CALL ABF    ( DD , B  , BM2, L1 , NK , N4 )                       01FE8  .
     CALL MFFV   ( D  , BM2, B   , L1 , L1 , NK )                      20MY8  .
C        CALCULATE RECURSION COEFFIECENT A                             04JA8  .
 280 CALL MFFV   ( E  , AM1, A   , L1 , 1  , NK )                      20MY8  .
     CALL MBFV   ( AA , AM2, ATM, L1 , 1  , NK , N1 )                  20MY8  .
     CALL ASFV   ( A  , ATM, AM2, L1 , 1  , NK , +1 )                  20MY8  .
     CALL ASFV   ( AM2, FF , ATM, L1 , 1  , NK , -1 )                  20MY8  .
     CALL MFFV   ( D  , ATM, A   , L1 , 1  , NK )                      20MY8  .
C        SAVE A COEFFICIENT ON TAPE 1                                  04JA8  .
     WRITE (1)   ( A(I), I = 1,NK )                                    02FE8  .
          IF( ML )400,600,500                                          04JA8  .
 400 READ (2)                                                          17JA8  .
          GO TO 1000                                                   04JA8  .
C        SAVE D AND E MULTIPLIERS ON TAPE 3                            17JA8  .
 500 WRITE (3)   (( D(I,K),E(I,K), I=1,NK), K=1,NK)                    02FE8  .
C        SAVE B AND C COEFFICIENTS ON TAPE 2                           17JA8  .
 600 WRITE (2)   (( B(I,K),C(I,K), I=1,NK), K=1,NK)                    02FE8  .
1000     CONTINUE                                                      04JA8  .
C                       . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
```

```
C***********************************************************************
C        BEGIN BACKWARD PASS  --  COMPUTE RECURSION EQUATION             04JA8
C***********************************************************************
      BACKSPACE 1                                                        20MY8
      BACKSPACE 2                                                        20MY8
      CALL RFV    ( W(NF,NL), A   , L1 , 1   , NK )                      01AG8
      BACKSPACE 1                                                        20MY8
      BACKSPACE 2                                                        20MY8
      READ (1)   ( A(I), I = 1,NK )                                      20MY8
      READ (2)   (( B(I,K) , C(I,K), I = 1,NK) , K = 1,NK)               20MY8
      BACKSPACE 1                                                        20MY8
      BACKSPACE 2                                                        20MY8
      CALL MFFV   ( B   , W(NF,NL), AM1, L1 , 1   , NK )                 01AG8
      CALL ASFV   ( A   , AM1, W(NF,NL-1)   , L1 , 1   , NK , +1 )       01AG8
               NLM2 = NL - 2                                             20MY8
C                     . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
        DO 2000  L = NF , NLM2                                           20MY8  .
               J = NLM2 + NF - L                                         20MY8  .
      BACKSPACE 1                                                        04JA8  .
      BACKSPACE 2                                                        17JA8  .
C        READ A COEFFICIENT FROM TAPE 1                                  04JA8  .
      READ  (1)   ( A(I), I = 1,NK )                                     02FE8  .
C        READ B AND C COEFFICIENTS FROM TAPE 2                           17JA8  .
      READ (2)   (( B(I,K) , C(I,K), I = 1,NK) , K = 1,NK)               02FE8  .
      BACKSPACE 1                                                        04JA8  .
      BACKSPACE 2                                                        17JA8  .
      CALL MFFV   ( B   , W(NF,J+1),AM1, L1 , 1   , NK )                 01AG8  .
      CALL MFFV   ( C   , W(NF,J+2),AM2, L1 , 1   , NK )                 01AG8  .
      CALL ASFV   ( AM1, AM2, AM1, L1 , 1   , NK , +1 )                  20MY8  .
      CALL ASFV   ( A   , AM1, W(NF,J) , L1 , 1   , NK , +1 )            01AG8  .
 2000     CONTINUE                                                       04JA8  .
C                 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
      RETURN                                                              4JA8
      END                                                                 4JA8
```

```
C******* NOTATION FOR FRIP 4
C        A     -    RECURSION COEFFICIENT ( A(I) )
C        AM1   -    RECURSION COEFFICIENT ( A(I-1) )
C        AM2   -    RECURSION COEFFICIENT ( A(I-2) )
C        ATM   -    TEMPORARY VECTOR
C        B     -    RECURSION COEFFICIENT ( B(I) )
C        BM1   -    RECURSION COEFFICIENT ( B(I-1) )
C        C     -    RECURSION COEFFICIENT ( C(I) )
C        CM1   -    RECURSION COEFFICIENT ( C(I-1) )
C        CM2   -    RECURSION COEFFICIENT ( C(I-2) )
C        D     -    RECURSION MULTIPLIER  ( D(I) )
C        E     -    RECURSION MULTIPLIER  ( E(I) )
C        EP1   -    RECURSION MULTIPLIER  ( E(I+1) )
C        ET2   -    SUB-MATRIX ( LITTLE E(I-2) TRANSPOSE )
C        DT    -    SUB-MATRIX ( LITTLE D(I) TRANSPOSE )
C        CC    -    SUB-MATRIX ( LITTLE C(I) )
C        ET1   -    SUB-MATRIX ( LITTLE E(I-1) TRANSPOSE )
C        EE    -    SUB-MATRIX ( LITTLE E(I) )
C        FF    -    SUB-MATRIX VECTOR ( LITTLE F(I) )
C        W     -    SOLUTION VECTOR ( STORED AS TWO-DIMENSIONAL )
C        N1    -    BAND WIDTH OF EE
C        N2    -    BAND WIDTH OF DD
C        N3    -    BAND WIDTH OF CC
C        ML    -    PROBLEM TYPE SWITCH     NEGATIVE FOR OFFSPRING
C                                           ZERO FOR STANDARD
C                                           POSITIVE FOR PARENT
C        NK    -    ORDER OF SUBMATRICES
C        NL    -    MATRIX ORDER OF OVERALL COEFFICIENT MATRIX
C        NF    -    STARTING VALUE FOR MAIN DO LOOP
C        L1    -    VARIABLE DIMENSION PARAMETER ( REQUIRED )
C        L2    -    VARIABLE DIMENSION PARAMETER ( OPTIONAL )
C        L3    -    VARIABLE DIMENSION PARAMETER ( OPTIONAL )
```

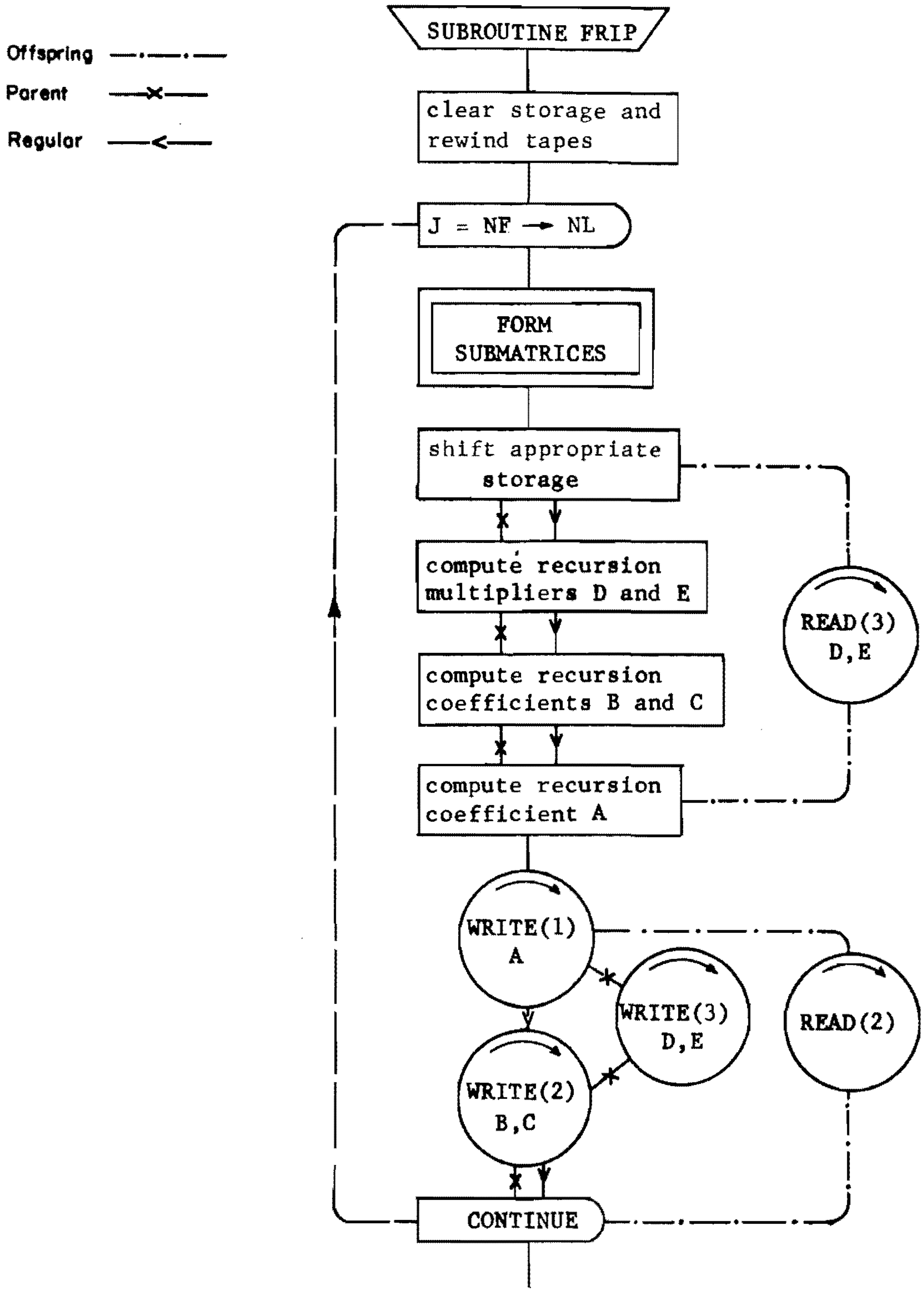
```
      SUBROUTINE FRIP4  ( L1,L2,L3,ML,A,AM1,AM2,ATM,B,BM1,EP1,C,CM1,        20MY8
     1                    D,E,ET2,ET1,CC,DT,EE,FF,W,N1,N2,N3 )               27MY8
C*******            THE LATEST REVISION DATE FOR THIS PROGRAM IS - - - -   01AG8
C******* THIS GROUP OF 15 SUBROUTINES PROVIDES THE USER WITH AN             20MY8
C        EFFICIENT GENERAL SPARSELY BANDED EQUATION SOLVER                  04JA8
C        (THE MATRIX IS ASSUMED TO BE SYMMETRIC AND POSITIVE DEFINITE)      12MR8
C        WHICH CAN HANDLE UP TO 5 GROUPS OF BANDS , EACH                    04JA8
C        OF ARBITRARY WIDTH                                                 04JA8
C******* THIS ROUTINE SUPERVISES 14 SUBROUTINES , 13 OF WHICH               20MY8
C        ARE SELF-SUFFICIENT AND COME AS A PACKAGE( SUMP 6 ), THE           01AG8
C        REMAINING ONE GENERATES AND PACKS THE STIFFNESS                    04JA8
C******* MATRIX AS OUTLINED IN IN THE APPENDIX OF THE RELATED REPORT        23MR8
C        THIS ROUTINE MUST BE SUPPLIED BY THE USER SINCE                    04JA8
C        IT DESCRIBES HIS PARTICULAR PROBLEM                                04JA8
C******* IN THE MAIN PROGRAM THE FOLLOWING CAN BE EQUIVALENCED              20MY8
C        ( ATM , DT )                                                       20MY8
C******* SCRATCH TAPES SHOULD BE REQUESTED FOR TAPES 1 AND 2                05MR8
C        TAPE 3 WORKS APPROPRIATELY AS A DISK FILE , BUT A SCRATCH          20MY8
C        TAPE CAN BE USED IF NECCESARY OR DESIRED.                          20MY8
      DIMENSION  A(L1    ) , AM1(L1    ) , AM2(L1    ) ,                    20MY8
     1           B(L1,L1)  , BM1(L1,L1)  , EP1(L1,L1)  , ATM(L1    ) ,      20MY8
     2           C(L1,L1)  , CM1(L1,L1)  ,   D(L1,L1)  ,                    20MY8
     3           E(L1,L1)  , W(L2,L3)    , ET2(L1,N1)  ,                    20MY8
     4          DT(L1,N2)  ,  CC(L1,N3)  , ET1(L1,N1)  ,  EE(N1,L1)  ,      23MR8
     5                       FF(L1)                                         23MR8
      COMMON /RI/ NK , NL , NF                                              02FE8
      REWIND 1                                                              04JA8
      REWIND 2                                                              04JA8
      REWIND 3                                                              17JA8
          IF( ML ) 140, 100, 100                                            04JA8
C        SET INITIAL CONDITIONS                                             04JA8
  100     DO 135  J = 1 , NK                                                01FE8
          DO 130  I = 1 , NK                                                01FE8
               B(I,J)  = 0.0                                                04JA8
               C(I,J)  = 0.0                                                04JA8
               CM1(I,J) = 0.0                                               04JA8
               EP1(I,J) = 0.0                                               23MR8
  130     CONTINUE                                                          04JA8
  135     CONTINUE                                                          04JA8
  140     DO 150  I = 1 , NK                                                01FE8
               A(I) = 0.0                                                   20MY8
               AM1(I) = 0.0                                                 20MY8
  150     CONTINUE                                                          04JA8
```

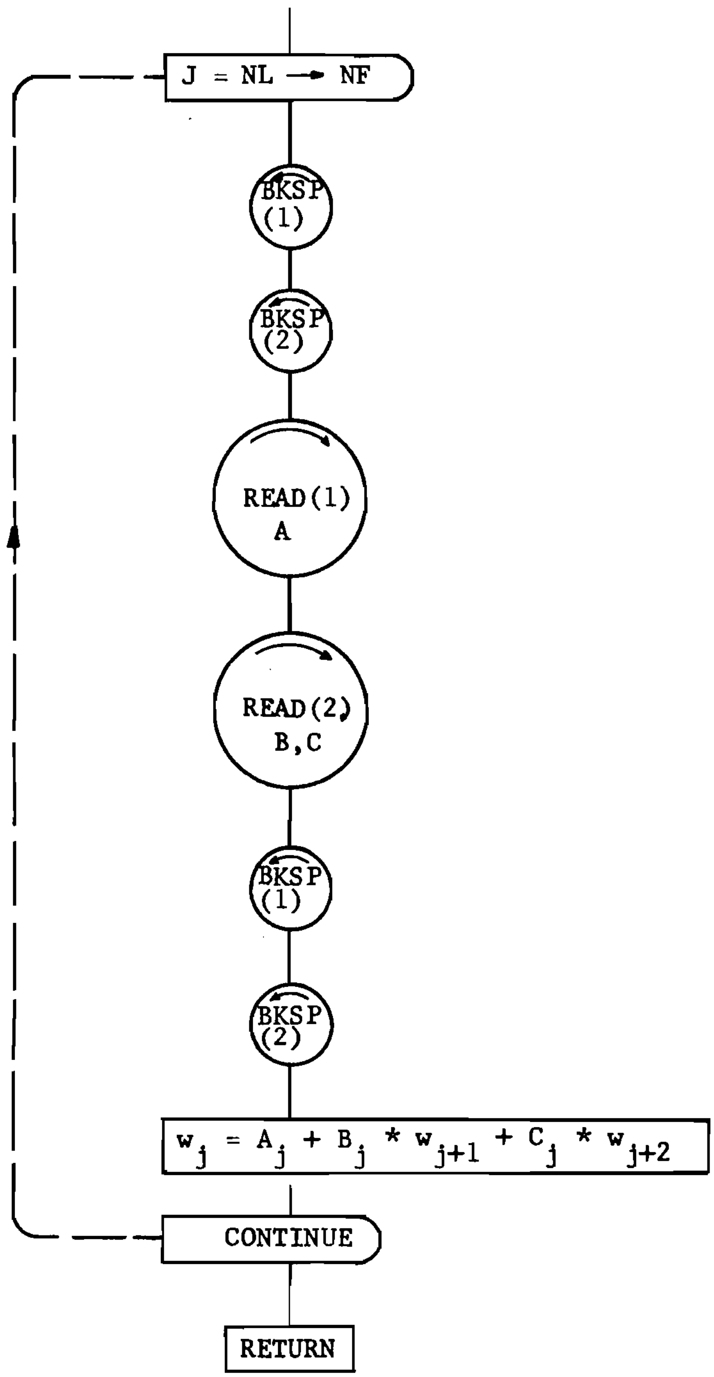
```
C***********************************************************************
C        BEGIN FORWARD PASS  --  SOLVE FOR RECURSION COEFFICIENTS      04JA8
C***********************************************************************
C                                   . . . . . . . . . . . . . . . . . . . . . . . . ..
          DO 1000  J = NF , NL                                         01FE8  .
               JJ = J                                                  04JA8  .
C        FORM SUB-MATRICES                                             04JA8  .
      CALL FSUB52 ( L1,L2,L3,ET2,ET1,CC,DT,EE,FF,ML,JJ,N1,N2,N3 )      27MY8  .
      CALL RFV    ( AM2, AM1, L1 , 1   , NK )                          20MY8  .
      CALL RFV    ( AM1, A   , L1 , 1   , NK )                         20MY8  .
          IF( ML ) 210, 180, 180                                       04JA8  .
  180 CALL RFV    ( BM1, B   , L1 , L1 , NK )                          20MY8  .
          GO TO 220                                                    04JA8  .
C        READ D AND E MULTIPLIERS FROM TAPE 3                          17JA8  .
  210 READ (3)   (( D(I,K) , E(I,K) , I = 1,NK) , K = 1,NK )           01FE8  .
          GO TO 280                                                    04JA8  .
C        CALCULATE RECURSION MULTIPLIER E                              04JA8  .
  220 CALL RFV    ( E   , EP1, L1 , L1 , NK )                          20MY8  .
C        CALCULATE RECURSION MULTIPLIER EP1                            23MR8  .
      CALL MBFV   ( ET1, BM1, EP1, L1 , L1 , NK , N1 )                 20MY8  .
      CALL ABF    ( DT , EP1, EP1, L1 , NK , N2 )                      23MR8  .
C        CALCULATE RECURSION MULTIPLIER D                              04JA8  .
      CALL SMFF   ( E   , BM1, D   , L1 , NK )                         05MR8  .
      CALL RFV    ( BM1, CM1, L1 , L1 , NK )                           20MY8  .
      CALL RFV    ( CM1, C   , L1 , L1 , NK )                          20MY8  .
      CALL MBFV   ( ET2, BM1, C   , L1 , L1 , NK , N1 )                20MY8  .
      CALL ASFV   ( D   , C   , D   , L1 , L1 , NK , +1 )              20MY8  .
      CALL ABF    ( CC , D   , D   , L1 , NK , N3 )                    01FE8  .
      CALL INVR6 ( D   , L1 , NK )                                     15MR8  .
      CALL CFV    ( D   , L1 , L1 , NK , -1.)                          20MY8  .
C        CALCULATE RECURSION COEFFIECENT C                             04JA8  .
      CALL MFB    ( D   , EE , C   , L1 , NK , N1 )                    20MR8  .
C        CALCULATE RECURSION COEFFIECENT B                             04JA8  .
      CALL MFFT   ( D   , EP1, B   , L1 , NK )                         23MR8  .
C        CALCULATE RECURSION COEFFIECENT A                             04JA8  .
  280 CALL MFFV   ( E   , AM1, A   , L1 , 1   , NK )                   20MY8  .
      CALL MBFV   ( ET2, AM2, ATM, L1 , 1   , NK , N1 )                20MY8  .
      CALL ASFV   ( A   , ATM, AM2, L1 , 1   , NK , +1 )               20MY8  .
      CALL ASFV   ( AM2, FF , ATM, L1 , 1   , NK , -1 )                20MY8  .
      CALL MFFV   ( D   , ATM, A   , L1 , 1   , NK )                   20MY8  .
C        SAVE A COEFFICIENT ON TAPE 1                                  04JA8  .
      WRITE (1)   ( A(I), I = 1,NK )                                   02FE8  .
          IF( ML )400,600,500                                          04JA8  .
  400 READ (2)                                                         17JA8  .
          GO TO 1000                                                    4JA8  .
C        SAVE D AND E MULTIPLIERS ON TAPE 3                            17JA8  .
  500 WRITE (3)   (( D(I,K),E(I,K), I=1,NK), K=1,NK)                   02FE8  .
C        SAVE B AND C COEFFICIENTS ON TAPE 2                           17JA8  .
  600 WRITE (2)   (( B(I,K),C(I,K), I=1,NK), K=1,NK)                   02FE8  .
 1000     CONTINUE                                                     04JA8  .
C                      . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
```

```
C*************************************************************************
C         BEGIN BACKWARD PASS  --  COMPUTE RECURSION EQUATION         04JA8
C*************************************************************************
      BACKSPACE 1                                                     20MY8
      BACKSPACE 2                                                     20MY8
      CALL RFV   ( W(NF,NL), A   , L1 , 1   , NK )                    01AG8
      BACKSPACE 1                                                     20MY8
      BACKSPACE 2                                                     20MY8
      READ (1)   ( A(I), I = 1,NK )                                   20MY8
      READ (2)   (( B(I,K) , C(I,K), I = 1,NK) , K = 1,NK)            20MY8
      BACKSPACE 1                                                     20MY8
      BACKSPACE 2                                                     20MY8
      CALL MFFV  ( B   , W(NF,NL), AM1, L1 , 1   , NK )               01AG8
      CALL ASFV  ( A   , AM1, W(NF,NL-1)   , L1 , 1   , NK , +1 )     01AG8
               NLM2 = NL - 2                                          20MY8
C                     . . . . . . . . . . . . . . . . . . . . . . . . . . ..
          DO 2000  L = NF , NLM2                                      20MY8  .
               J = NLM2 + NF - L                                      20MY8  .
      BACKSPACE 1                                                     04JA8  .
      BACKSPACE 2                                                     17JA8  .
C        READ A COEFFICIENT FROM TAPE 1                               04JA8  .
      READ  (1)   ( A(I), I = 1,NK )                                  02FE8  .
C        READ B AND C COEFFICIENTS FROM TAPE 2                        17JA8  .
      READ (2)   (( B(I,K) , C(I,K), I = 1,NK) , K = 1,NK)            02FE8  .
      BACKSPACE 1                                                     04JA8  .
      BACKSPACE 2                                                     17JA8  .
      CALL MFFV  ( B   , W(NF,J+1),AM1, L1 , 1   , NK )               01AG8  .
      CALL MFFV  ( C   , W(NF,J+2),AM2, L1 ,'1   , NK )               01AG8
      CALL ASFV  ( AM1, AM2, AM1, L1 , 1   , NK , +1 )                20MY8  .
      CALL ASFV  ( A   , AM1, W(NF,J) , L1 , 1   , NK , +1 )          01AG8  .
 2000     CONTINUE                                                    04JA8  .
C                     . . . . . . . . . . . . . . . . . . . . . . . . . . ..
      RETURN                                                           4JA8
      END                                                              4JA8
```

Flow Chart for Five-Wide Recursion Inversion Process

FRIP 3 , FRIP 4




J = NL → NF
BKSP (1)
BKSP (2)
READ(1) A
READ(2) B,C
BKSP (1)
BKSP (2)
$w_j = A_j + B_j * w_{j+1} + C_j * w_{j+2}$
CONTINUE
RETURN

APPENDIX B

FLOW CHARTS, LISTINGS, AND COMMENTS ON SUBORDINATE ROUTINES FOR MATRIX MANIPULATIONS

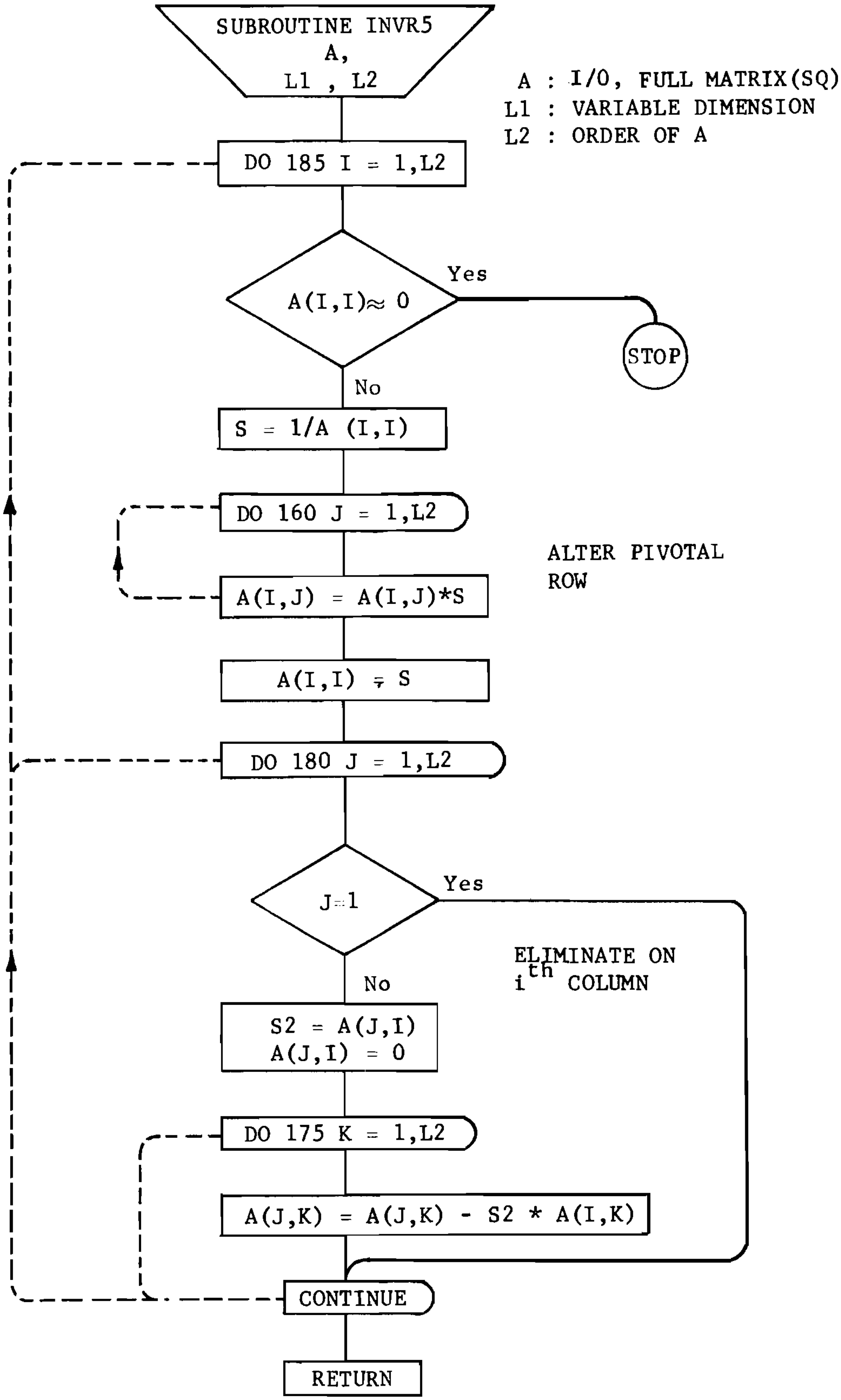
APPENDIX B. FLOW CHARTS, LISTINGS, AND COMMENTS ON SUBORDINATE SUBPROGRAMS FOR MATRIX MANIPULATIONS

Listings and Flow Charts of Subordinate Subroutines*

INVR5		-	Takes inverse of general positive definite matrix
INVR6	DCOM1 INVLT1 MLTXL	-	Takes inverse of symmetric positive definite matrix
MFFV		-	Multiplies full (square) matrix times a full (square) matrix or a vector
SMFF		-	Symmetric multiplication of a full times a full matrix
MFFT		-	Multiplies a full times the transpose of a full matrix
MBFV		-	Multiplies a banded (packed) matrix times a full matrix or a vector
MFB		-	Multiplies a full matrix times a banded (packed) matrix
ABF		-	Adds a banded matrix to a full matrix
ASFV		-	Adds or subtracts two full matrices or two vectors
RFV		-	Replaces a full matrix or a vector by another
CFV		-	Multiplies a full matrix or a vector by a constant

*In all the above subroutines, all matrices are either square or column vectors, except in the banded routines where banded square matrices are represented by packed matrices that are $K \times J$ where K is the order of the square matrix, and J is the band width.

```
      SUBROUTINE INVR5 ( A , L1 , L2 )                                   31JA8
C******* THIS ROUTINE TAKES THE INVERSE OF A GENERAL POSTIVE             05MR8
C         DEFINITE MATRIX , A SUPERIMPOSED AUGMENTED MATRIX METHOD       05MR8
C         IS EMPLOYED                                                    05MR8
      DIMENSION A(L1,L1)                                                 28DE7
   20 FORMAT ( /85X,* NON-POSITIVE DEFINITE MATRIX ENCOUNTERED * )       08AP8
   30 FORMAT ( 1X,10E10.3 )                                              10OC7
               EP = 1.0E-10                                              10OC7
C                    . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
          DO 185  I = 1 , L2                                             10OC7  .
          IF ( ABS(A(I,I) ) - EP ) 990, 990, 150                         31JA8  .
  150          S = 1.0 / A(I,I)                                          06FE8  .
          DO 160  J = 1 , L2                                             28DE7 ..
               A(I,J) = A(I,J) * S                                       10OC7 ..
  160     CONTINUE                                                       10OC7 ..
               A(I,I) = S                                                28DE7  .
          DO 180  J = 1 , L2                                             10OC7 ..
          IF ( J-I ) 170, 180, 170                                       10OC7 ..
  170          S = A(J,I)                                                06FE8 ..
               A(J,I) = 0.0                                              28DE7 ..
          DO 175  K = 1 , L2                                             28DE7...
               A(J,K) = A(J,K) - S * A(I,K)                              06FE8...
  175     CONTINUE                                                       10OC7...
  180     CONTINUE                                                       10OC7 ..
  185     CONTINUE                                                       10OC7  .
C              . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
      RETURN                                                             10OC7
  990 PRINT 20                                                           10OC7
      PRINT 30,(( A(I,J),J=1,L2),I=1,L2)                                 28DE7
      END                                                                10OC7
```



SUBROUTINE INVR5
A,
L1 , L2
A : I/O, FULL MATRIX(SQ)
L1 : VARIABLE DIMENSION
L2 : ORDER OF A
DO 185 I = 1,L2
A(I,I)≈ 0
Yes
STOP
No
S = 1/A (I,I)
DO 160 J = 1,L2
ALTER PIVOTAL
ROW
A(I,J) = A(I,J)*S
A(I,I) = S
DO 180 J = 1,L2
J=1
Yes
ELIMINATE ON
i th COLUMN
No
S2 = A(J,I)
A(J,I) = 0
DO 175 K = 1,L2
A(J,K) = A(J,K) - S2 * A(I,K)
CONTINUE
RETURN

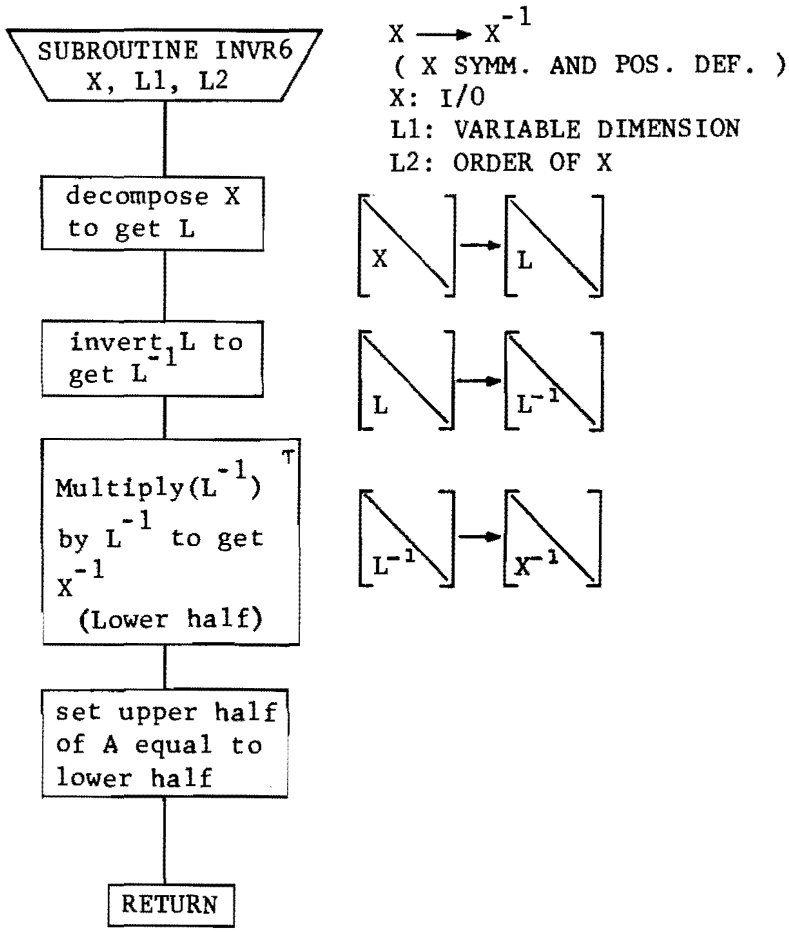
```
      SUBROUTINE INVR6 ( X , L1 , L2 )                                  19FE8
C******* THIS ROUTINE TAKES THE INVERSE OF A SYMMETRIC POSITIVE - DEF    05MR8
C        MATRIX USING A COMPACTED CHOLESKI DECOMPOSITION PROCEDURE ,    05MR8
C        A FULL DIMENSIONED MATRIX IS REQUIRED BUT ONLY THE LOWER       05MR8
C        HALF IS USED BY THE 3 ROUTINES DRIVEN BY INVR6                 05MR8
      DIMENSION X(L1,L1)                                                19FE8
         IF ( L2 - 1 ) 600, 10, 20                                      01OC8
   10    IF ( ABSF(X).LT.E-10 )      GO TO  600                         01OC8
             X = 1./ X                                                  01OC8
         GO TO  500                                                     01OC8
   20    IF ( L2 - 2 ) 30, 30, 40                                       01OC8
   30        S1 = X * X(2,2) - X(1,2) * X(2,1)                          01OC8
         IF ( ABSF(S1).LT.E-10 )      GO TO  600                        01OC8
             S1 = 1./ S1                                                01OC8
             S = X                                                      01OC8
             X = S1 * X(2,2)                                            01OC8
             X(2,2) = S1 * S                                            01OC8
             X(1,2) = -S1 * X(1,2)                                      01OC8
             X(2,1) = -S1 * X(2,1)                                      01OC8
         GO TO 500                                                      01OC8
C  40 CALL FIX1 ( X, L1, L2 )                                           04MR0
   40    CONTINUE                                                       19OC0
      CALL DCOM1  ( X , L1, L2 )                                        04MR0
      CALL INVLT1 ( X , L1 , L2 )                                       19FE8
      CALL MLTXL  ( X , L1 , L2 )                                       05MR8
C     CALL FIX1 ( X, L1, L2 )                                           29JL9
         DO 100  I = 2 , L2                                             19FE8
             KC = I - 1                                                 19FE8
         DO 50  J = 1 , KC                                              19FE8
             X(J,I) = X(I,J)                                            19FE8
   50    CONTINUE                                                       19FE8
  100    CONTINUE                                                       19FE8
  500 RETURN                                                            01OC8
  600 PRINT 601,((X(I,J), J=1,L2), I=1,L2)                              01OC8
  601 FORMAT ( 1H1,30H   SINGULAR MATRIX ENCOUNTERED ,/,2(5X,2F15.7) )  01OC8
      END                                                               19FE8
```

```
      SUBROUTINE DCOM1 ( X , L1 , L2 )                                          19FE8
      DIMENSION  X(L1,L1) , T(100)                                              12MR8
      DOUBLE PRECISION S , S1                                                   29MY8
C*************************************************************************** B
C                                                                          29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS          29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLES     29MY8 A
C        S AND S1 . CARE SHOULD BE TAKEN TO INSURE THIS IS DONE PROPERLY29MY8 R
C*************************************************************************** E
   10 FORMAT ( /85X,* NON-POSITIVE DEFINITE MATRIX ENCOUNTERED * )          12MR8
   15 FORMAT ( /,5X,13E10.3 )                                                20MR9
          DO 20  I = 1 , L2                                                 12MR8
               T(I) = X(I,I)                                                12MR8
   20     CONTINUE                                                          12MR8
          IF ( X(1,1) .LE. 0.0 )  GO TO 4000                                05MR8
               S1 = X(1,1)                                                  24JE8
               S1 = DSQRT( S1 )                                             24JE8
               X(1,1) = S1                                                  24JE8
               S1 = 1.0 / S1                                                24JE8
          DO 50  I = 2 , L2                                                 19FE8
               X(I,1) = X(I,1) * S1                                         19FE8
   50     CONTINUE                                                          19FE8
               L2M1 = L2 - 1                                                19FE8
C                        . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
          DO 200  J = 2 , L2M1                                              19FE8   .
               S = 0.0                                                      19FE8   .
               JM1 = J - 1                                                  19FE8   .
          DO 120  K = 1 , JM1                                               19FE8 ..
               S = S + X(J,K) * X(J,K) * 1.0D0                              24JE8 ..
  120     CONTINUE                                                          19FE8 ..
          IF ( X(J,J) .LE. S )  GO TO 4000                                  05MR8   .
               S1 = X(J,J) - S                                              24JE8   .
               S1 = DSQRT( S1 )                                             24JE8   .
               X(J,J) = S1                                                  24JE8   .
               S1 = 1.0 / S1                                                24JE8   .
              JP1 = J + 1                                                   19FE8   .
          DO 190  I = JP1 , L2                                              19FE8 ..
               S = 0.0                                                      19FE8 ..
          DO 180  K = 1 , JM1                                               19FE8...
               S = S + X(I,K) * X(J,K) * 1.0D0                              24JE8...
  180     CONTINUE                                                          19FE8...
               X(I,J) = ( X(I,J) - S ) * S1                                 19FE8 ..
  190     CONTINUE                                                          19FE8 ..
  200     CONTINUE                                                          19FE8   .
C                . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . ..
               S = 0.0                                                      19FE8
          DO 250  K = 1, L2M1                                               19FE8
               S = S + X(L2,K) * X(L2,K) * 1.0D0                            24JE8
  250     CONTINUE                                                          19FE8
               S = X(L2,L2) - S                                             05MR8
          IF ( S .LE. 0.0 )  GO TO 4000                                     05MR8
               X(L2,L2) = DSQRT( S )                                        05MR8
      RETURN                                                                19FE8
 4000 PRINT 10                                                              12MR8
               X(1,1) = T(1)                                                12MR8
```

```
      DO 400  I = 2 , L2                                      09AP8
           K = I - 1                                          12MR8
           X(I,I) = T(I)                                      12MR8
      DO 350  J = 1 , K                                       12MR8
           X(I,J) = X(J,I)                                    12MR8
350   CONTINUE                                                12MR8
400   CONTINUE                                                12MR8
      DO 500  I = 1, L2                                       20MR9
500 PRINT 15, ( X(I,J), J = 1,L2 )                            20MR9
    END                                                       19FE8
```

```
      SUBROUTINE INVLT1 ( X , L1 , L2 )                                 19FE8
      DIMENSION X(L1,L1)                                                19FE8
      DOUBLE PRECISION SUM                                              29MY8
C************************************************************************B
C                                                                       29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS       29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE   29MY8 A
C        SUM . CARE SHOULD BE TAKEN TO INSURE THIS IS DONE PROPERLY .   29MY8 R
C************************************************************************E
          DO 50  I = 1 , L2                                             19FE8
               X(I,I) = 1.0 / X(I,I)                                    19FE8
   50     CONTINUE                                                      19FE8
               L2M1 = L2 - 1                                            19FE8
          DO 200  J = 1 , L2M1                                          19FE8
               JP1 = J+1                                                19FE8
          DO 150  I = JP1 , L2                                          19FE8
               IM1 = I-1                                                19FE8
               SUM = 0.0                                                19FE8
          DO 120  K = J , IM1                                           19FE8
               SUM = SUM - X(I,K) * X(K,J) * 1.0D0                      24JF8
  120     CONTINUE                                                      19FE8
               X(I,J) = X(I,I) * SUM                                    19FE8
  150     CONTINUE                                                      19FE8
  200     CONTINUE                                                      19FE8
      RETURN                                                            19FE8
      END                                                               19FE8
```

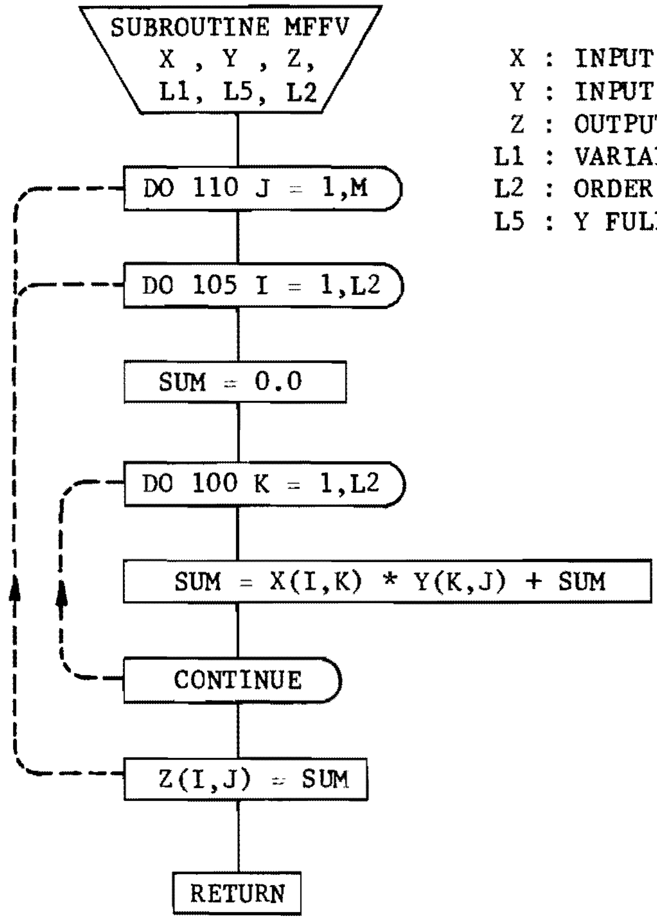
```
      SUBROUTINE MLTXL ( X , L1 , L2 )                                  19FE8
      DIMENSION X(L1,L1)                                                19FE8
      DOUBLE PRECISION SUM                                              29MY8
C************************************************************************B
C                                                                       29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS       29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE   29MY8 A
C        SUM . CARE SHOULD BE TAKED TO INSURE THIS IS DONE PROPERLY .   29MY8 R
C************************************************************************E
          DO 200  I = 1 , L2                                            19FE8
          DO 150  J = 1 , I                                             19FE8
              SUM = 0.0                                                 19FE8
          DO 100  K = I , L2                                            19FE8
              SUM = SUM + X(K,I) * X(K,J) * 1.0D0                       24JE8
  100     CONTINUE                                                      19FE8
              X(I,J) = SUM                                              19FE8
  150     CONTINUE                                                      19FE8
  200     CONTINUE                                                      19FE8
      RETURN                                                            19FE8
      END                                                               19FE8
```



$$X = L \cdot L^T$$

$$X^{-1} = (L \cdot L^T)^{-1} = (L^T)^{-1} \cdot L^{-1} = (L^{-1})^T \cdot L^{-1}$$

```
      SUBROUTINE MFFV ( X , Y , Z , L1 , L5 , L2 )                      03MR8
C******* THIS ROUTINE MULTIPLIES A FULL MATRIX                           13DE7
C         TIMES A FULL MATRIX OR A VECTOR                               13DE7
C         ( X * Y = Z )                                                 13DE7
      DIMENSION X(L1,L1) , Y(L1,L5) , Z(L1,L5)                          13DE7
      DOUBLE PRECISION SUM                                              29MY8
C*************************************************************************B
C                                                                       29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS       29MY8 W
C         BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE  29MY8 A
C         SUM . CARE SHOULD BE TAKED TO INSURE THIS IS DONE PROPERLY .  29MY8 R
C*************************************************************************E
              M = 1                                                     20MY8
          IF( L1 .EQ. L5 ) M = L2                                       20MY8
          DO 110  J = 1,M                                               13DE7
          DO 105  I = 1,L2                                              13DE7
              SUM = 0.0                                                 03MR8
          DO 100  K = 1,L2                                              13DE7
              SUM = SUM + X(I,K) * Y(K,J) * 1.0D0                       24JE8
  100     CONTINUE                                                      13DE7
              Z(I,J) = SUM                                              03MR8
  105     CONTINUE                                                      13DE7
  110     CONTINUE                                                      13DE7
      RETURN                                                            13DE7
      END                                                               13DE7
```



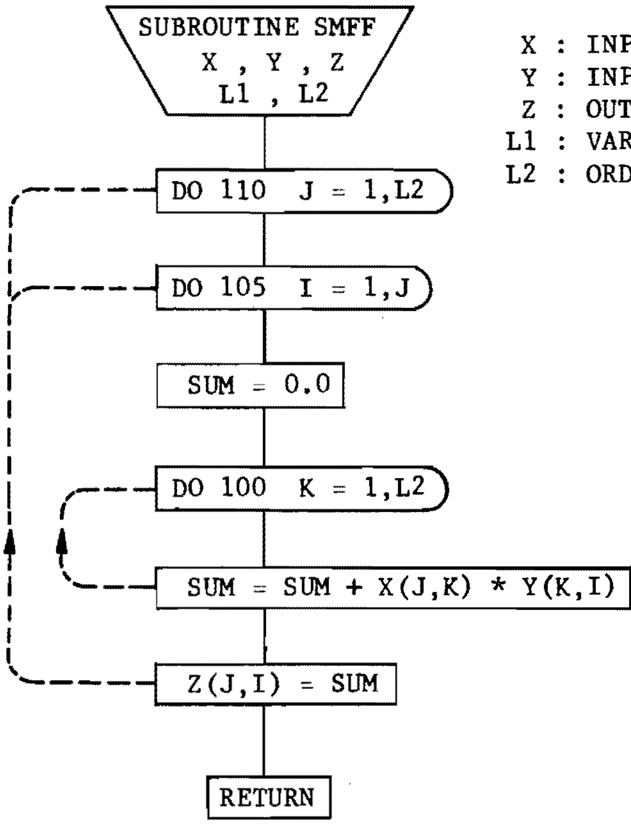
X : INPUT, FULL(Sq)
Y : INPUT, FULL OR VEC
Z : OUTPUT, FULL OR VEC
L1 : VARIABLE DIMENSION
L2 : ORDER OF X
L5 : Y FULL OR VECTOR ?

$$\begin{bmatrix} X \end{bmatrix} \cdot \begin{bmatrix} Y \end{bmatrix} = \begin{bmatrix} Z \end{bmatrix} \qquad L5 = L1$$

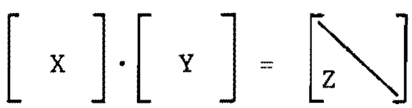
OR

$$\begin{bmatrix} X \end{bmatrix} \cdot \begin{bmatrix} Y \end{bmatrix} = \begin{bmatrix} Z \end{bmatrix} \qquad L5 = 1$$

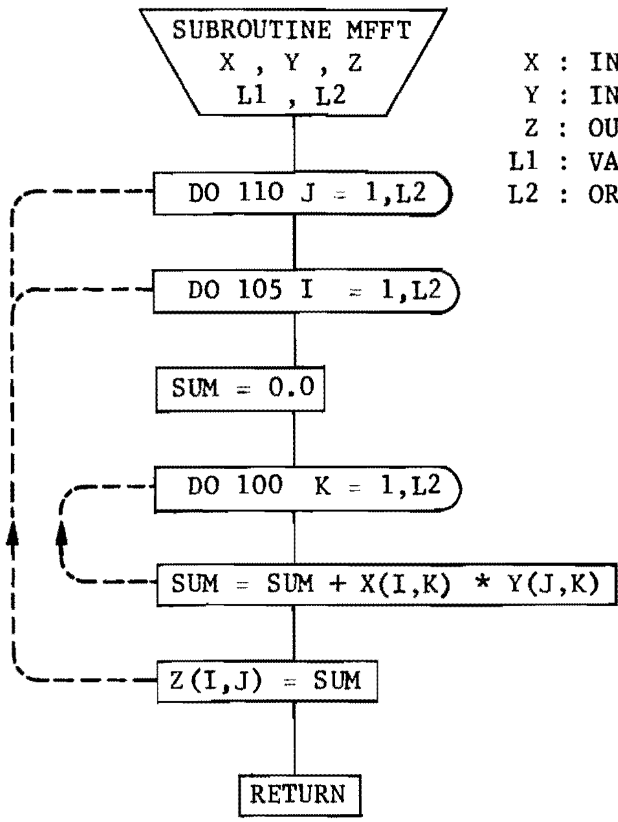
```
      SUBROUTINE SMFF  ( X , Y , Z , L1, L2 )                           19FE8
C******* THIS ROUTINE MULTIPLIES TWO FULL MATRICES UNDER THE ASSUMPTION 05MR8
C        THAT THEIR PRODUCT WILL BE SYMMETRIC ( X,Y, AND Z ARE FULL     05MR8
C        DIMENSIONED BUT ONLY THE LOWER HALF OF EACH IS USED )          05MR8
      DIMENSION  X(L1,L1) , Y(L1,L1) , Z(L1,L1)                         19FE8
      DOUBLE PRECISION SUM                                              29MY8
C************************************************************************B
C                                                                       29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS       29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE   29MY8 A
C        SUM . CARE SHOULD BE TAKED TO INSURE THIS IS DONE PROPERLY .   29MY8 R
C************************************************************************E
          DO 110  J = 1 , L2                                            19FE8
          DO 105  I = 1 , J                                             19FE8
               SUM = 0.0                                                19FE8
          DO 100  K = 1 , L2                                            19FE8
               SUM = SUM + X(J,K) * Y(K,I) * 1.0D0                      24JE8
  100     CONTINUE                                                      19FE8
               Z(J,I) = SUM                                             19FE8
  105     CONTINUE                                                      19FE8
  110     CONTINUE                                                      19FE8
      RETURN                                                            19FE8
      END                                                               19FE8
```



X : INPUT, FULL(sq)
Y : INPUT,
Z : OUTPUT, LOWER HALF
L1 : VARIABLE DIMENSION
L2 : ORDER OF X , Y , Z



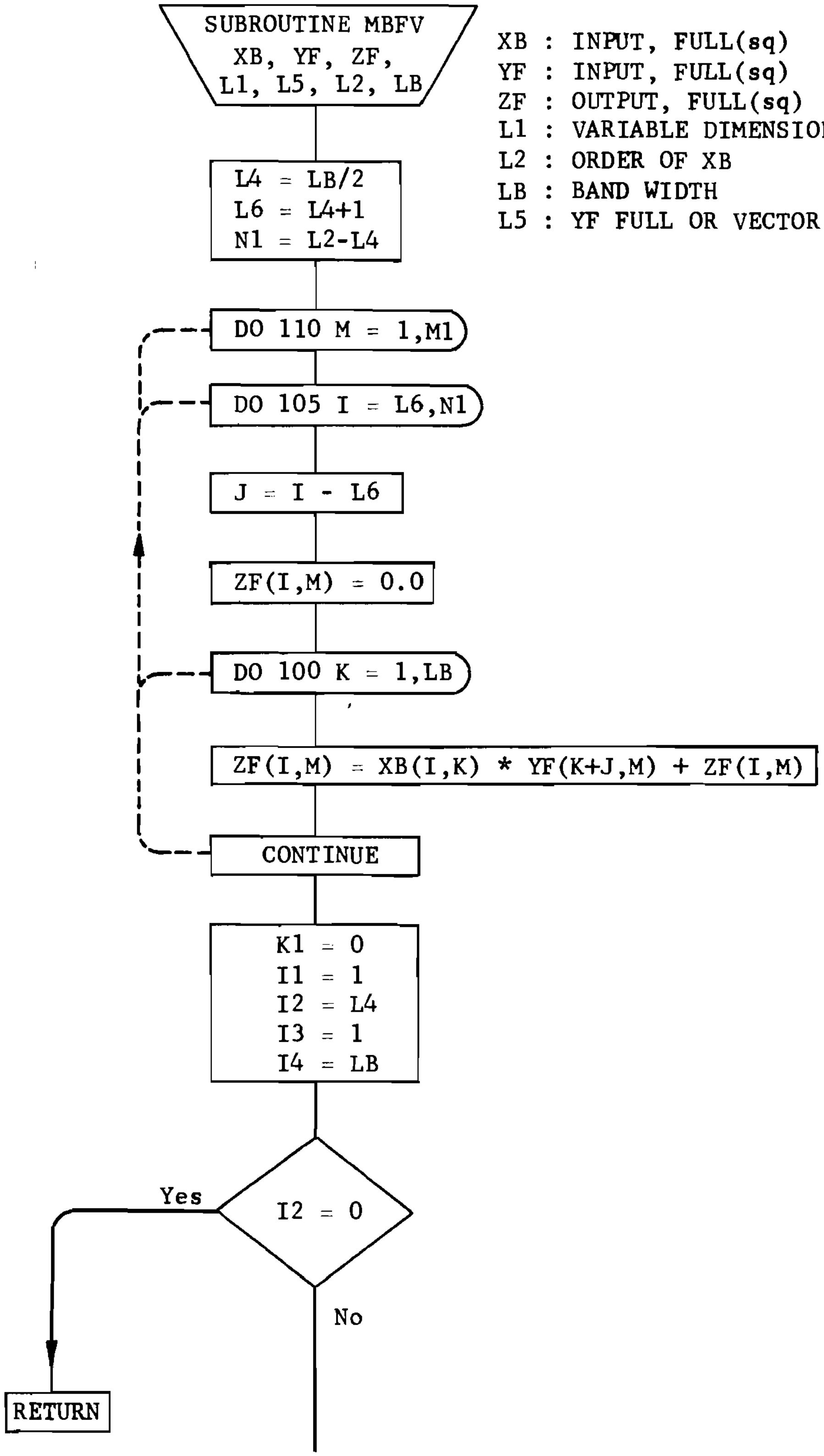
```
      SUBROUTINE MFFT ( X   , Y   , Z   , L1 , L2 )                       18MR8
C******* THIS ROUTINE MULTIPLIES A FULL MATRIX                           18MR8
C        TIMES THE TRANSPOSE OF A SECOND FULL MATRIX                     18MR8
C        ( X * YT = Z )                                                  18MR8
      DIMENSION  X(L1,L1) , Y(L1,L1) , Z(L1,L1)                         18MR8
      DOUBLE PRECISION SUM                                               29MY8
C**************************************************************************B
C                                                                        29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS        29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE    29MY8 A
C        SUM . CARE SHOULD BE TAKED TO INSURE THIS IS DONE PROPERLY .    29MY8 R
C**************************************************************************E
          DO 110  J = 1 , L2                                             18MR8
          DO 105  I = 1 , L2                                             18MR8
               SUM = 0.0                                                 18MR8
          DO 100  K = 1 , L2                                             18MR8
               SUM = SUM + X(I,K) * Y(J,K) * 1.0D0                       24JE8
  100     CONTINUE                                                       18MR8
               Z(I,J) = SUM                                              18MR8
  105     CONTINUE                                                       18MR8
  110     CONTINUE                                                       18MR8
      RETURN                                                             18MR8
      END                                                                18MR8
```



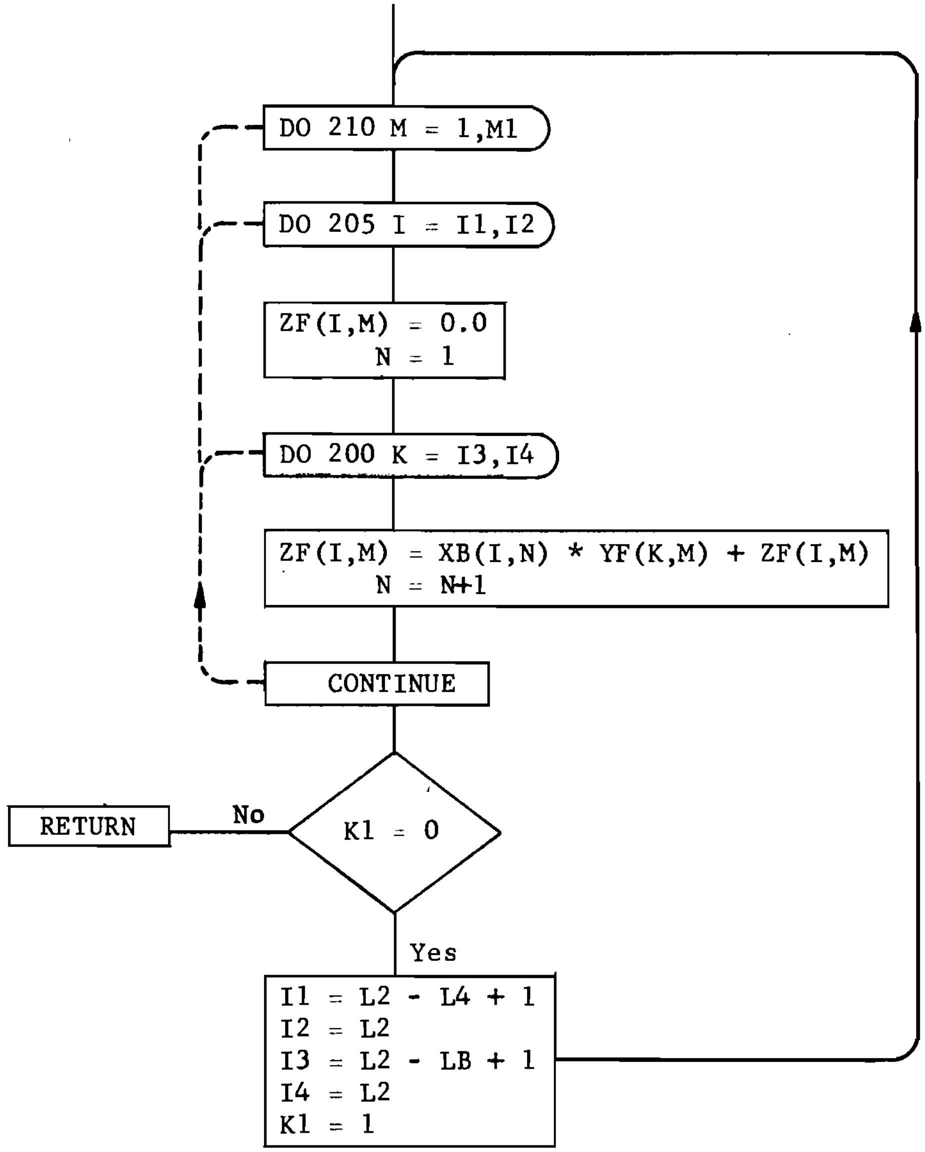
X : INPUT, FULL(sq)
Y : INPUT, FULL(sq)
Z : OUTPUT, FULL(sq)
L1 : VARIABLE DIMENSION
L2 : ORDER OF X , Y , Z

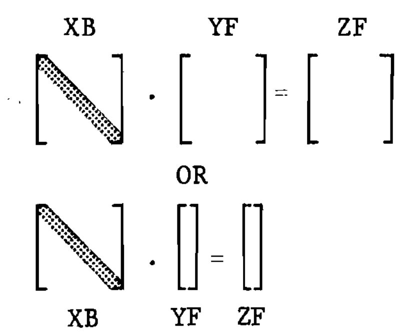
$$\begin{bmatrix} X \end{bmatrix} \cdot \begin{bmatrix} Y \end{bmatrix}^T = \begin{bmatrix} Z \end{bmatrix}$$

```
      SUBROUTINE MBFV ( XB , YF , ZF , L1 , L5 , L2 , LB )              07DE7
C******* THIS ROUTINE MULTIPLIES A BANDED MATRIX                         07DE7
C        TIMES A FULL MATRIX OR A VECTOR                                 07DE7
C        ( XB * YF = ZF )                                                07DE7
      DIMENSION  XB( L1,LB ) , YF( L1,L5 ) , ZF( L1,L5 )                 07DE7
      DOUBLE PRECISION SUM                                               29MY8
C************************************************************************B
C                                                                        29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS        29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE    29MY8 A
C        SUM . CARE SHOULD BE TAKED TO INSURE THIS IS DONE PROPERLY .    29MY8 R
C************************************************************************E
           M1 = 1                                                        20MY8
        IF( L1 .EQ. L5 ) M1 = L2                                         20MY8
           L4 = LB/2                                                     07DE7
           L6 = L4 + 1                                                   07DE7
           N1 = L2 - L4                                                  07DE7
        DO 110  M = 1,M1                                                 13DE7
        DO 105  I = L6,N1                                                13DE7
           J = I - L6                                                    07DE7
           SUM = 0.0                                                     06MY8
        DO 100  K = 1,LB                                                 07DE7
           SUM = SUM + XB(I,K) * YF(K+J,M) * 1.0D0                       24JE8
  100   CONTINUE                                                         10NO7
           ZF(I,M) = SUM                                                 06MY8
  105   CONTINUE                                                         10NO7
  110   CONTINUE                                                         10NO7
           K1 = 0                                                        10NO7
           I1 = 1                                                        10NO7
           I2 = L4                                                       07DE7
           I3 = 1                                                        07DE7
           I4 = LB                                                       07DE7
        IF( I2 ) 150, 900, 150                                           07DE7
  150   DO 210  M = 1,M1                                                 13DE7
        DO 205  I = I1,I2                                                13DE7
           SUM = 0.0                                                     06MY8
           N = 1                                                         07DE7
        DO  200 K = I3, I4                                               07DE7
           SUM = SUM + XB(I,N) * YF(K,M) * 1.0D0                         24JE8
           N = N + 1                                                     07DE7
  200   CONTINUE                                                         10NO7
           ZF(I,M) = SUM                                                 06MY8
  205   CONTINUE                                                         10NO7
  210   CONTINUE                                                         10NO7
        IF( K1 ) 900,300,900                                             10NO7
  300      I1 = L2 - L4 + 1                                              07DE7
           I2 = L2                                                       07DE7
           I3 = L2 - LB + 1                                              07DE7
           I4 = L2                                                       07DE7
           K1 = 1                                                        10NO7
        GO TO 150                                                        10NO7
  900 RETURN                                                             10NO7
      END                                                                10NO7
```



XB : INPUT, FULL(sq)
YF : INPUT, FULL(sq)
ZF : OUTPUT, FULL(sq)
L1 : VARIABLE DIMENSION
L2 : ORDER OF XB
LB : BAND WIDTH
L5 : YF FULL OR VECTOR ?

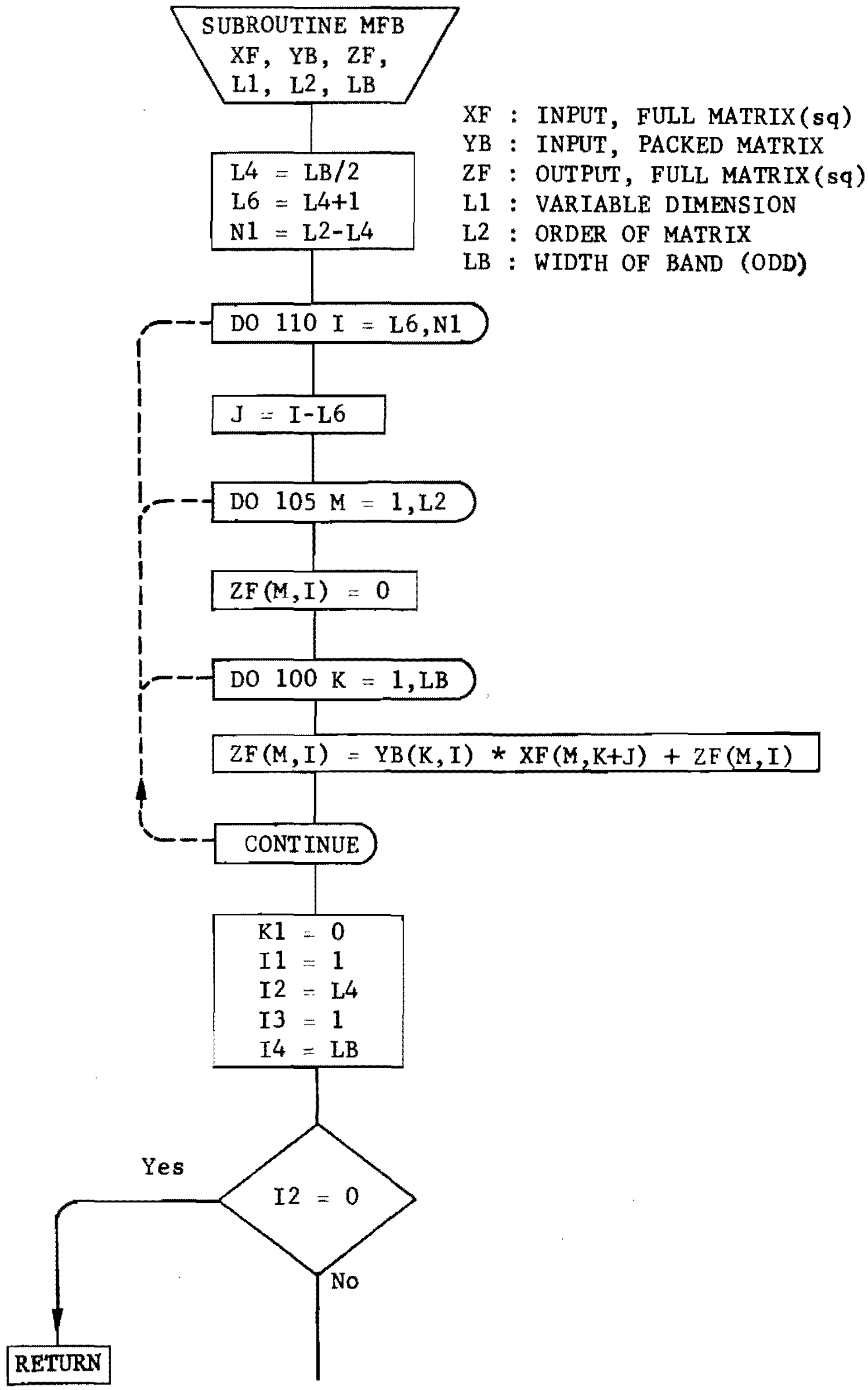

DO 210 M = 1,M1
DO 205 I = I1,I2
ZF(I,M) = 0.0
N = 1
DO 200 K = I3,I4
ZF(I,M) = XB(I,N) * YF(K,M) + ZF(I,M)
N = N+1
CONTINUE
K1 = 0
No
RETURN
Yes
I1 = L2 - L4 + 1
I2 = L2
I3 = L2 - LB + 1
I4 = L2
K1 = 1


XB
YF
ZF
OR
XB
YF
ZF

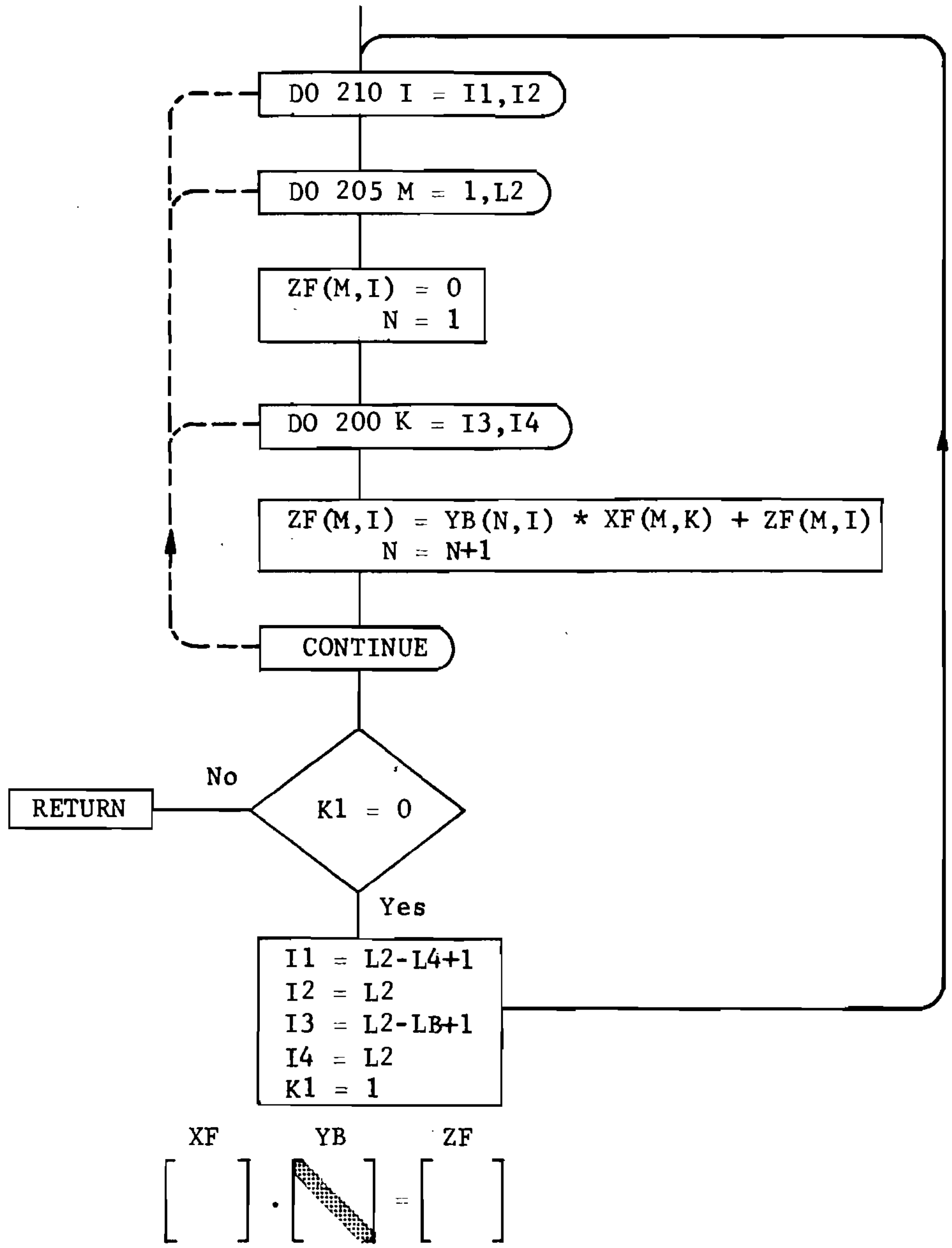
L5 = L1

L5 = 1

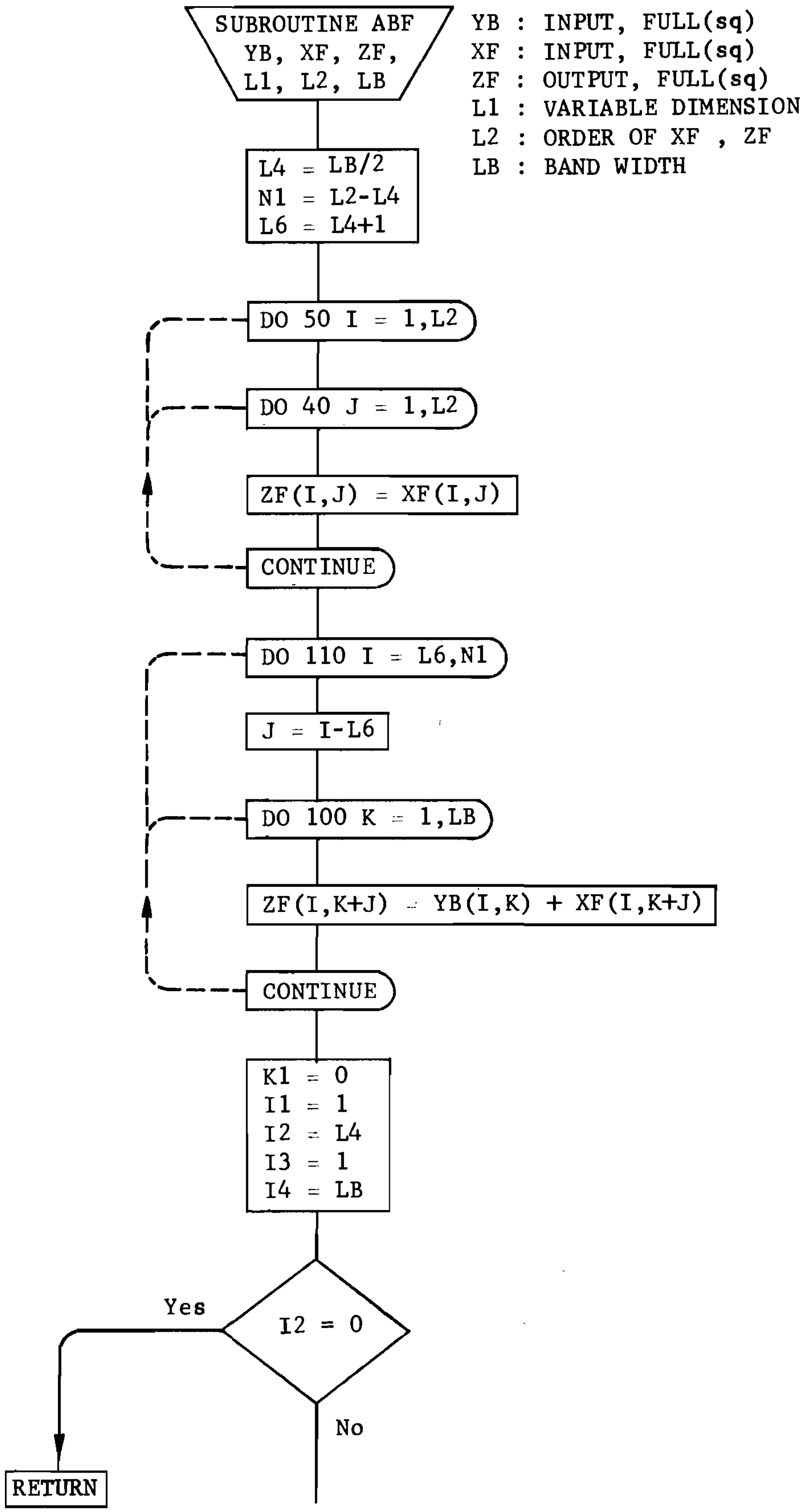
```
      SUBROUTINE MFB ( XF , YB , ZF , L1 , L2 , LB )                    07DE7
C******* THIS ROUTINE MULTIPLIES A FULL MATRIX                           07DE7
C        TIMES A BANDED MATRIX                                          07DE7
C        ( XF * YB = ZF )                                               07DE7
      DIMENSION  XF( L1,L1 ) , YB( LB,L1 ) , ZF( L1,L1 )                07DE7
      DOUBLE PRECISION SUM                                              29MY8
C***********************************************************************B
C                                                                       29MY8 E
C******* CAUTION - THE ACCURATE ACCUMULATION OF INNER PRODUCTS IS       29MY8 W
C        BEING ATTEMPTED THRU THE DOUBLE PRECISIONING OF THE VARIABLE   29MY8 A
C        SUM . CARE SHOULD BE TAKED TO INSURE THIS IS DONE PROPERLY .   29MY8 R
C***********************************************************************F
              L4 = LB/2                                                 07DE7
              L6 = L4 + 1                                               07DE7
              N1 = L2 - L4                                              07DE7
         DO 110  I = L6,N1                                              07DE7
              J = I - L6                                                07DE7
         DO 105  M = 1,L2                                               07DE7
              SUM = 0.0                                                 06MY8
         DO 100  K = 1,LB                                               07DE7
              SUM = SUM + YB(K,I) * XF(M,K+J) * 1.0D0                   24JE8
  100    CONTINUE                                                       10NO7
              ZF(M,I) = SUM                                             06MY8
  105    CONTINUE                                                       10NO7
  110    CONTINUE                                                       10NO7
              K1 = 0                                                    10NO7
              I1 = 1                                                    10NO7
              I2 = L4                                                   07DE7
              I3 = 1                                                    08DE7
              I4 = LB                                                   07DE7
         IF( I2 ) 150, 900, 150                                         07DE7
  150    DO  210 I = I1, I2                                             10NO7
         DO 205  M =  1,L2                                              07DE7
              SUM = 0.0                                                 06MY8
              N = 1                                                     08DE7
         DO  200 K = I3, I4                                             08DE7
              SUM = SUM + YB(N,I) * XF(M,K) * 1.0D0                     24JE8
              N = N + 1                                                 08DE7
  200    CONTINUE                                                       10NO7
              ZF(M,I) = SUM                                             06MY8
  205    CONTINUE                                                       10NO7
  210    CONTINUE                                                       10NO7
         IF( K1 ) 900,300,900                                           10NO7
  300         I1 = L2 - L4 + 1                                          07DE7
              I2 = L2                                                   07DE7
              I3 = L2 - LB + 1                                          07DE7
              I4 = L2                                                   07DE7
              K1 = 1                                                    10NO7
         GO TO 150                                                      10NO7
  900 RETURN                                                            10NO7
      END                                                               10NO7
```

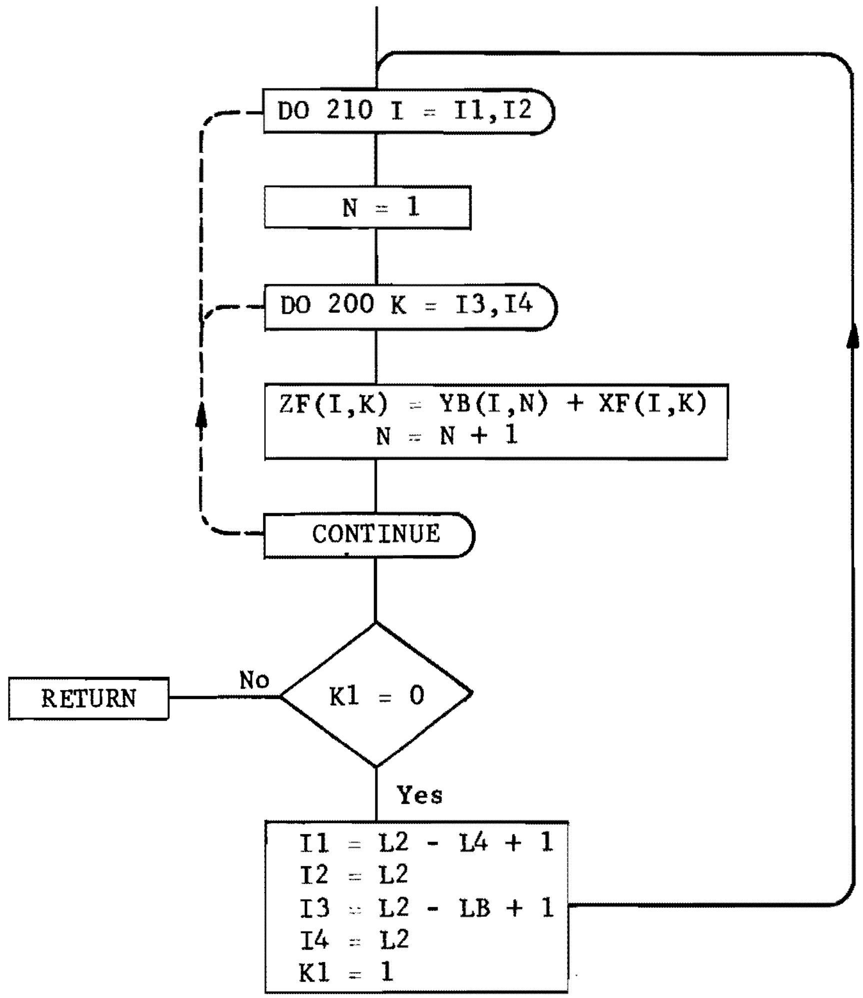


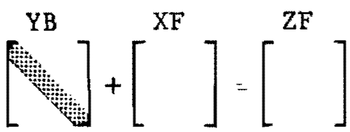
SUBROUTINE MFB
XF, YB, ZF,
L1, L2, LB
XF : INPUT, FULL MATRIX(sq)
YB : INPUT, PACKED MATRIX
ZF : OUTPUT, FULL MATRIX(sq)
L1 : VARIABLE DIMENSION
L2 : ORDER OF MATRIX
LB : WIDTH OF BAND (ODD)
L4 = LB/2
L6 = L4+1
N1 = L2-L4
DO 110 I = L6,N1
J = I-L6
DO 105 M = 1,L2
ZF(M,I) = 0
DO 100 K = 1,LB
ZF(M,I) = YB(K,I) * XF(M,K+J) + ZF(M,I)
CONTINUE
K1 = 0
I1 = 1
I2 = L4
I3 = 1
I4 = LB
I2 = 0
Yes
No
RETURN


DO 210 I = I1,I2
DO 205 M = 1,L2
ZF(M,I) = 0
N = 1
DO 200 K = I3,I4
ZF(M,I) = YB(N,I) * XF(M,K) + ZF(M,I)
N = N+1
CONTINUE
K1 = 0
No
RETURN
Yes
I1 = L2-L4+1
I2 = L2
I3 = L2-LB+1
I4 = L2
K1 = 1
XF
YB
ZF

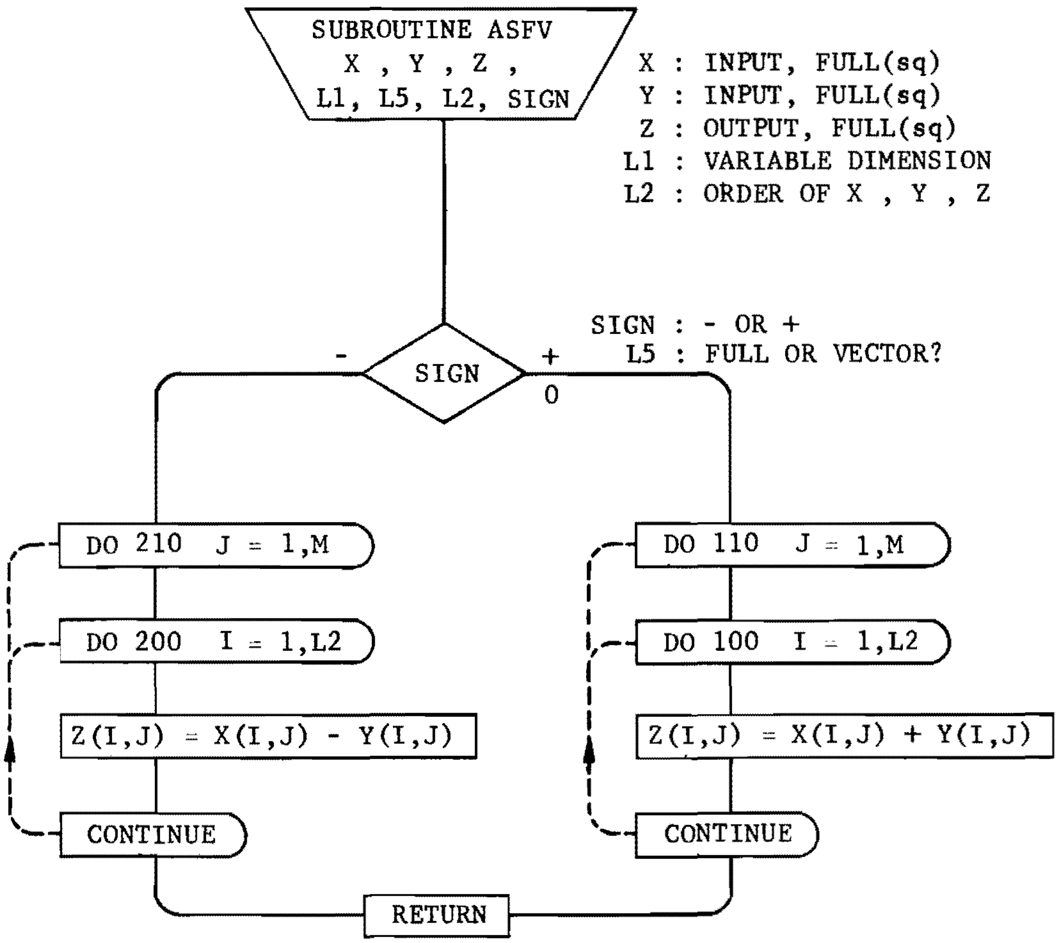
```
      SUBROUTINE ABF ( YB , XF , ZF , L1 , L2 , LB )                   07DE7
C******* THIS ROUTINE ADDS A BANDED MATRIX                             07DE7
C        TO A FULL MATRIX                                              07DE7
C        ( YB + XF = ZF  OR  XF + YB = ZF )                            07DE7
      DIMENSION YB( L1,LB ) , XF( L1,L1 ) , ZF(L1,L1)                  07DE7
              L4 = LB/2                                                07DE7
              N1 = L2 - L4                                             07DE7
              L6 = L4 + 1                                              07DE7
         DO 50  I = 1,L2                                               07DE7
         DO 40  J = 1,L2                                               07DE7
              ZF(I,J) = XF(I,J)                                        07DE7
   40    CONTINUE                                                      07DE7
   50    CONTINUE                                                      07DE7
         DO 110  I = L6,N1                                             07DE7
              J = I - L6                                               07DE7
         DO 100  K = 1,LB                                              07DE7
              ZF(I,K+J) = YB(I,K) + XF(I,K+J)                          11DE7
  100    CONTINUE                                                      10NO7
  110    CONTINUE                                                      10NO7
              K1 = 0                                                   10NO7
              I1 = 1                                                   10NO7
              I2 = L4                                                  07DE7
              I3 = 1                                                   08DE7
              I4 = LB                                                  07DE7
         IF( I2 ) 150, 900, 150                                        07DE7
  150    DO 210 I = I1,I2                                              10NO7
              N = 1                                                    08DE7
         DO  200 K = I3, I4                                            08DE7
              ZF(I,K) = YB(I,N) + XF(I,K)                              11DE7
              N = N + 1                                                08DE7
  200    CONTINUE                                                      10NO7
  210    CONTINUE                                                      10NO7
         IF(K1) 900, 300, 900                                          10NO7
  300         I1 = L2 - L4 + 1                                         07DE7
              I2 = L2                                                  07DE7
              I3 = L2 - LB + 1                                         07DE7
              I4 = L2                                                  07DE7
              K1 = 1                                                   07DE7
         GO TO 150                                                     10NO7
  900    RETURN                                                        10NO7
         END                                                           10NO7
```


SUBROUTINE ABF
YB, XF, ZF,
L1, L2, LB
YB : INPUT, FULL(sq)
XF : INPUT, FULL(sq)
ZF : OUTPUT, FULL(sq)
L1 : VARIABLE DIMENSION
L2 : ORDER OF XF , ZF
LB : BAND WIDTH
L4 = LB/2
N1 = L2-L4
L6 = L4+1
DO 50 I = 1,L2
DO 40 J = 1,L2
ZF(I,J) = XF(I,J)
CONTINUE
DO 110 I = L6,N1
J = I-L6
DO 100 K = 1,LB
ZF(I,K+J) = YB(I,K) + XF(I,K+J)
CONTINUE
K1 = 0
I1 = 1
I2 = L4
I3 = 1
I4 = LB
I2 = 0
Yes
No
RETURN


DO 210 I = I1,I2
N = 1
DO 200 K = I3,I4
ZF(I,K) = YB(I,N) + XF(I,K)
N = N + 1
CONTINUE
K1 = 0
No
RETURN
Yes
I1 = L2 - L4 + 1
I2 = L2
I3 = L2 - LB + 1
I4 = L2
K1 = 1

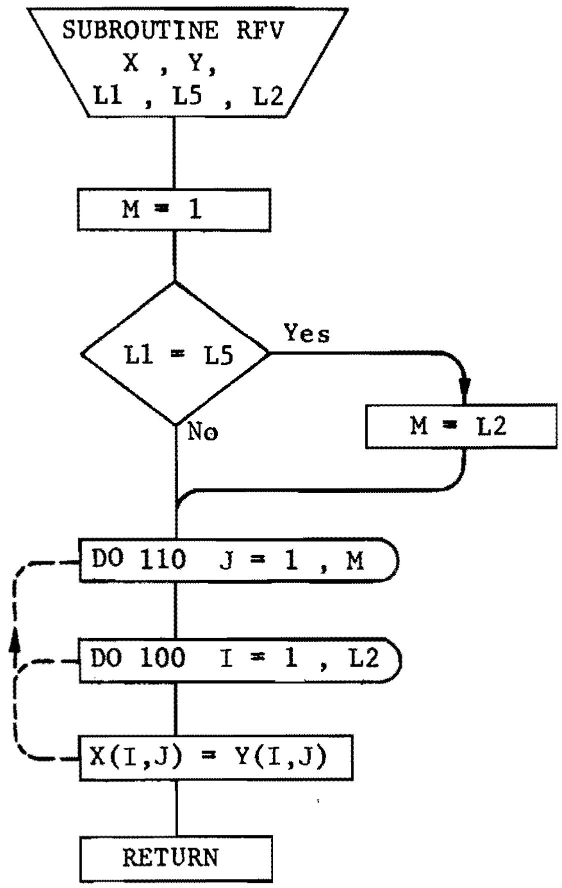

YB
XF
ZF

```
      SUBROUTINE ASFV ( X , Y , Z , L1 , L5 , L2 , SIGN )              20MY8
C******* THIS ROUTINE ADDS OR SUBTRACTS 2 FULL MATRICES OR 2 VECTORS    20MY8
C         ( X - Y = Z  OR  X + Y = Z )                                 13DE7
      DIMENSION X(L1,L5) , Y(L1,L5) , Z(L1,L5)                          13DE7
              M = 1                                                     20MY8
          IF( L1 .EQ. L5 ) M = L2                                       20MY8
          IF ( SIGN ) 190, 50, 50                                       13DE7
   50     DO 110  J = 1,M                                               13DE7
          DO 100  I = 1,L2                                              13DE7
              Z(I,J) = X(I,J) + Y(I,J)                                  13DE7
  100     CONTINUE                                                      13DE7
  110     CONTINUE                                                      13DE7
          GO TO 300                                                     13DE7
  190     DO 210  J = 1,M                                               13DE7
          DO 200  I = 1,L2                                              13DE7
              Z(I,J) = X(I,J) - Y(I,J)                                  13DE7
  200     CONTINUE                                                      13DE7
  210     CONTINUE                                                      13DE7
  300 RETURN                                                            13DE7
      END                                                               13DE7
```



$$\begin{bmatrix} X \end{bmatrix} \pm \begin{bmatrix} Y \end{bmatrix} = \begin{bmatrix} Z \end{bmatrix}$$

```
      SUBROUTINE RFV ( X , Y , L1 , L5 , L2 )                            23MR8
C******* THIS ROUTINE REPLACES A FULL MATRIX OR A VECTOR                 23MR8
C          ( X = Y )                                                     23MR8
      DIMENSION  X(L1,L5) , Y(L1,L5)                                     23MR8
               M = 1                                                     20MY8
          IF( L1 .EQ. L5 ) M = L2                                        20MY8
          DO 110  J = 1,M                                                23MR8
          DO 100  I = 1 , L2                                             23MR8
               X(I,J) = Y(I,J)                                           23MR8
  100     CONTINUE                                                       23MR8
  110     CONTINUE                                                       23MR8
      RETURN                                                             20MY8
      END                                                                23MR8
```

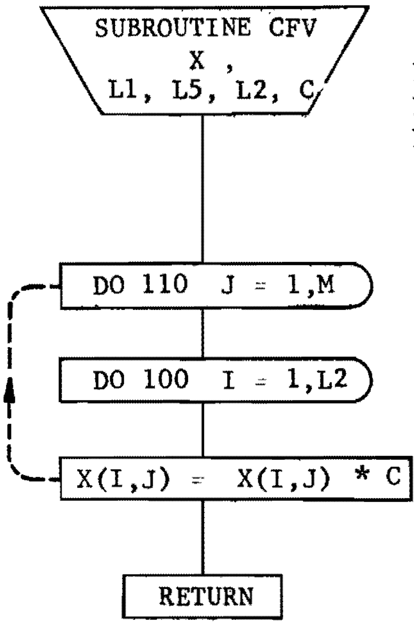


$$\left[\ X\ \right] = \left[\ Y\ \right] \qquad L5 = L1$$

OR

$$\overset{X}{\left[\ \right]} = \overset{Y}{\left[\ \right]} \qquad L5 = 1$$

```
      SUBROUTINE CFV ( X , L1 , L5 , L2 , C )                          20MY8
C******* THIS ROUTINE MULTIPLIES A FULL MATRIX OR A VECTOR BY A CONSTANT13DE7
C        ( X = C*X )                                                   13DE7
      DIMENSION X(L1,L5)                                               13DE7
            M = 1                                                      20MY8
        IF( L1 .EQ. L5 ) M = L2                                        20MY8
        DO 110  J = 1,M                                                13DE7
        DO 100  I = 1,L2                                               13DE7
            X(I,J) = X(I,J) * C                                        13DE7
  100   CONTINUE                                                       13DE7
  110   CONTINUE                                                       13DE7
      RETURN                                                           13DE7
      END                                                              13DE7
```



X : I/O
L1 : VARIABLE DIMENSION
L2 : ORDER OR LENGTH OF X
L5 : FULL OR VECTOR
C : CONSTANT MULTIPLIER

$$\begin{bmatrix} X \end{bmatrix} = C \cdot \begin{bmatrix} X \end{bmatrix}$$

OR

$$\begin{array}{c} X \\ \begin{bmatrix} \ \\ \ \end{bmatrix} \end{array} = C \cdot \begin{array}{c} X \\ \begin{bmatrix} \ \\ \ \end{bmatrix} \end{array}$$

APPENDIX C

USE OF SUBROUTINES FSUB 3 AND FSUB 5

APPENDIX C. USE OF SUBROUTINES FSUB 3 AND FSUB 5

Example Using FSUB 32 for TRIP 4 (for problem with symmetric coefficient matrix with a three-wide partitioning)

The call statement requires that FSUB 32 contain the following variables in its parameter list:

```
SUBROUTINE FSUB 32 (L1, L2, L3, BB, CC, DD, FF, ML, JJ, N2, N3)
```

where

L1 - variable dimension,

L2, L3 - variable dimension parameters for data, necessary for coefficient matrix generation if they are to be handled as such. If so, they too must be included in the parameter list.

BB, CC, DD, FF - dummy parameters representing the packed submatrices

$$d^T_{i-1} , \quad c_i , \quad d_i , \text{ and } \quad f_i ,$$

ML - switch indicating whether problem is parent, regular, or offspring (+, 0, -1). For this routine we are concerned only with whether it is an offspring or not. Remember for the symmetric case

$$b_i = d^T_{i-1} ,$$

JJ - index of what partition we are at, JJ = 1, 2, ..., L ,

N2 - band width of BB and DD , and

N3 - band width of CC

The dimension statement will be as follows (if generation data are added and variable dimensioning is used, then the additional variables should be added on):

DIMENSION BB(L1, N2) , CC(L1, N3) , DD(N1, L1) , FF(L1)

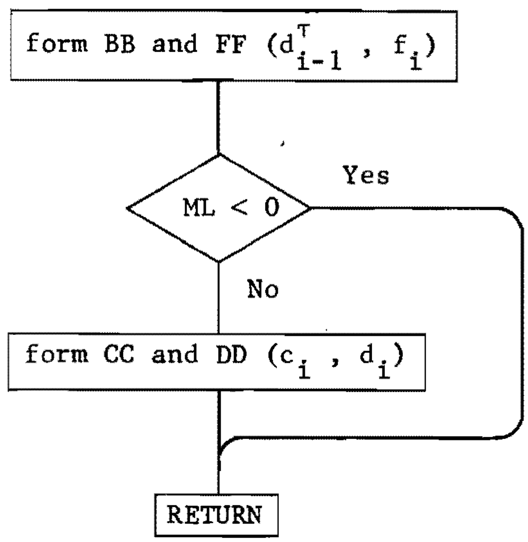
Common block RI is needed and appears as follows:

COMMON /RI/ NK , NL , NF

This block is common to both the solution driver (TRIP 4) and the main driving routine utilizing the solution package and therefore will be explained in the following appendix.

In general FSUB 32 will have the following form:

```
SUBROUTINE FSUB 32 (L1, L2, L3, BB, CC, DD, FF, ML, JJ, N2, N3)
DIMENSION BB(L1, N1) , CC(L1, N2) , DD(N1, L1) , FF(L1)
COMMON /RI/ NK , NL , NF
```



As it has been mentioned before, the submatrices are packed, and it is here that the packing must be done. Figures C.1 and C.2 explain the packing procedure for the specific routines involved. For FSUB 31 and FSUB 32, BB and CC (b_i , c_i) will be packed according to Fig C.1 and DD (d_i) according to Fig C.2. For FSUB 51 and FSUB 52, AA , BB , CC , and DD (a_i , b_i , c_i , d_i) are done as in Fig C.1 and EE (e_i) as in Fig C.2. In both cases FF (f_i) is a vector. Figures C.3 and C.4 demonstrate a specific application of the packing procedure.

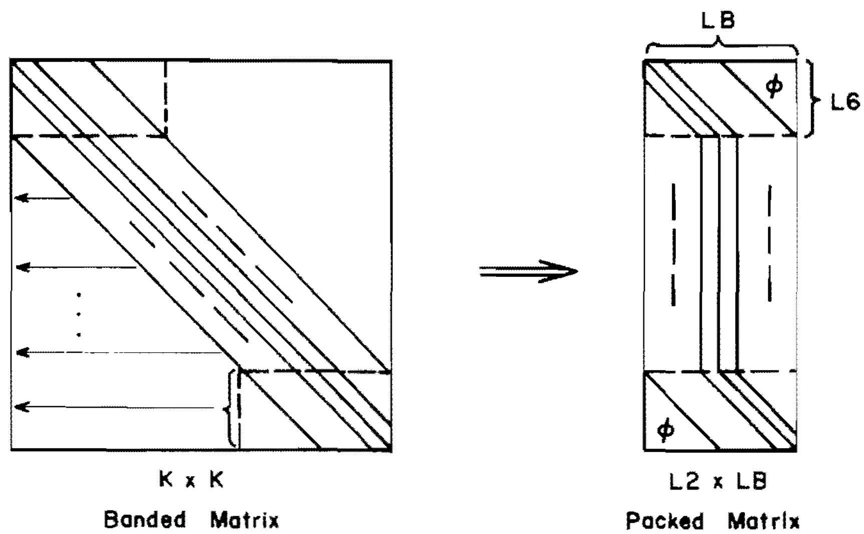


Fig C.1. Packing of the banded matrices that are multiplied by a full matrix.

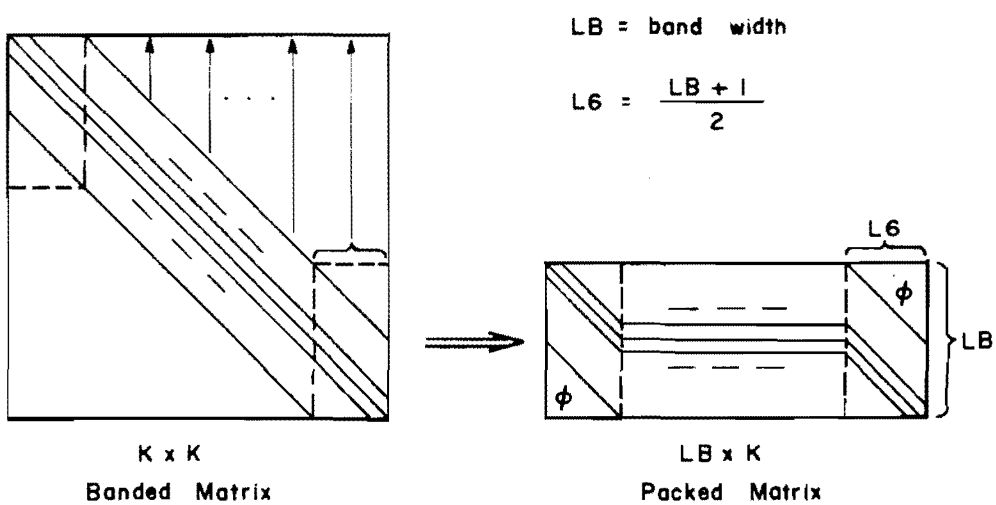
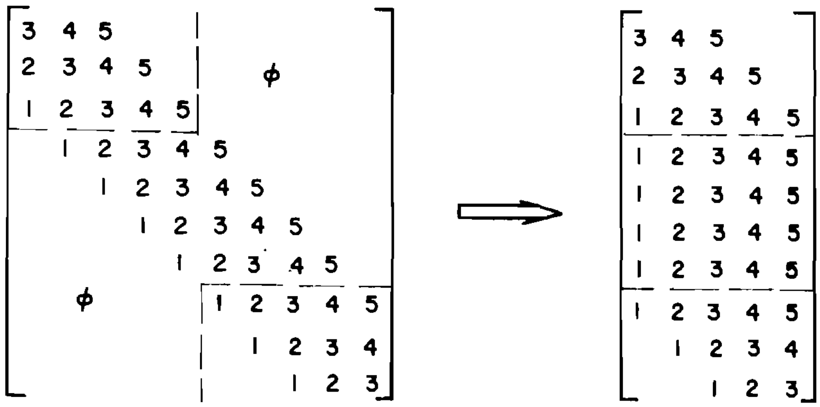


Fig C.2. Packing of the banded matrices that are multiplied by a full matrix.



LB = 5 $L6 = \frac{5+1}{2} = 3$

Fig C.3. Example of packing for subroutines ABF and MBFV.

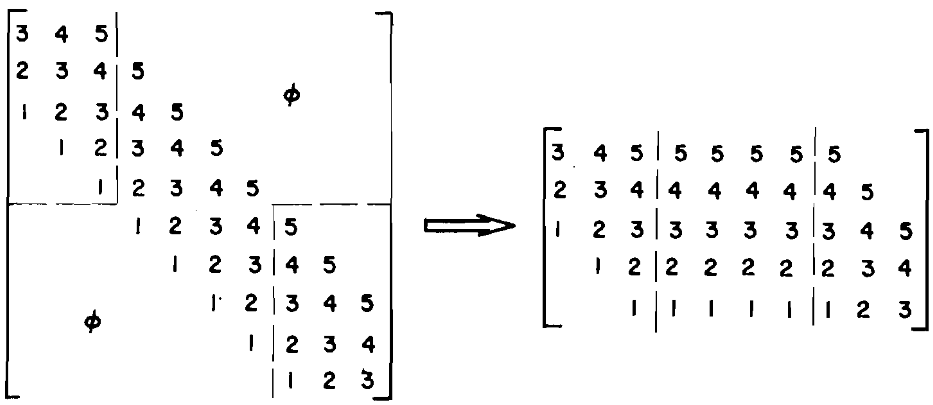


Fig C.4. Example of packing for subroutine MFB.

Example Using FSUB 52 for FRIP 4

The call statement requires that FSUB 52 contain the following variables in its parameter list:

```
      SUBROUTINE FSUB 52 (L1, L2, L3, AA, BB, CC, DD, EE, FF, ML, JJ,
     1                    N1, N2, N3)
```

where

L1, L2, L3 - as in FSUB 32,

AA, BB, CC, DD, EE, FF - dummy parameters representing the packed matrices e^T_{i-2} , e^T_{i-1} , c_i , d^T_i , e_i , f_i .
For the symmetric case $a_i = e^T_{i-2}$, $b_i = e^T_{i-1}$, and we need the transpose of d_i .

ML and JJ - as in FSUB 32,

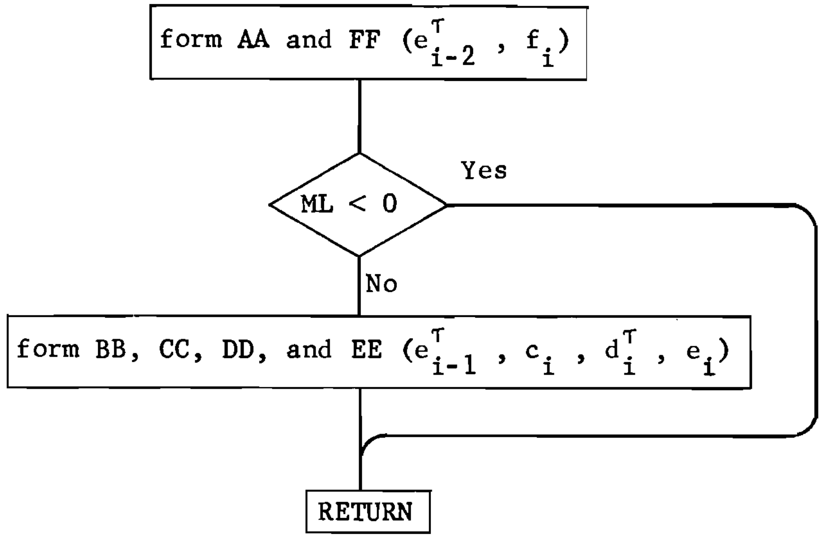
N1 - band width of AA and EE ,

N2 - band width of BB and DD , and

N3 - band width of CC .

In general FSUB 52 will have the following form:

```
      SUBROUTINE FSUB 52 (L1, L2, L3, AA, BB, CC, DD, EE, FF, ML, JJ,
     1                    N1, N2, N3)
      DIMENSION  AA(L1, N1) , BB(L1, N2) , CC(L1, N3) ,  BD(L1, N2) ,
     1           EE(N1, L1) , FF(L1)
      COMMON /RI/ NK , NL , NF
```



For the nonsymmetric cases either TRIP3 or FRIP3 would be used and therefore FSUB 31 or FSUB 51 would have to be created. The only difference in FSUB 31 as opposed to FSUB 32 would be that b_i is not necessarily equal to d_{i-1}^T ; therefore, it would have to be formed independently and therefore the additional band width parameter N4 is needed, and the calling would be as follows:

```
        CALL FSUB 31 (L1, L2, L3, BB, CC, DD, FF, ML, JJ, N2, N3, N4)
```

The analogous situation exists for the five-wide nonsymmetric case. The five parameters AA through EE would now stand for a_i through e_i , respectively and again the band width parameters must be added to the calling statement and the subroutine parameter list.

```
        CALL FSUB 51 (L1, L2, L3, AA, BB, CC, DD, EE, FF, ML, JJ,
       1              N1, N2, N3, N4, N5)
```

This means we would now dimension DD with the variable N4 and EE with N5 whereas in the symmetric, since it is necessary that N5 = N1 and N4 = N2 , they were not needed.

APPENDIX D

USE OF OVERALL PACKAGE

APPENDIX D. USE OF OVERALL PACKAGE

To demonstrate what is necessary to use the solution package, we will take TRIP 4 to use as an example.

First of all, the user's main program must dimension the following variables thusly:

```
      DIMENSION  A(α) , AM1(α) , ATM(α) , C(α, α) , D(α, α) ,
     1           W(β, ρ) , DT1(α, 1) , CC(α, 1) , DD(1, α) , FF(α)   (1)
```

For all four routines the dimension statement necessary in the main program is identical with that in the appropriate routine, with the variable dimensions replaced by the actual values the variable dimension parameters have been set equal to. In the example below we have chosen $\alpha = 10$ which limits the maximum value K (the order of our submatrices) to 10. Also we have arbitrarily chosen $N2 = 1$, $N3 = 1$ which is where the one's come from in the above equation. Next we have the following common block:

```
      COMMON /RI/ NK , NL , NF                                     (2)
```

Since variable dimensioning has been used throughout, it is necessary that certain variables be defined prior to the call statement. In the event that data information needed for FSUB 32 is in common or is read and programmed in, as is the case in this example, the variables L2 and L3 are extraneous. They were included solely as variable dimension definers for any necessary data arrays that should be variably dimensioned for practical purposes (in all routines we have used the L2 and L3 parameters for the W array). We use L1 to define α of the above dimension statement.

```
      L1 = α
      L2 = β
      L3 = ρ                                                       (3)
```

Generally NF will be either 1, 2, or 3 and is merely the starting value for the main do loop that carries us through the L partitions. Due to internal subscripting (biased so as to avoid zero and negative subscripts) it is sometimes advantageous to run from 2 to L + 1 , or 3 to L + 2 , instead of the standard 1 to L . NK is the order of the submatrices for the particular problem, and NL is the matrix order of the overall coefficient matrix (there are NL^2 submatrices in the coefficient matrix). NK and NL will generally be changing from problem to problem and therefore will most likely be read in or be a function of some input parameter. For simplicity in our rather restrictive example, we have explicitly given them values but all that is necessary is that they be defined in some fashion.

To distinguish between parent, standard, and offspring problems, the variable ML must be set to either a positive, zero, or negative value, respectively.

NF - starting value for main do loop

$\left.\begin{array}{l}\text{NK}\\ \text{NL}\end{array}\right\}$ - defined as a function of the particular problem (see above)

ML - -, 0, + number depending on type of problem

The only remaining necessity in the driving program is the call statement itself.

```
      CALL TRIP 4 (L1, L2, L3, ML, A, AM1, ATM, C, CM1, D, DT1, CC,
     1            DD, FF, W, 1, 3)
```

All the variables have been explained except the last three. The last two are the respective band widths of the outside and center submatrices, and W is our output parameter containing the solution vector. The NL partitions of W have been stacked as columns of a rectangular array, thereby being more directly related to the type of problems this procedure was written for.

Combining this with the appropriate FSUB routine, we have the necessary information to use the solution procedure. On the next page is an example use of TRIP 4 using a restrictive but sufficient FSUB 32 to demonstrate the minimal requirements necessary to use the procedure.

The FSUB 32 routine written for our example with $NK = 3$, $NL = 4$ generates the following coefficient matrix.

$$A = \left[\begin{array}{ccc|ccc|ccc|ccc}
10 & & & 1 & & & & & & & & \\
 & 10 & & & 1 & & & \phi & & & \phi & \\
 & & 10 & & & 1 & & & & & & \\
\hline
1 & & & 10 & & & 1 & & & & & \\
 & 1 & & & 10 & & & 1 & & & \phi & \\
 & & 1 & & & 10 & & & 1 & & & \\
\hline
 & & & 1 & & & 10 & & & 1 & & \\
 & \phi & & & 1 & & & 10 & & & 1 & \\
 & & & & & 1 & & & 10 & & & 1 \\
\hline
 & & & & & & 1 & & & 10 & & \\
 & \phi & & & \phi & & & 1 & & & 10 & \\
 & & & & & & & & 1 & & & 10
\end{array}\right]$$

$$f_1 = \left[\begin{array}{c} 1\\1\\1\\ \hline 1\\1\\1\\ \hline 1\\1\\1\\ \hline 1\\1\\1 \end{array}\right] \quad
f_2 = \left[\begin{array}{c} 2\\2\\2\\ \hline 2\\2\\2\\ \hline 2\\2\\2\\ \hline 2\\2\\2 \end{array}\right] \quad
f_3 = \left[\begin{array}{c} 1\\2\\3\\ \hline 4\\5\\6\\ \hline 7\\8\\9\\ \hline 10\\11\\12 \end{array}\right] \quad
f_4 = \left[\begin{array}{c} 0\\0\\0\\ \hline 0\\0\\5\\ \hline 5\\0\\0\\ \hline 0\\0\\0 \end{array}\right] \quad
f_5 = \left[\begin{array}{c} 10\\0\\0\\ \hline 0\\0\\0\\ \hline 0\\0\\0\\ \hline 0\\0\\0 \end{array}\right]$$

The solution vectors to the equation $A w_i = f_i$, $i = 1, 2, \ldots, 5$ where A and f_i are defined as above.

The solution to problem 100 was obtained as a parent problem whereas 101 through 104 were obtained as offsprings since the coefficient matrix was the same in all five cases. We have introduced no generalities or versalities into FSUB 32 since its purpose for this example was merely to generate a suitable coefficient matrix with which to demonstrate the use of TRIP 4. Below is the actual driver described above with the associated output.

```
      PROGRAM DRIV4 (INPUT,OUTPUT,TAPE1,TAPE2,TAPE3 )                     27MR8
      DIMENSION  A(10)     , AM1(10)     , ATM(10)     ,                  27MR8
     1            C(10,10) ,   D(10,10) ,   W(10,10) ,                    19AP8
     2          DT1(10, 1) ,  CC(10, 1) ,  DD(1 ,10) ,  FF( 10  )         19AP8
      DIMENSION  PROB(5)                                                  19AP8
      EQUIVALENCE  ( ATM , CC )                                           29MR8
      COMMON /RI/ NK , NL , NF                                            27MR8
    5 FORMAT ( 5A5,I5 )                                                   27MR8
    6 FORMAT (1H1)                                                        27MR8
    7 FORMAT ( //,10X,5A5,//,10X,* ML = *,I5 )                            27MR8
    8 FORMAT ( 5X,* CONSTANT VECTOR *// )                                 27MR8
    9 FORMAT ( //,5X,*SOLUTION VECTOR PARTITIONED AND PRINTED *           27MR8
     1         ,/,5X,*OUT AS A K BY L MATRIX *// )                        27MR8
   10 FORMAT ( 5X,10F9.4 )                                                27MR8
               L1 = 10                                                    27MR8
               L2 = 10                                                    27MR8
               L3 = 10                                                    27MR8
               NF = 1                                                     27MR8
               NK = 3                                                     27MR8
               NL = 4                                                     27MR8
   20 READ 5 ,(PROB(I), I =1,5), ML                                       27MR8
      PRINT 6                                                             27MR8
      PRINT 7 ,(PROB(I),I=1,5), ML                                        27MR8
      PRINT 8                                                             27MR8
      CALL TRIP4 ( L1,L2,L3,ML,A,AM1,ATM,C,D,DT1,CC,DD,FF,W,1,1 )         21MY8
      PRINT 9                                                             27MR8
          DO 100  I = 1 , NK                                              27MR8
      PRINT 10 , ( W(I,J), J=1,NL )                                       27MR8
  100     CONTINUE                                                        27MR8
          GO TO 20                                                        27MR8
      END                                                                 27MR8
```

```
      SUBROUTINE FSUB32 ( L1,L2,L3,BB,CC,DD,FF,ML,JJ,N1,N2 )            27MR8
      DIMENSION  BB(L1,N1) , CC(L1,N2) , DD(N1,L1) , FF(L1)              27MR8
      COMMON /RI/ NK , NL , NF                                           27MR8
          DO 100  I = 1 , NK                                             27MR8
               BB(I) = 1.0                                               27MR8
 100      CONTINUE                                                       27MR8
      READ 10 , ( FF(I),I=1,NK)                                          27MR8
  10  FORMAT (  3F5.3)                                                   27MR8
      PRINT 11, JJ , ( FF(I),I=1,NK )                                    20AP8
  11  FORMAT ( 5X,* F(*I1,*) = *,10F6.3 )                                20AP8
          IF ( ML.LT.0 ) GO TO 1000                                      27MR8
          DO 200  I = 1 , NK                                             27MR8
               CC(I) = 10.0                                              27MR8
 200      CONTINUE                                                       27MR8
          IF ( JJ.EQ.NL )  GO TO 900                                     27MR8
          DO 300  I = 1 , NK                                             27MR8
               DD(I) = 1.0                                               27MR8
 300      CONTINUE                                                       27MR8
          GO TO 1000                                                     27MR8
 900      DO 500 I = 1 , NK                                              27MR8
               DD(I) = 0.0                                               27MR8
 500      CONTINUE                                                       27MR8
1000      CONTINUE                                                       27MR8
      RETURN                                                             27MR8
      END                                                                27MR8
```

```
      PROB 100 - PARENT -

       ML =       1
 CONSTANT VECTOR


 F(1) =  1.000 1.000 1.000
 F(2) =  1.000 1.000 1.000
 F(3) =  1.000 1.000 1.000
 F(4) =  1.000 1.000 1.000

SOLUTION VECTOR PARTITIONED AND PRINTED
OUT AS A K BY L MATRIX


    .0917     .0826     .0826     .0917
    .0917     .0826     .0826     .0917
    .0917     .0826     .0826     .0917
```

```
      PROB 101 - OFFSPRING -

       ML =      -1
 CONSTANT VECTOR


 F(1) =  1.000 1.000 1.000
 F(2) =  1.000 1.000 1.000
 F(3) =  1.000 1.000 1.000
 F(4) =  1.000 1.000 1.000

SOLUTION VECTOR PARTITIONED AND PRINTED
OUT AS A K BY L MATRIX


    .0917     .0826     .0826     .0917
    .0917     .0826     .0826     .0917
    .0917     .0826     .0826     .0917
```

```
     PROB 102 - OFFSPRING -

       ML =     -1
 CONSTANT VECTOR


 F(1) =  1.000 2.000 3.000
 F(2) =  4.000 5.000 6.000
 F(3) =  7.000 8.000 9.000
 F(4) = 10.00011.00012.000


SOLUTION VECTOR PARTITIONED AND PRINTED
OUT AS A K BY L MATRIX


    .0664     .3362     .5721      .9428
    .1581     .4187     .6547     1.0345
    .2499     .5013     .7372     1.1263




     PROB 103 - OFFSPRING -

       ML =     -1
 CONSTANT VECTOR


 F(1) = -0.    -0.    -0.
 F(2) = -0.    -0.    5.000
 F(3) =  5.000-0.    -0.
 F(4) = -0.    -0.    -0.


SOLUTION VECTOR PARTITIONED AND PRINTED
OUT AS A K BY L MATRIX


    .0052    -.0515     .5103    -.0510
   0.        0.        0.        0.
   -.0510     .5103    -.0515     .0052
```

```
     PROB 104 - OFFSPRING -

       ML =     -1
 CONSTANT VECTOR


 F(1) = 10.000-0.    -0.
 F(2) = -0.    -0.   -0.
 F(3) = -0.    -0.   -0.
 F(4) = -0.    -0.   -0.


SOLUTION VECTOR PARTITIONED AND PRINTED
OUT AS A K BY L MATRIX

   1.0102   -.1021    .0103   -.0010
   0.       0.        0.      0.
   0.       0.        0.      0.
```

THE AUTHORS

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