THEORY OF MOISTURE MOVEMENT IN EXPANSIVE CLAYS

by

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Study of Expansive Clays in
Roadway Structural Systems

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PREFACE

This report is the first in a series of reports from Research Project 3-8-68-118 entitled "Study of Expansive Clays in Roadway Structural Systems." The report presents a theoretical discussion of moisture movement in clay soil. The theory presented in this report is essential to the development and understanding of the computer programs to be described in subsequent reports and will ultimately lead to practical applications of the results. The major topics discussed in this report include permeability, soil suction, and the unsaturated soil stress coefficients α and X . These topics are combined in a derivation of the differential equation for moisture movement in an unsaturated porous material such as expansive clay.

This project is a part of the Cooperative Highway Research Program of the Center for Highway Research, The University of Texas at Austin, and the Texas Highway Department in cooperation with the U. S. Department of Transportation, Bureau of Public Roads. The Texas Highway Department contact representative is Larry J. Buttler.

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LIST OF REPORTS

Report No. 118-1, "Theory of Moisture Movement in Expansive Clays" by Robert L. Lytton, presents a theoretical discussion of moisture movement in clay soil.

Report No. 118-2, "Continuum Theory of Moisture Movement and Swell in Expansive Clays" by R. Ray Nachlinger and Robert L. Lytton, presents a theoretical study of the phenomena of expansive clay.

ABSTRACT

Moisture flows through clay under the influence of a variety of potentials; the size of these potentials is important in determining the ease with which the water travels. The permeability of the soil is treated as a concentration-dependent tensor. The weighted sum of driving potentials is termed suction. A comprehensive discussion of the suction-moisture content relation is presented, together with some speculative comments on the reasons for the shape of the characteristic suction and permeability curves. The amount of stress transferred to the soil mineral skeleton by the water is discussed in detail and some fine distinctions are drawn concerning the unsaturated soil stress parameters α and χ .

The differential equation for flow of moisture in an unsaturated medium is derived and two alternative forms of this equation are considered: one describes a "diffusion" of suction and the other uses a diffusion of volumetric moisture content. Advantages of each procedure are discussed.

Moisture flow in clay is a complicated subject and this theoretical development forms the basis for the prediction tools to be described in subsequent reports.

KEY WORDS: clays, soil mechanics, permeability, pore water pressure, soil suction, swelling water, climatology, theory (expansion/swelling).

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CHAPTER 1. INTRODUCTION

This report is to present a theoretical background for use of the computer programs to predict moisture distribution and soil swell developed in Project 3-8-68-118, "Study of Expansive Clays in Roadway Structural Systems," sponsored by the Texas Highway Department and the Bureau of Public Roads. The information presented in this report refers primarily to the process of moisture movement in clay, although this cannot, in reality, be separated from the volume change processes, which will be covered in another report.

Clay swells when it accumulates water and it shrinks when water leaves; this volumetric activity causes widespread damage and serious economic loss; an attempt to reduce this loss through better understanding of the phenomenon is the reason for this study of the mechanics of water transfer in clay.

To illustrate the parts played by the subjects treated in Chapters 2, 3, and 4, an analogy may be drawn with electricity. Voltage is a potential; it furnishes the energy to drive the electric current through a wire. The current is what flows and it is held back by a property of the material and its geometry known as resistance. The formula used in elementary physics textbooks to describe the relation between the electrical quantities is

$$E = IR \tag{1.1}$$

where

E = the voltage,

I = the current,

R = the resistance.

This relation can be given in another way, which is more appropriate to the present study:

$$I = \frac{1}{R} \times E \qquad (1.2)$$

In this case, the reciprocal of resistance is the conductance. In water flow through porous media, the head gradient is the analog of voltage, permeability is analogous to conductance, and filter velocity is similar to current.

Darcy's law connects these quantities as follows:

$$v = -k \frac{dH}{dL} \qquad (1.3)$$

where

v = the velocity, positive in the direction of falling head gradient,

k = the permeability,

$\frac{dH}{dL}$ = the head gradient.

The water that flows in natural clay deposits may be supplied by a number of sources such as rain, condensation, groundwater, and leakage from man-made water carriers.

Dry clay attracts water and furnishes its own head gradients. The size of the head depends on how dry the clay is. Clay also furnishes its own resistance to flow, which varies with how tightly the water is bound to clay particles. So, the flow of water in swelling clay is largely self-induced. These autogenous driving and resistance characteristics, the suction, and the permeability of clay describe the mechanisms of moisture transfer in clay. For this reason, Chapter 2 is concerned with permeability of clay; and Chapter 3 covers a summary of the topic of soil suction. Chapter 4 includes a further discussion regarding the effect of overburden pressure on the suction at a point in a soil mass and on the empirical unsaturated soil factors α and χ, which are important in the problem of water flow in soil.

Any attempt to describe the properties of soils must first define the size of the fundamental particle which is being considered. Current literature deals with soils on at least four levels of fundamental particle size:

(1) atomic,
(2) microscopic,
(3) macroscopic, and
(4) systemic.

Atomic properties reported in the literature include the arrangement of atoms in crystals, chemical properties, and physical effects of different ions and molecules acting on individual or pairs of particles.

Microscopic properties could be considered to be the effects of one integration of all of the atomic properties within the particle of interest. At this level, observations are made of individual soil grains and water menisci. Particle size, shape, weight, and specific gravity are important in these studies.

Macroscopic properties are the results of another integration of all microscopic properties within an "element" of soil which is large enough to see with the naked eye, to feel with the hands, and to test in the laboratory. Unlike numerical integration, these "property" integrations produce not smoother data but more erratic. The properties measured and reported at the macroscopic level are really statistical averages. Once they are regarded as such, there is no difficulty in using the results to solve practical problems, despite the inexact nature of the properties used. The only concern is the range of error in the final result. Macroscopic-level properties include grain size distribution; permeability; water-content relations; soil-suction, water-content relations; compressibility and expansibility coefficients; Atterberg limits; and others. Engineers are interested in these properties; they correlate directly or intuitively with observed field behavior of a soil mass.

A soil mass is a system composed of a number of macroscopic elements. It has passive properties that are described as effects and responses rather than causes. Settlement, heave, differential swell, and moisture distribution are measurable effects in a soil mass and can be explained ultimately only by "differentiation" to the atomic particle size. A more convenient approach is to deal with the macroscopic elements and their statistical properties. By integrating once, the properties of the soil mass are achieved. These must be checked with what actually occurs and reasonable agreement should be expected. The degree of agreement will be compatible with the degree of accuracy of macroscopic soil properties used.

In 1907, Buckingham (Ref 10) published his ideas of "capillary potential" and "capillary conductivity," which have served as fundamental concepts to the prediction of water movement in soil. Gardner (Ref 22) further developed the approach which was treated as an analogy to a diffusion or heat flow problem. Childs and Collis-George (Ref 12), in a widely quoted paper on permeability, showed that the "diffusivity" or diffusion "constant" was really dependent on the moisture content of the soil. Philip (Ref 42) made these concepts most generally applicable to isothermal flow of water by recognizing that water diffuses through the soil in liquid, vapor, and adsorbed phases. He incorporated all of these into a diffusion equation. Most recent experimental work which gives hope that the diffusion approach is a valid one has been done by Aitchison, Blight, B. G. Richards, and Croney, Coleman, and Black.

Theoretical objections to the diffusion approach have been considered by Philip, who showed that inertia-effects due to changing momentum of soil moisture are of negligible significance in most practical problems. However, a theoretical development of the physics of moisture movement which has been presented by de Wet (Ref 16) considers soil and moisture momentum in a mixture theory approach. Overall, derivations and experimental work to date indicate that the diffusion approach is practical and shows promise of producing answers that are reasonable.

In considering any theoretical development, the words of H. F. Winterkorn (Ref 61) are good to remember:

> Water and soil are not only the most important materials in this world, but also the most complex ones, each in its own category, water as a liquid and soil as a multi-phase dispersed system. As a matter of fact, this very complexity of water and soil is the reason for their importance in engineering as well as in biology. If water and soil are combined . . . then the complexity of each is multiplied by that of the other. At this point, the learned coward gives up; the conscientious scientist and engineer goes on. He observes and describes his observations; later, he classifies and systematizes facts found by himself and others; still later, certain simple mechanisms can be recognized and, if they are simple enough, be described by means of mathematical formulae. The total complexity, however, remains and any theory or

> hypothesis that is sufficiently simple to permit mathematical formulation is either so general as to be obvious, or it is applicable only to a limited area of the total field.

Little can be added to this statement other than to note that since this remark was made in 1958, computers have increased in size and speed, so that more complex mechanisms may be considered now than formerly. This progression of computational ability may permit the use of some fairly satisfactory mathematical models sometime in the future.

The Swelling Problem

The study of the swelling phenomenon is a valid engineering problem. The damage done by swelling clay to residential and light commercial foundations as well as to highway and freeway pavements, retaining walls, drilled piers, masonry veneer and bearing walls, and gypsum board interior walls represents economic loss that has probably never been fully estimated. The Building Research and Advisory Board of the National Academy of Sciences prepared design recommendations for residential slabs-on-ground for the Federal Housing Administration (Ref 11) which recognize the importance of such variables as:

(1) the climate - drought probability,

(2) the plasticity index of the soil - a measure of the potential swell, and

(3) the type of construction (Frame buildings are more flexible and can permit greater differential movement without developing serious "utility" cracks. Brick construction requires a more rigid foundation. Utility cracks do not affect structural integrity of the building but are unsightly and sometimes greatly inconvenient. Their mere presence lowers property values).

The recommended design method assumes uniform soil support over an area which is assumed smaller as climate and soil conditions grow worse. The climate rating is apparently a probability of drought not occurring. A paper by Barger and Thom (Ref 5) outlines the criteria for evaluating drought hazard which was used as a basis for establishing the climate rating. The drought hazard is a probability derived from statistical data on:

(1) annual precipitation,

(2) uniformity of precipitation distribution throughout the year,

(3) number of times precipitation occurs,

(4) duration of the occurrence, and
(5) amount of precipitation at each occurrence.

In the judgment of the Building Research Advisory Board, the drier the soil conditions before construction, the more likely damage would occur. The drought hazard determination provided a quantitative evaluation of the factors they considered most important.

The engineering problem became apparent in the application of the design recommendations. As reported in Engineering News-Record (Ref 27), a group of home builders in San Antonio computed that a 26 by 68-foot "floating" grid house foundation slab designed according to the BRAB recommendations would cost \$1.68 per square foot as opposed to \$1.04 per square foot, the cost of current standard construction. No claim is made here as to the accuracy of these figures, but they do serve to point up the fact that designing foundations to withstand the destructive effects of swelling clays can be expensive. The engineer's task is to provide the most economical design which can be expected to perform well under the given foundation and climatic conditions.

In an effort to study the problem of soil support, Meyer and Lytton (Ref 34) investigated the interaction problem of typical grid beams resting on a swollen foundation. Two different types of beams in common construction use and three different strengths of soil were studied for a house and a warehouse foundation. For the least stiff soil tried, soil support under the beams was substantial and could not be called uniform. The stiffer the supporting soil, the less uniform was the soil support.

The more one delves into this subject the more complicated it becomes. It appears that at least the following characteristics of a foundation problem should be considered:

(1) climate,
(2) soil strength,
(3) soil swelling capacity,
(4) distribution of soil moisture,
(5) structural characteristics of overlying structures,
(6) load distribution, and
(7) initial soil conditions.

All of these are technical conditions. Economic factors must be weighed as well. A flow chart of the technical problem is shown in Fig 1. This

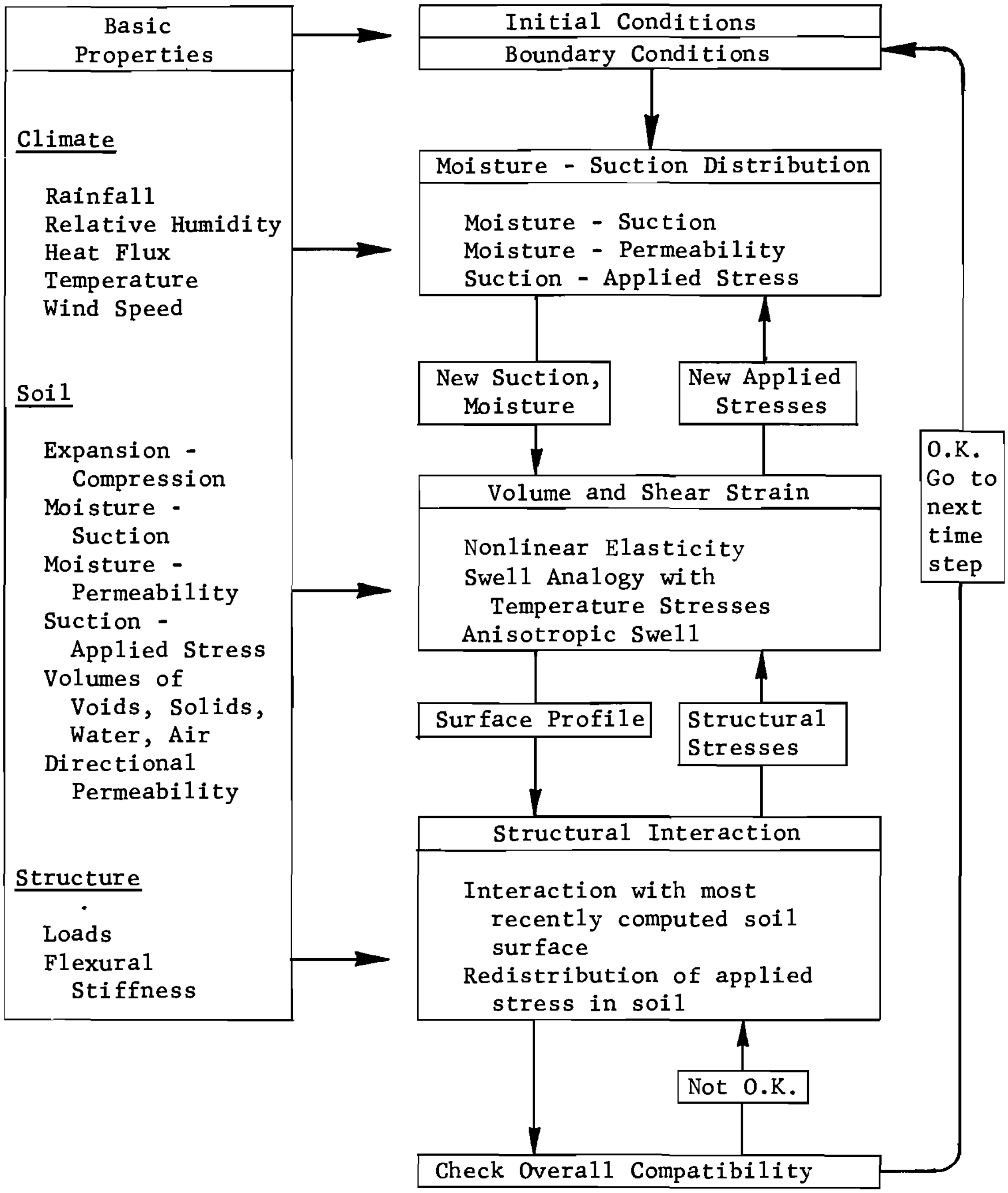


Fig 1. A flow chart representation of the swelling clay problem.

indicates that a rather complete solution of a swelling clay problem would involve a knowledge of the basic soil properties, initial soil conditions which include suction and moisture content, and the conditions on the boundary of a soil region of interest at each step in time subsequent to the first. This solution presupposes an ability to predict moisture-suction redistribution with time, a capability of coupling this moisture-suction change to the volume and shape change characteristics of each soil element, and capacity for interacting the soil mass with overlying structures and loading patterns. The nonlinear stress-strain characteristics of soil would require an iterative solution in a computer in which an assumed elastic modulus would be tried and revised until imposed stress and computed strain fall on the specified stress-strain curve. Structural interaction with nonlinear supporting soil involves the same kind of process. The volume change of a swelling clay may be different in different directions. This phenomenon has been at least partially explained by Nalezny and Li (Ref 35) as being related to particle orientation. The swelling process under such conditions may be explained by analogy with volume change due to temperature change. As water (or heat) enters a solid, it freely changes volume and develops no internal stresses unless it is restrained by the pressure due to the weight of solids above it. Under this restraint, it changes volume by a lesser amount. It is possible to apply enough pressure to prevent expansion entirely. The volume change due to the increase of water content (or heat content) may be different in different directions due to the anisotropic particle (or crystalline) arrangement of the solid.

There may be no computer large enough at the present time to store all of the information required to solve a practical problem of the sort outlined in Fig 1. It may be possible to work the pieces separately, however. If some measure of prediction can be achieved, this information can be used by engineers in establishing more economical design procedures.

It appears that the central problem of predicting the swell of a clay mass is tied to the prediction of moisture redistribution with time due to altered boundary conditions.

Relation of This Report to the Purpose of Project 118

This report furnishes the theoretical background for the flow equations developed and solved numerically by the moisture distribution computer programs to be described in Research Report No. 118-3. This theoretical background will also apply to later reports on the computer programs to predict the swelling of clay.

The important soil properties that are considered include:

(1) permeability - Chapter 2,
(2) soil "suction" - Chapter 3, and
(3) empirical factors α and χ which have to do with the volume change characteristics of the soil - Chapter 4.

These properties are combined in Chapter 5 to give the differential equations of flow in rectangular and cylindrical coordinate systems.

The moisture distribution computer programs currently do not provide the capability of calculating the effect of temperature gradients on moisture flow. While the effect of thermal gradients on moisture is evident in experimental tests, it is judged to be small in comparison with those effects induced by the isothermal soil water potentials for most problems involving swelling clays.

The problem of "hydrogenesis," the development of water in the base course, has been explained by Winterkorn (Ref 60) to be due to condensation and would be expected to be most effective in supplying water where the night-to-day temperature range is greater. Moist night air condenses on cooler base course aggregate. The greater temperature during the next day decreases surface tension, which in turn decreases the thickness of water film that can be supported on the aggregate. Gradually, a drop forms at the base of each pebble. When it drops, it adds to the water supplied to the clay subgrade beneath it.

Thus, although the soil beneath the pavement is hotter than the soil outside and the "temperature gradient" would imply a transfer of water away from the hotter zone, in fact, more water develops under the pavement. This mechanism, well-known to desert travelers, may explain much of what has been termed the "membrane effect," the collection of water beneath a membrane normally thought to be because of an interrupted natural "transpiration" of moisture from the ground.

The fact that water actually collects under pavements where thermal gradients would dictate otherwise forms some of the basis for judging that these gradients do not produce large effects in moisture transfer, and their consequent exclusion from consideration in the moisture distribution computer programs.

Cases in which thermal effects are important include foundations affected by permafrost and frost heave and foundations for kilns and blast furnaces.

In Chapter 6, conclusions are presented with regard to the theory developed, its limitations, and its capabilities, and the need for further theoretical work is discussed.

CHAPTER 2. PERMEABILITY OF CLAY SOIL

Permeability of clay is determined by the properties of the fluids contained in its pores and by the geometric properties of the porous medium. This chapter presents an analysis of each of these properties, regarding fluid flow as a rate process and the porous medium as a collection of particles having anisotropic geometric properties. The fact that clays change in volume as water content changes also introduces complications in the analysis of clay permeability. An interpretation of the results of changing geometry is presented.

Permeability was originally considered by Darcy (Ref 15), in 1856, to be a constant of proportionality between the flow rate of liquid and the loss of pressure head along the length of a sample of porous material.

This linear proportion between flow rate and pressure gradient has been found experimentally to be valid for low rates of flow. Reynolds numbers based on average velocity and grain diameter are required to be less than one (1) for the linear relation to hold. Above this threshold there is a nonlinear relation.

An analogy between experimental observations on flow through porous media and flow through pipes can be made. In pipe flow, three flow regions are recognized: laminar, mixed, and turbulent. In laminar flow there is a linear relation between flow rate and pressure gradient, but at this point similarity between flow processes ends. Laminar pipe flow occurs in straight lines, is irrotational, and is quite predictable. Linear flow through porous media occurs along tortuous paths, is highly rotational, and is perhaps macroscopically turbulent. The linear empirical relation mentioned above may be established only if porous media flow is viewed microscopically. In 1930, Nutting (Ref 37) noted that permeability defined in this manner is dependent on the viscosity of the flowing liquid and introduced the idea of specific permeability used by petroleum engineers. In civil engineering work, the fluid is usually water and its viscosity and density do not vary enough to make it necessary to consider them. However, the concept is valuable because it separates permeability into its two components:

(1) fluid properties: viscosity and density, and
(2) geometric properties of the porous medium.

The permeability ordinarily used by civil engineers is

$$k = \frac{\gamma}{\mu} k_s \tag{2.1}$$

where

γ = the density of the liquid,

μ = the viscosity of the liquid,

k_s = the "specific" permeability of the medium.

Gardner's Empirical Relation

Gardner (Ref 21) has related the permeability of clays to the suction causing water movement. Gardner's relation is as follows:

$$k = \frac{a}{|\tau|^n + b} \tag{2.2}$$

where

$\frac{a}{b}$ = the saturated permeability,

$|\tau|$ = the absolute value of the suction causing water movement,

n = an exponent which varies with grain size (larger for coarser grains).

Suction is an isothermal energy quantity, usually expressed in centimeters of water. It is a measure of the total energy with which soil at a given moisture content will attract or imbibe pure water at standard temperature and pressure. A negative energy gradient causes the flow of moisture in approximately the same direction. Olsen and Langfelder (Ref 39) have given a good explanation of the components of suction: matrix suction, osmotic suction, and pore pressures. Matrix suction includes adsorption and capillary potentials and is related to the geometrical configuration of soil. Osmotic suction results from a difference of ion concentration. Because soil will draw pure water, there must be a negative gradient which points into the soil. Because of this and because the free energy of pure water is taken as the zero suction datum, soil suctions are negative quantities.

Water is a polar liquid because of the dipole arrangement of the oxygen and hydrogen atoms in the water molecule. When water contacts a wettable solid surface, the physical properties of the water close to the surface are different from those of the free liquid. The amount of this difference is determined primarily by the types of surface and dissolved ions in the water.

Macey (Ref 33) found that nonpolar organic liquids passed through clays at rates 10^5 to 10^6 times as high as that of water. Lutz and Kemper (Ref 32) made tests on several clays to determine the effect of different electrolytes on permeability and found that most of those tried resulted in higher permeabilities than that of water. It has been noted in other experiments that permeability increases with applied pressure gradients.

These facts may be explained in a macroscopic sense which views the adsorptive bonding of polar liquids as an increase in viscosity. Near the surface of a particle, viscosity is higher than at a distance of 100 A from the surface. Thus, pressure gradient flow between particles can be regarded as flow of a non-Newtonian stratified liquid. For some average flow velocity, there can be some average viscosity which relates that velocity to the pressure gradient. In this rough manner, Darcy's law can be said to apply to the flow of water through clay.

Gardner's relation is useful because it fits much of the measured permeability-suction data. However, it does not appear to account for the changing geometry of the soil, as well as the known increase of average viscosity with higher values of suction. The empirical coefficients a , b , and n can be interpreted in these terms; but before doing so some peculiarities of Gardner's relation must be observed. Bouwer (Ref 9) presented the results of many observations of unsaturated permeability, all of which have characteristic shapes as shown in Fig 2. The designation τ_m refers to the suction at which the maximum slope of the curve occurs. A higher n in Gardner's relation gives a more square shape to the curve. The higher n is associated with coarser-grained and poorly graded soils.

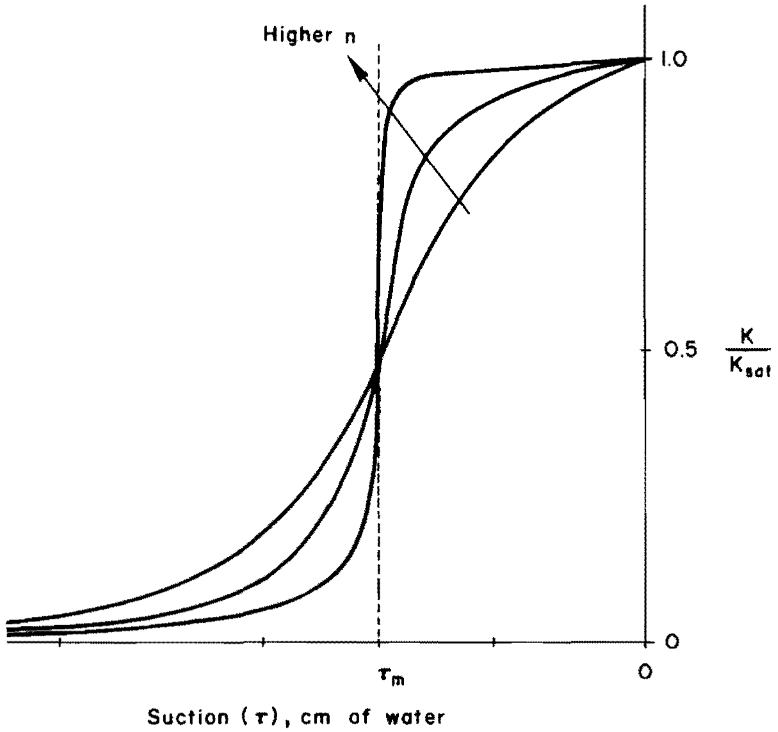


Fig 2. The permeability-suction relation.

One of the most interesting relations is observed in the $\frac{dk}{d\tau}$ curve. This equation is

$$\frac{dk}{d|\tau|} = -k_u \frac{n}{|\tau| + \frac{b}{|\tau|^{n-1}}} \tag{2.3}$$

That is, the derivative of permeability with respect to the absolute value of suction is the product of current permeability and a weighting function. This weighting function is of the form

$$y = \frac{a}{x + \frac{b}{x^{m-1}}} \tag{2.4}$$

Several members of the family of curves generated by varying m are shown in Fig 3. For the purposes of simplification, a and b have been held constant at 1.0. In general, the maximum value of y is

$$y_{max} = \frac{a}{b}\left[b(m - 1) \right]^{\frac{m-1}{m}} \tag{2.5}$$

and it occurs at

$$x = \left[b(m - 1) \right]^{\frac{1}{m}} \tag{2.6}$$

The function y is similar to the weighting factor on the permeability, and it has at least four pertinent characteristics:

(1) The weighting function is zero when x (or $|\tau|$) is zero.

(2) The weighting function reaches a maximum (or minimum) at some finite value of x (or $|\tau|$).

(3) The weighting function approaches zero as x (or $|\tau|$) increases.

(4) As m (or n) increases, there is a more abrupt change in the value of the weighting function near the point where it reaches its maximum.

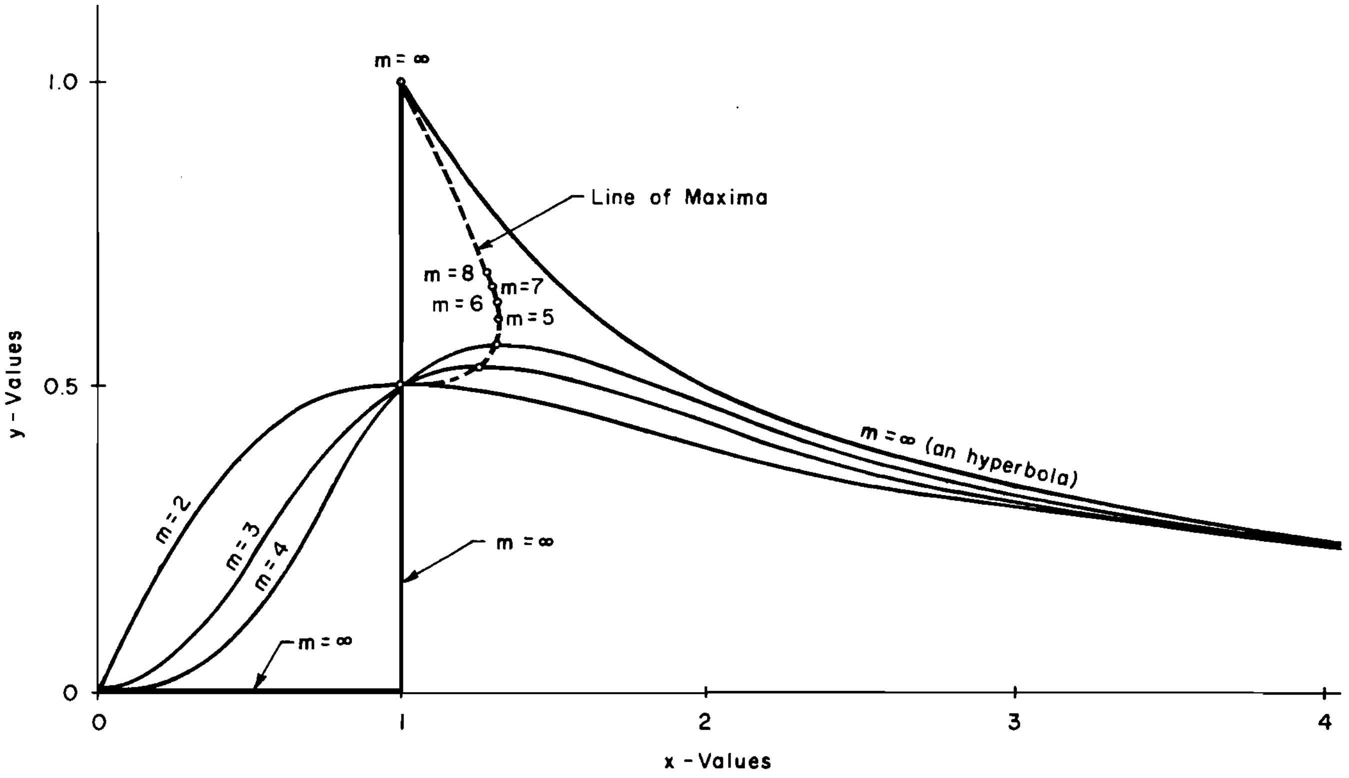


Fig 3. Gardner's weighting function.

The initial part of the weighting function will closely resemble the $\frac{dk}{d\tau}$ curve. Beyond the maximum value of the weighting function $\frac{dk}{d\tau}$ will drop more sharply than the weighting function. The net effect of this is seen in Fig 4.

When clay is nearly saturated and suction is nearly zero, water flows mainly in the liquid phase, adsorptive forces do not inhibit water movement significantly, and the change of permeability with a slight increase of suction is very small. Little geometric rearrangement of particles occurs, the degree of saturation remains relatively constant, and a specified amount of energy gradient will produce an ample flow rate through a specified area of soil.

As will be shown subsequently, the ordinate of the $\frac{dk}{d|\tau|}$ curve gives a measure of the importance of suction in the flow of water through soil. The derivation of Darcy's law with respect to suction will illustrate the point. A constant gradient is assumed and is designated by M , a negative number.

$$k = -\frac{\left(\frac{Q}{A}\right)}{\frac{\partial\tau}{\partial x}} = -\frac{1}{M}\left(\frac{Q}{A}\right) \tag{2.7}$$

$$\frac{\partial k}{\partial|\tau|} = -\frac{1}{M}\frac{\partial}{\partial|\tau|}\left(\frac{Q}{A}\right) \tag{2.8}$$

The shape of the $\frac{dk}{d|\tau|}$ curve is the same as that of the change of flow rate with respect to suction. As the soil becomes less saturated, suction increases in magnitude and the flow rate decreases.

The physical significance of the exponent n is apparent from the $\frac{dk}{d|\tau|}$ curve. In the coarse grained, poorly graded soils, the adsorbed layer is small and has little significance in changing flow rate until the point of minimum $\frac{dk}{d|\tau|}$ is reached. At this level of suction, maximum drainage of pores occurs and the surface tension component of suction has its greatest significance. It is at this point where air pressure and water pressure become equal and occluded bubbles make the transition into pendular water. Other peculiarities occur at this point, and they have been considered in depth by A. B. Walsh-Sparks (Ref 58). The $\frac{dk}{d|\tau|}$ curve for a uniform size, coarse-grained soil would have zero ordinate except for a spike at the pore drainage suction level.

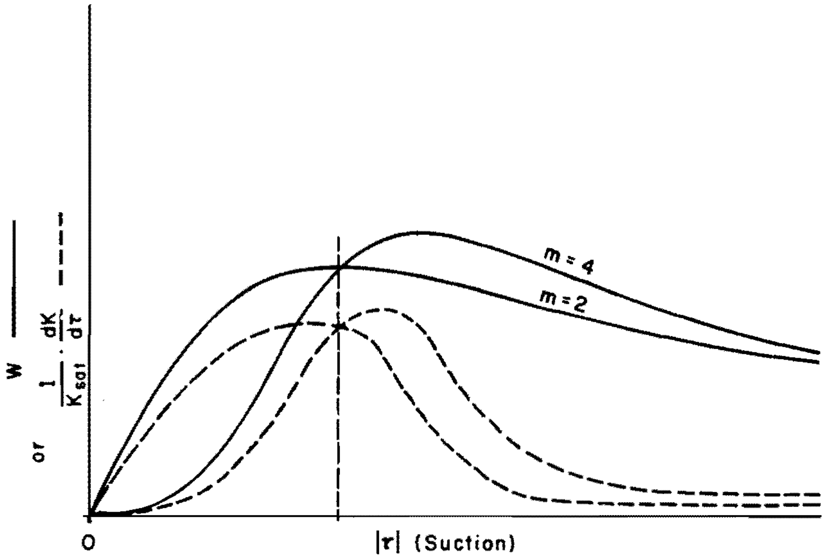


Fig 4. Comparison of Gardner's weighting function with the rate of change of permeability with respect to suction.

In clays or other fine-grained soils and well-graded sands, the transition from occluded bubbles to pendular water is more gradual. This is reflected by a lower n exponent and a smooth transition from saturated to unsaturated flow condition. The lower n indicates two facts: a smooth gradation of grain sizes and a more even influence of soil suction on decreasing flow rate.

Beyond the point where $\frac{dk}{d|\tau|}$ and $\frac{d}{d|\tau|}\left(\frac{Q}{A}\right)$ reach a minimum, water flows mainly in the vapor phase, adsorptive forces begin to predominate, and an increase in the magnitude of suction is less effective in decreasing flow rate.

In this range of saturation, suction is normally defined as a function of the relative vapor pressure of water vapor in the soil, assuming water vapor to be an ideal gas.

$$\tau = \frac{RT_e}{mg} \ln \frac{p}{p_o} \tag{2.9}$$

where

R = universal gas constant,

T_e = absolute temperature,

g = acceleration due to gravity,

m = weight of one molar volume of water (equal to molecular weight in grams),

p = vapor pressure,

p_o = saturated vapor pressure,

and τ is a negative quantity. This formula presumes that there is no interaction among the molecules of water vapor. If there is substantial attractive and repulsive force between water molecules, the suction would have to be found from

$$\tau = \int_{p_o}^{p} V dp \tag{2.10}$$

where V is a volume of water vapor. In order to evaluate the integral, V must be found as a function of p. If Van der Waal's equation of state is presumed valid, then the following equation must be solved for V in terms of p:

$$V^3 - \left(nb + \frac{nRT_e}{p} \right) V^2 + \frac{an^2}{p} V - \frac{n^3ab}{p} = 0 \tag{2.11}$$

where a is related to the attractive force and nb is the volume that cannot be occupied because of repulsion between particles. Assuming no excluded volume in Van der Waal's equation of state (nb = 0) and that only attractive forces act between particles of water vapor, another formula of suction may be derived:

$$\tau = \frac{RT_e}{mg} \left[\frac{1}{2} \ln \frac{p}{p_o} + \frac{1}{2} \ln \frac{[I(p) - 1][I(p_o) + 1]}{[I(p) + 1][I(p_o) - 1]} \right.$$

$$\left. + I(p) - I(p_o) \right] \tag{2.12}$$

where

$$I(p) = \left(1 - \frac{4a}{R^2T_e^2} p \right)^{\frac{1}{2}} \tag{2.13}$$

This formula gives values of suction that are lower than those computed from the ideal gas formula. Thus, part of the explanation of the apparent decreased effectiveness of suction in reducing flow rate is that the suction computed with the ideal gas formula may not represent the actual intensity with which the soil attracts water. A change of the theoretical ideal gas suction may be larger than the actual change of suction.

A more important factor is the fact that in the higher suction ranges, water flows in the vapor phase much more readily than in the liquid or adsorbed phase. Provided a water molecule has sufficient excess energy to escape one adsorbed state, it can travel toward and be adsorbed by water in a more firmly adsorbed state. The probability that a molecule will have sufficient energy to escape the surface of the adsorbed liquid gets smaller in the layers closer to the surface of a clay particle.

These notions are similar to ideas developed in quantum mechanics. It is not entirely surprising, then, to see that the $\frac{dk}{d|\tau|}$ curve has a shape that is greatly similar to the curve of spectral distribution of energy density in black-body radiation first explained by Planck's law which is of the form

$$u = C \frac{E}{e^{\frac{E}{KT_e}} - 1} = CE \frac{1}{-1 + e^{E/KT_e}} \qquad (2.14)$$

where

e = base of natural logarithm,
u = energy density,
C = a constant,
E = a certain quantum of energy,
K = Boltzmann's constant,
T_e = absolute temperature.

This idea is not entirely applicable to soil because of the geometry changes that occur in the soil as a consequence of an increase of suction.

Fluid Properties

It would seem that permeability should be some function of geometry, saturation, unit weight of water, and viscosity. This relation is expressed in all rationally derived permeability formulas as

$$k = \frac{\gamma}{\mu} G$$

where

γ = the unit weight of water,
μ = the viscosity,
G = a collection of geometry and saturation factors.

According to the theory of Eyring (Ref 23), et al, viscosity is given as a property of the energy, entropy, and absolute temperature of the fluid by the following equation:

$$\mu = \frac{hN}{V} e^{\frac{-T_e \Delta S + E}{RT_e}} \tag{2.16}$$

where

h = Planck's constant,

V = molar volume of water,

N = Avogadro's number,

T_e = absolute temperature,

R = universal gas constant,

ΔS = change of entropy when a molecule breaks free from one thermal energy state and enters another (called activation entropy),

E = change of energy required when a molecule breaks free from its surroundings and moves to another (called activation energy).

Note that for a water molecule in soil to break free and travel into pure water requires a potential energy equal to the suction of the soil. The local activation energy in soil will be assumed to be minus the suction τ (a negative quantity) plus τ' which is the suction of the immediate surroundings. Thus, for this development

$$\mu = \frac{hN}{V} e^{\frac{-T_e \Delta S - \tau + \tau'}{RT_e}} \tag{2.17}$$

where τ will vary but τ' will remain constant. The quantity $\tau - \tau'$ must be equal to E before molecular movement takes place. The formula for permeability becomes

$$k = G \frac{\gamma V}{hN} e^{\frac{\tau - \tau' + T_e \Delta S}{RT_e}} \qquad (2.18)$$

Also, when soil is saturated

$$k_{sat} = G_o \frac{\gamma V}{hN} e^{\frac{T_e \Delta S - \tau'}{RT_e}} \qquad (2.19)$$

where G_o is the value of G at saturated conditions. The derivative of k with respect to τ permitting G to vary with τ is

$$\frac{dk}{d\tau} = k \left[\frac{1}{G} \left(\frac{dG}{d\tau} \right) + \frac{1}{RT_e} \right] \qquad (2.20)$$

The corresponding equations in Gardner's relations are as follows:

$$k = \frac{a}{|\tau|^n + b} \qquad (2.21)$$

$$k_{sat} = \frac{a}{b} \qquad (2.22)$$

$$\frac{dk}{d\tau} = - \frac{dk}{d\,|\tau|} = +k \frac{n}{|\tau| + \frac{b}{|\tau|^{n-1}}} \qquad (2.23)$$

Setting equivalent terms equal and solving for the empirical coefficients a , b , and n give

$$a = bG_o \frac{\gamma V}{hN} e^{\frac{T_e \Delta S - \tau'}{RT_e}} \qquad (2.24)$$

$$b = \frac{|\tau|^n}{\left(\frac{G_o}{G}\right) e^{-\frac{H}{RT_e}} - 1} \qquad (2.25)$$

$$n = \frac{a_1}{k_1 RT_e}\left[\frac{RT_e}{G_1}\left(\frac{dG}{d\tau}\right)_1 + 1\right] \qquad (2.26)$$

where a_1 , G_1 , k_1 , and $\left(\frac{dG}{d\tau}\right)_1$ are those quantities evaluated at a suction of -1 cm.

As can be seen from these equations, b and n are the only empirical constants involved in the $\frac{dk}{d\tau}$ weighting function. It is not surprising to find that the expressions for b and a have the same form as Planck's law but have an adjustment for changing geometry, $\frac{G_o}{G}$. This suggests that extraordinary changes of geometry will require b to change as suction changes.

In addition, n may change and the fact that a single value has been used by Gardner and others to describe a permeability-suction relation may be somewhat fortuitous. The general relation for n comes from a solution of the equation

$$\ln n + (n - 1)\ln|\tau| = \ln \frac{a_\tau}{k_\tau RT_e}\left[\frac{RT_e}{G_\tau}\left(\frac{dG}{d\tau}\right)_\tau + 1\right] \qquad (2.27)$$

where a_τ , G_τ , k_τ , and $\left(\frac{dG}{d\tau}\right)_\tau$ are evaluated at the value of suction indicated. In order for n to remain constant, the following relation must hold for all values of τ :

$$1 = \left(\frac{1}{n}\right)^{\frac{1}{n-1}}\left(\frac{a}{kG\,|\tau|^{n-1}}\frac{dG}{d\tau} + \frac{a}{k\,|\tau|^{n-1}RT_e}\right)^{\frac{1}{n-1}} \qquad (2.28)$$

This means that

$$\frac{a}{kG\,|\tau|^{n-1}}\,\frac{dG}{d\tau} + \frac{a}{k\,|\tau|^{n-1}RT_e} \;=\; n \;=\; \text{constant} \qquad (2.29)$$

where n is a positive number. This leads to the differential equation

$$\frac{dG}{d\tau} + \frac{G}{RT_e} - \frac{kG}{a}\,n\,|\tau|^{n-1} \;=\; 0 \qquad (2.30)$$

which may be solved with numerical methods using a marching solution beginning with the initial condition that $G(0)$ is G_o when τ is zero.

If k and a are presumed to be stationary in some small range of τ, the differential equation may be integrated to give

$$G \;=\; G_o e^{-\left[\frac{|\tau|}{RT_e} + \frac{k}{a}|\tau|^n\right]} \qquad (2.31)$$

which is obtained recognizing that $d\tau = -d\,|\tau|$. Some experimental evidence has indicated that n for some compressible clays is 2.0. Thus, in these cases

$$G \;=\; G_o e^{-\left[\frac{|\tau|}{RT_e} + \frac{k\tau^2}{a}\right]} \qquad (2.32)$$

These equations show that for a constant value of n and a locally stationary value of k and a , the geometry and saturation terms vary exponentially with a power of the suction. Some typical geometry and saturation terms are given below:

$$G \;=\; \frac{(S - S_o)^3(1 - c)^3}{c^2} \qquad \text{(Irmay (Ref 28)} \quad (2.33)$$

$$G \;=\; \xi(c)\,\frac{(1 - c)^2}{c}\,\frac{d^2}{B}\,e^{-kc} \qquad \text{(Steinour)(Ref 55)} \quad (2.34)$$

$$G = \xi(c)\,\frac{(1-c)^2}{c}\,\frac{d^2}{B}\,e^{-\left[\left(\frac{\beta}{T_e}+\gamma\right)\frac{c}{1-c}\right]} \quad \text{(Powers, et al)(Ref 44)} \tag{2.35}$$

where

c = the volume concentration of solids and is equal to 1 - (the porosity n),

S = the degree of saturation,

S_o = a threshold saturation below which flow does not occur,

d = an effective diameter of soil grains,

$\xi(c)$ = a shape factor to account for deviation of particle shapes from that of a sphere, and

A, B, k, β, γ = constants.

The first equation was derived for use with sands and covers a wide range of saturation and permeability with small or negligible changes in geometry. The latter two equations were developed for the saturated condition in which both k and a would be stationary. In these cases the similarity of form between Eq 2.31 and these equations is striking. The last equation is used by the Portland Cement Association in describing the permeability of hardened cement paste. The work of Powers, et al may be extended to the case of other types of materials with stationary permeability values by equating their constants β and γ to the following:

$$\beta = \frac{|\tau|}{R}\left(\frac{1-c}{c}\right) \tag{2.36}$$

$$\gamma = \frac{k}{a}\,|\tau|^n\left(\frac{1-c}{c}\right) \tag{2.37}$$

The more general case of a material that changes volume, saturation, and permeability simultaneously but which maintains a constant value of n can be described by the differential equation in Eq 2.30, which can be solved numerically once the value of n is known.

It is conceivable that n could vary with suction. In this case Gardner's relation is invalid and some other mathematical model must be proposed. The relations given in this analysis will permit the rational construction of a more complex model which accounts for the effect of a changing n .

The broad generality that may be achieved with Gardner's model, coupled with the physical understanding it can convey when interpreted from a rate process point of view, makes it a very useful empirical concept which future work on permeability may fruitfully explore.

Geometry Factors

Most formulas derived for specific permeability assume a certain distribution of pore sizes and a tortuosity of paths taken by water particles.

Several elementary models have been proposed; all have a certain degree of validity and none is generally applicable. A list of such models and writers who have proposed typical equations for these models is given below:

Ordered Models

Flow through Straight Capillary - Kruger - 1918
Flow through Parallel Capillary - Purcell - 1949
Flow through Serial Capillary - Scheidegger - 1953
Hydraulic Radius of Collection of Channels - Kozeny - 1927
Drag on Elements of Medium - Emersleben - 1924, 1925

Disordered Models

Average Particle Motion Parallel to Head Gradient - Hubbert - 1940
Pore-Size Distribution - Probability of any Two Pore Sizes Occurring in Series - Childs and Collis-George - 1950

Reference to these authors is given in Scheidegger's (Ref 52) book.

More advanced work has been done on the concept of permeability, which, in effect, constitutes a revision of Darcy's law. Scheidegger (Ref 51) uses a probability distribution to describe the possible location of a particle of water which has moved through a statistically homogeneous medium for a period of time. The probable positions of this particle fall into a Gaussian distribution pattern. Thus, a collection of water particles starting out together would disperse with time as they move through the medium.

The amount of dispersion will depend on the "dispersivity" of the medium and, of course, the properties of the liquid itself. Bear (Ref 6) investigated the flow of tracer fluid through homogeneous media and found that the normal distribution of the tracer (a second rank tensor) is related to the displacement tensor by a fourth rank tensor of dispersivity of the medium. Scheidegger (Ref 50) extended the treatment to the case of an anisotropic medium and found that there are 36 independent dispersivity constants of a completely anisotropic porous medium. As a contrast there are 21 elastic constants to be found for a triclinic elastic crystal.

Thus, the concepts of permeability can be made as mathematically sophisticated as is felt necessary and they all have one aim in common: to approximate the actual behavior of porous media. The adequacy of the model to be used will depend upon the accuracy required of the computing process. For example, if it is required to calculate the dispersion of salt tracer in porous medium to match carefully controlled laboratory experiments, then the tensor form of dispersivity might be used. If, on the other hand, mass movement of water through soil is to be predicted, then some less precise form of permeability might be adapted. The latter approach is taken in this development.

Permeability as a Tensor

The idea of a tensor was conceived when mathematicians realized that valid physical laws must be independent of any particular coordinate systems used in describing them mathematically. Thus, in discussing the tensor concept of permeability, an attempt is made to discover those properties which are independent of coordinate systems to which they are referred.

Scheidegger (Ref 52) makes two conclusions about permeability in anisotropic porous media:

(1) In general, the force potential gradient and filter velocity are not parallel.

(2) There are three orthogonal axes in space along which the force potential gradient and the velocity do have the same direction. These axes are termed the "principal axes" of the permeability tensor.

Ferrandon (Ref 20) described anisotropic permeability as a second order tensor with N^2 elements where N is the number of dimensions being considered and 2 is the order of the tensor. In matrix form, the equations for the three-dimensional case are written as follows:

$$\begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = - \begin{bmatrix} k_{xx} & k_{xy} & k_{xz} \\ k_{yx} & k_{yy} & k_{yz} \\ k_{zx} & k_{zy} & k_{zz} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \\ \frac{\partial H}{\partial z} \end{bmatrix} \tag{2.38}$$

where

v_x = the velocity in the x-direction,

$\frac{\partial H}{\partial x}$ = the force potential head gradient in the x-direction,

k_{xy} = the proportion of v_x contributed by $\frac{\partial H}{\partial y}$. This is a cross-permeability factor.

The three by three matrix of permeabilities is the most general linear relation between velocity and head gradient that can be written unless a reversal of direction of head gradient component causes a different velocity contribution. Experimental results indicate that this is not the case and thus the three by three matrix is judged to be sufficient.

Attempts have been made to show the permeability tensor to be symmetric by an appeal to the principle of the conservation of energy of microscopic, irreversible thermodynamic processes. This is called Onsager's (Ref 40) principle. The fact that flow through porous media is fundamentally an energy-losing process makes this "proof" seem intuitively ill-founded.

Another approach is to assume that three orthogonal axes of principal permeability exist. Experimental evidence appears to support this assumption. Once this assumption is made, a derivation such as shown below will establish that the permeability tensor is symmetric based on purely geometric considerations. This development is made in two dimensions to maintain simplicity although the technique is valid in three dimensions.

Referring to Fig 5 it is evident that the components of head gradient $\frac{\partial H}{\partial n}$ in the directions of the principal permeabilities k_1 and k_2 are

$$\frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial n} \cos (n,1) \tag{2.39}$$

$$\frac{\partial H}{\partial x_2} = \frac{\partial H}{\partial n} \cos (n,2) \tag{2.40}$$

Velocities in these directions are

$$v_1 = -k_1 \frac{\partial H}{\partial x_1} \tag{2.41}$$

$$v_2 = -k_2 \frac{\partial H}{\partial x_2} \tag{2.42}$$

Velocity components in the x-y coordinate system are

$$v_x = v_1 \cos (x,1) + v_2 \cos (x,2) \tag{2.43}$$

$$v_y = v_1 \cos (y,1) + v_2 \cos (y,2) \tag{2.44}$$

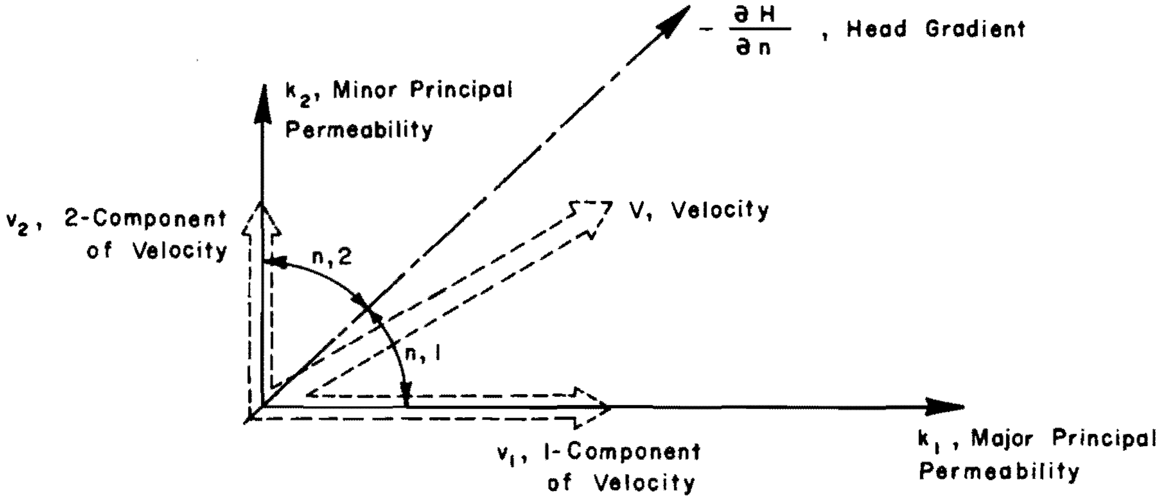
Similarly, the components of head gradient are

$$\frac{\partial H}{\partial x_1} = \frac{\partial H}{\partial x} \cos (x,1) + \frac{\partial H}{\partial y} \cos (y,1) \tag{2.45}$$

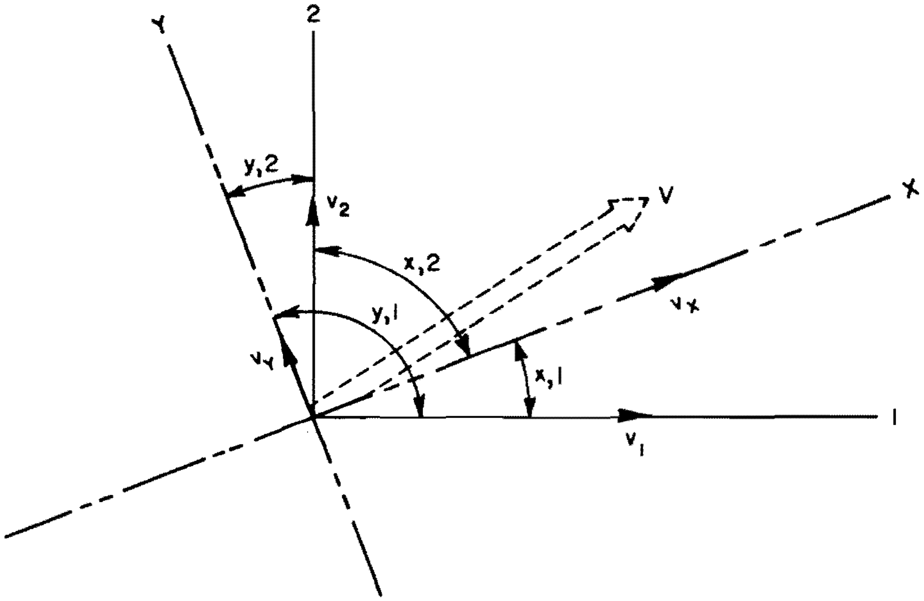
$$\frac{\partial H}{\partial x_2} = \frac{\partial H}{\partial x} \cos (x,2) + \frac{\partial H}{\partial y} \cos (y,2) \tag{2.46}$$

Thus, we see that

$$v_x = -k_1 \frac{\partial H}{\partial x_1} \cos (x,1) - k_2 \frac{\partial H}{\partial x_2} \cos (x,2) \tag{2.47}$$



(a) Anisotropic permeability, gradient, and velocity.



(b) Velocity components.

Fig 5. Components of head gradient.

$$v_y = -k_1 \frac{\partial H}{\partial x_1} \cos (y,1) - k_2 \frac{\partial H}{\partial x_2} \cos (y,2) \tag{2.48}$$

Substitution of the head gradient transformation gives the following equation:

$$v_x = -k_1 \frac{\partial H}{\partial x} \cos^2 (x,1) - k_1 \frac{\partial H}{\partial y} \cos (x,1) \cos (y,1)$$

$$-k_2 \frac{\partial H}{\partial x} \cos^2 (x,2) - k_2 \frac{\partial H}{\partial y} \cos (x,2) \cos (y,2) \tag{2.49}$$

The velocity in the y-direction is given by

$$v_y = -k_1 \frac{\partial H}{\partial x} \cos (x,1) \cos (y,1) - k_1 \frac{\partial H}{\partial y} \cos^2 (y,1)$$

$$-k_2 \frac{\partial H}{\partial x} \cos (x,2) \cos (y,2) - k_2 \frac{\partial H}{\partial y} \cos^2 (y,2) \tag{2.50}$$

As can be seen from the equations just derived, the velocity in the x-direction receives contributions from the head gradient in the y-direction and conversely for the y-direction velocity. If the velocity is defined as

$$v_x = -k_{xx} \frac{\partial H}{\partial x} - k_{xy} \frac{\partial H}{\partial y} \tag{2.51}$$

$$v_y = -k_{yx} \frac{\partial H}{\partial x} - k_{yy} \frac{\partial H}{\partial y} \tag{2.52}$$

then it is evident that

$$k_{xx} = k_1 \cos^2 (x,1) + k_2 \cos^2 (x,2) \tag{2.53}$$

$$k_{xy} = k_1 \cos (x,1) \cos (y,1) + k_2 \cos (x,2) \cos (y,2) \tag{2.54}$$

$$k_{yx} = k_1 \cos(x,1)\cos(y,1) + k_2 \cos(x,2)\cos(y,2) \tag{2.55}$$

$$k_{yy} = k_1 \cos^2(y,1) + k_2 \cos^2(y,2) \tag{2.56}$$

Term for term comparison of the expressions for k_{xy} and k_{yx} shows that the terms are identical. Since these are the only off-diagonal terms in the two-dimensional permeability tensor, the tensor is thus shown to be symmetric. Only geometric relations have been used to prove this property of permeability.

The result can be extended to the general case in three dimensions, and it is found that the permeability tensor transforms in the same way that the stress tensor transforms.

The analogy between stress and permeability can be extended to an investigation of the quantities which do not vary with rotation of axes. As in the stress tensor, three invariants of the permeability tensor can be derived.

$$K_1 = k_{11} + k_{22} + k_{33} \tag{2.57}$$

$$K_2 = -(k_{11}k_{22} + k_{22}k_{33} + k_{33}k_{11} - k_{12}^2 - k_{23}^2 - k_{13}^2) \tag{2.58}$$

$$K_3 = (k_{11}k_{22}k_{33} - k_{11}k_{23}^2 - k_{22}k_{13}^2 - k_{33}k_{12}^2 + 2k_{12}k_{23}k_{13}) \tag{2.59}$$

These same quantities can be expressed in terms of the principal permeabilities as

$$K_1 = k_1 + k_2 + k_3 \tag{2.60}$$

$$K_2 = -(k_1k_2 + k_2k_3 + k_3k_1) \tag{2.61}$$

$$K_3 = k_1k_2k_3 \tag{2.62}$$

These suggest a treatment similar to that of the stress tensor of defining a mean permeability, a mean cross-permeability, and a cross-permeability distribution factor. The mean permeability is

$$k = \frac{k_1 + k_2 + k_3}{3} = \frac{1}{3} K_1 \tag{2.63}$$

The mean cross-permeability is

$$c = \frac{1}{3} \left[(k_1 - k_2)^2 + (k_2 - k_3)^2 + (k_3 - k_1)^2 \right]^{\frac{1}{2}} \tag{2.64}$$

or

$$c = \frac{1}{3} \left[2(K_1^2 + 3K_2) \right]^{\frac{1}{2}} \tag{2.65}$$

The cross-permeability distribution factor is

$$\mu = - \frac{n(9 - n^2)}{(3 + n^2)^{\frac{3}{2}}} = \sqrt{2}\, \frac{(k_1 - k)(k_2 - k)(k_3 - k)}{c^3} \tag{2.66}$$

where n is as defined in Eq 2.67 indicates the size of k_2 with respect to k_1 and k_3 .

$$k_2 = \frac{k_1 + k_3}{2} + n\, \frac{k_1 - k_3}{2} \tag{2.67}$$

This is a quantity that is analogous to the shearing stress distribution factor proposed by Juarez-Badillo (Ref 30).

Table 1 gives values for n and μ for selected values of k_2 . Figure 6 shows a graph of n versus μ . The directions of the three principal

TABLE 1. VALUES FOR k_2 , n , AND μ

k_2	n	μ
k_1	1	-1
$\frac{k_1 + k_3}{2}$	0	0
k_3	-1	1

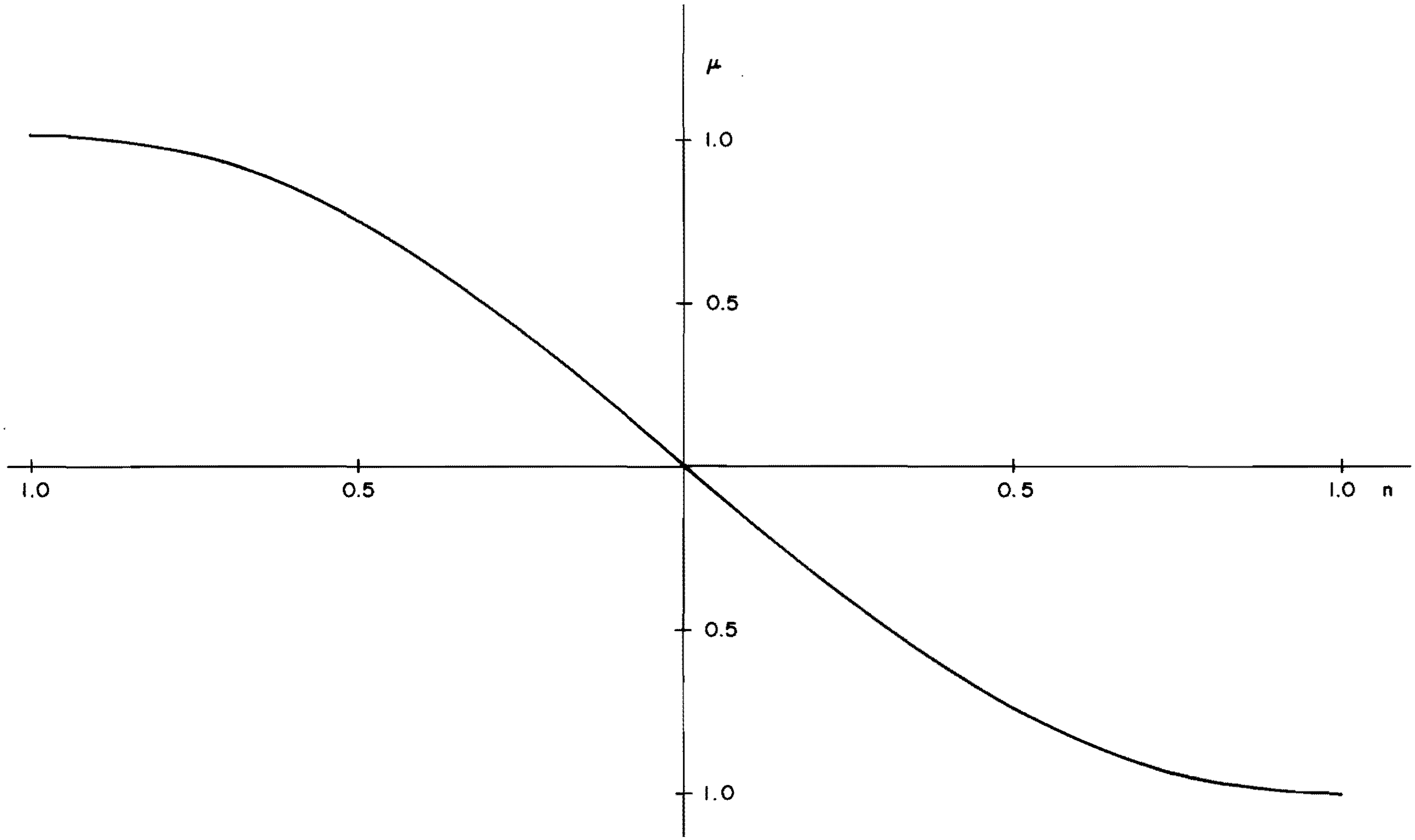


Fig 6. Intermediate principal permeability factor n versus cross-permeability distribution factor μ .

permeabilities and the values of the three derived invariants k, c, and μ will uniquely determine the permeability at a point.

In addition to the tensor properties of permeability, a graphical representation of permeability at a point is taken from the analogy with the stress tensor. A Mohr's circle of permeability may be drawn with direct permeabilities as abscissas and cross-permeabilities as ordinates. The largest principal permeability is termed k_1 and the smallest principal permeability is k_3 with k_2 having the intermediate value. Typical Mohr's circles of permeability are shown in Figs 7(a) and 7(b). As is seen in these figures, cross-permeability terms may be either positive or negative. For flow in two dimensions, with the positive x-direction to the right and the positive y-direction upward, a positive cross-permeability term indicates that the velocity vector is clockwise from the head gradient.

In two dimensions, the formulas for direct and cross-permeability are as follows:

$$k_{11} = \frac{k_1 + k_3}{2} + \frac{k_1 - k_3}{2} \cos 2\alpha \qquad (2.68)$$

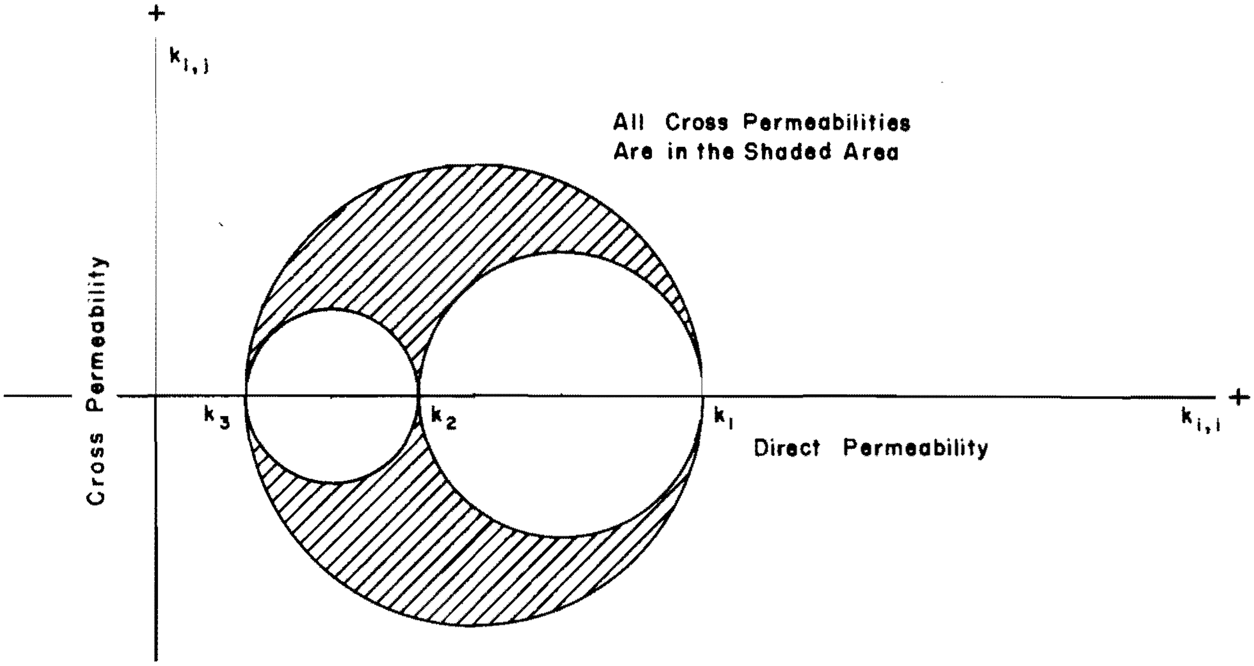
$$k_{13} = k_{31} = - \frac{k_1 - k_3}{2} \sin 2\alpha \qquad (2.69)$$

$$k_{33} = \frac{k_1 + k_3}{2} - \frac{k_1 - k_3}{2} \cos 2\alpha \qquad (2.70)$$

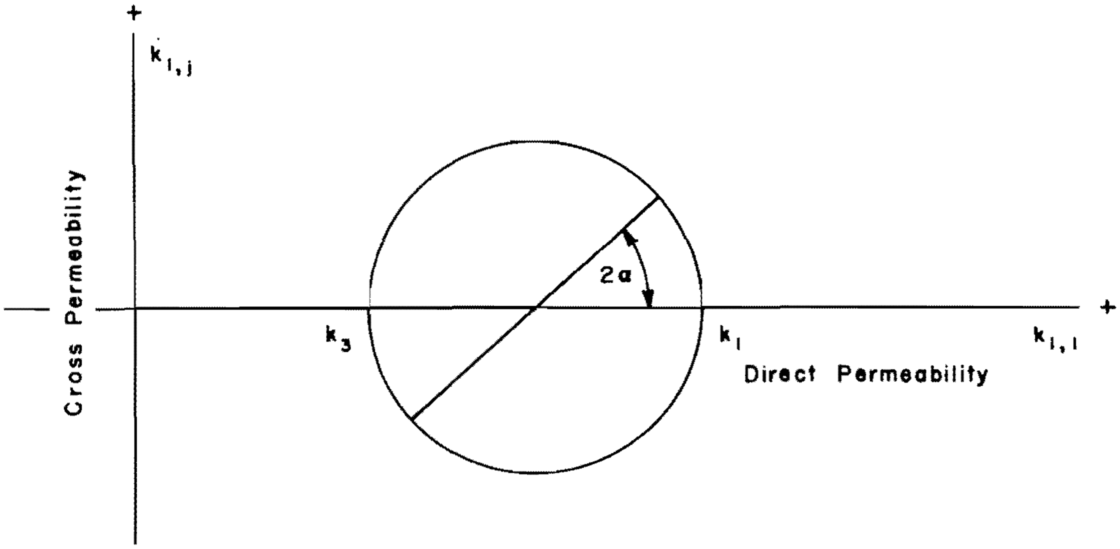
which can be derived from Eqs 2.53 through 2.56 by trigonometric manipulation.

The transformation equations for three-dimensional permeability may be generated from Table 2. For example,

$$\begin{aligned} k_{xx} = {} & k_{11} \cos^2 (1,x) + k_{22} \cos^2 (2,x) + k_{33} \cos^2 (3,x) \\ & + 2k_{12} \cos (1,x) \cos (2,x) + 2k_{23} \cos (2,x) \cos (3,x) \\ & + 2k_{13} \cos (1,x) \cos (3,x) \end{aligned} \qquad (2.71)$$



a. Three-dimensional Mohr's permeability circle.



b. Two-dimensional Mohr's permeability circle.

Fig 7. Typical Mohr's circles of permeability.

TABLE 2. TABLE FOR TRANSFORMATION OF COORDINATES

	k_1	k_2	k_3
k_x	cos (1,x)	cos (2,x)	cos (3,x)
k_y	cos (1,y)	cos (2,y)	cos (3,y)
k_z	cos (1,z)	cos (2,z)	cos (3,z)

The same pattern may be followed in deriving expressions for all other components of the permeability tensor.

These properties of permeability are related to the geometrical properties of the porous medium. For example, the mean permeability factor k is related to the average porosity of the medium. The mean cross-permeability would indicate an average of the directional deviations from the average porosity. If particles are all shaped the same, c would be very small. Thus, c is probably related to a particle shape and orientation factor. The cross-permeability distribution factor indicates the degree of deviation of porosity in a particular direction from the mean and average deviation values. In a perfectly homogeneous medium, this factor becomes $\frac{0}{0}$, undefined. But in anisotropic soils, μ will be a function of directional gradation and tortuosity.

Effect of Geometrical Properties on Permeability

The following approximate study shows the effect of the geometrical structure of an idealized soil on the permeability properties of the medium. Two geometrical structures are considered: a dispersed and a flocculated arrangement of particles. These are shown in Figs 9 and 10. In the approximate study involving these geometrical arrangements, only the water phase is considered to be flowing. In addition, the validity of the form of Poiseuille's equation is assumed:

$$Q_i = u_i A_i = A_i \frac{R_i^2}{m\mu} \frac{dP}{dx_i} \qquad i = 1,2,3 \qquad (2.72)$$

where

Q_i = discharge rate in the i^{th} direction,

u_i = average water velocity in the i^{th} direction,

A_i = area open for flow in the i^{th} direction,

R_i = hydraulic radius in the i^{th} direction,

μ = viscosity,

m = a constant; for round tubes $m = 2$,

$\frac{dP}{dx_i}$ = pressure gradient in the i^{th} direction.

If a pressure gradient exists in the n direction, then

$$\frac{dP}{dx_i} = \frac{dP}{dn} \cos (n,i) \tag{2.73}$$

where

$\cos (n,i)$ = the cosine of the angle between n and i .

The discharge velocity is measured in terms of the overall area rather than the open flow area and thus

$$v_i = \frac{Q_i}{A_{T_i}} = u_i \frac{A_i}{A_{T_i}} \tag{2.74}$$

Because the hydraulic radius is the ratio of area to wetted perimeter, Eq 2.74 becomes

$$v_i = \frac{A_i^3}{A_{T_i} WP_i^2} \cos (n,i) \frac{1}{m\mu} \frac{dP}{dn} \tag{2.75}$$

where

A_{T_i} = total area in the i^{th} direction,

v_i = discharge velocity in the i^{th} direction.

The approximate study shows values of velocity which are proportional to $\frac{1}{m\mu}$ and is based on assuming a unit pressure gradient in the n-direction. The velocity along the pressure gradient is given by

$$v_n = v_1 c_{n1} + v_2 c_{n2} + v_3 c_{n3} \tag{2.76}$$

where

v_1 = velocity in the 1-direction,

c_{n1} = cosine of the angle between n and 1 ,

and the remainder follow the same pattern. The tangential velocity component v_t is that component of resulting velocity at right angles to the plane containing the 1-direction and the pressure gradient $\frac{dP}{dn}$. The formula for the tangential velocity is

$$v_t = \frac{c_{n2}}{s} v_3 + \frac{c_{n3}}{s} v_2 \tag{2.77}$$

where

$$s = \sqrt{c_{n2}^2 + c_{n3}^2} \tag{2.78}$$

The perpendicular velocity component v_p is at right angles with both the pressure gradient $\frac{dP}{dn}$ and the tangential direction and is given by the following formula:

$$v_p = v_1 s - v_2 \frac{c_{n2} c_{n1}}{s} - v_3 \frac{c_{n3} c_{n1}}{s} \tag{2.79}$$

An illustration of these velocity components is given in Fig 8. When the pressure gradient is directed along one of the principal permeability directions, v_t and v_p are both zero. Figures 9 and 10 give relative values of v_1 , v_2 , v_3 , and v_n , v_t , and v_p for a pressure gradient with direction cosines $(1/\sqrt{3}$, $1/\sqrt{3}$, and $1/\sqrt{3})$ from the 1, 2, and 3 principal directions for dispersed and flocculated soil structures.

Relative sizes of soil particles are used. Particles with side ratios of 1:1:0.02 to 1:1:0.5 represent kaolinite particles. Particles with side ratios of 1:1:0.003 represent montmorillonite particles.

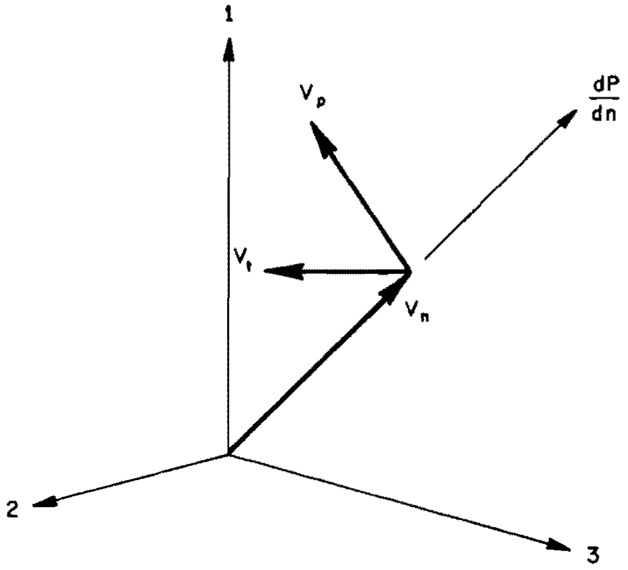
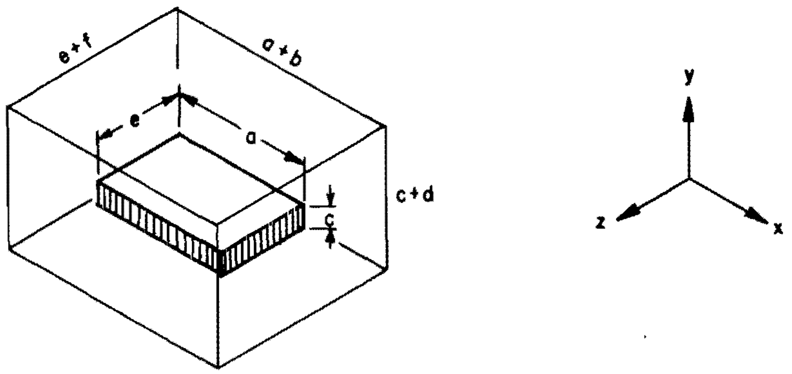


Fig 8. Velocity components parallel, tangential, and perpendicular to head gradient direction.



a	b	c	d	e	f	v_x	v_y	v_z	v_n	v_t	v_p
1	1	.02	.5	1	1	.245	.422	.245	.304	.072	-.042
1	1	.02	.1	1	1	.011	.422	.011	.148	.168	-.097
1	1	.03	.5	1	1	.243	.422	.242	.303	.073	-.042
1	1	.03	.1	1	1	.011	.422	.011	.148	.168	-.097
1	1	.05	.5	1	1	.239	.422	.239	.300	.075	-.043
1	1	.05	.1	1	1	.012	.422	.012	.149	.167	-.097
1	1	.003	.05	1	1	.0026	.422	.0026	.142	.171	-.098
1	1	.003	.01	1	1	.0001	.422	.0001	.141	.172	-.099

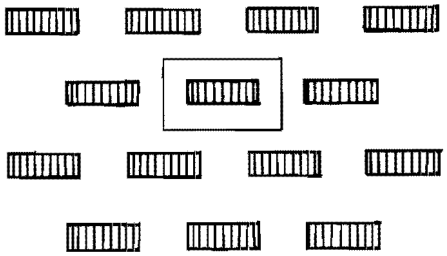
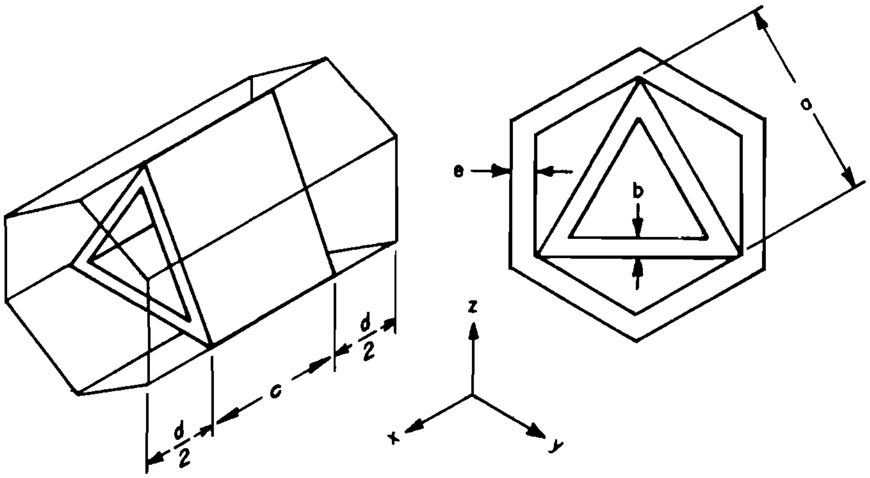


Fig 9. Relative discharge velocities
for dispersed structure.



a	b	c	d	e	v_x	v_y	v_z	v_n	v_t	v_p
1	.003	1	.1	.05	.0894	.0186	.0005	.0362	.0074	.0377
1	.003	1	.1	0.00	.0511	.0093	.0001	.0202	.0038	.0219
1	.16	1	.2	.1	.0817	.0515	.0037	.0456	.0195	.0255
1	.16	1	.2	0.00	.0102	.0181	.0004	.0096	.0072	.0004
1	.50	1	.2	.1	4.251	.0515	.0037	1.436	.0195	1.991
1	.50	1	.2	0.00	.9677	.0181	.0004	.3287	.0072	.4518

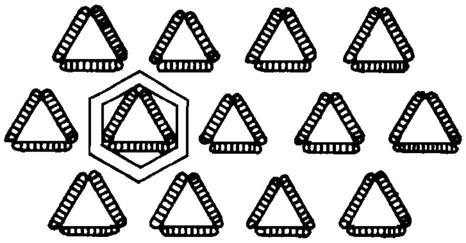


Fig 10. Relative discharge velocities for flocculated structure.

It is significant to note that when a head gradient is directed away from the principle directions, flow tends to move toward the larger opening.

Multi-Phase Unsaturated Permeability

Soil is permeable to water in all of its phases, liquid, vapor, and adsorbed. The rate of movement of water in each phase may be linearly related to the total potential energy head gradient applied to a length of soil. This is essentially what was shown by Philip (Ref 41). The total flow of water through a unit area in a unit time may thus be expressed in tensor form as

$$v_i = -(k_\ell + k_v + k_a)_{ij} \frac{\partial H}{\partial x_j} \qquad i,j = 1,2,3 \qquad (2.80)$$

where

k_ℓ = liquid phase permeability,

k_v = vapor phase permeability,

k_a = adsorbed phase permeability,

H = a total potential energy head.

The sum of the terms in the parentheses is a single number k, the gross permeability of the soil to the passage of water. All of the assumptions inherent in the development of the permeability tensor idea are retained in this concept of a multi-phase permeability. Because it is virtually impossible to make experimental measurements of the permeabilities of the different phases, it is the gross permeability which is normally reported in current literature regarding suction-induced water flow in clay soils.

De Wet (Ref 16) used the theory of mixtures to describe the three-dimensional moisture movement of water in the liquid and vapor phases through an isotropic porous medium under the influence of temperature and energy gradients.

The theory of mixtures is a recently developed technique for studying the properties of mixtures. Clay is a mixture composed of clay mineral, water, and a gas which is itself a mixture of air and water vapor. Tensor notation is used in the theory of mixtures as a shorthand convenience. For example, if ϕ

has three orthogonal components ϕ_1 , ϕ_2 , and ϕ_3 , then the divergence of ϕ is

$$\frac{\partial\phi_1}{\partial x_1} + \frac{\partial\phi_2}{\partial x_2} + \frac{\partial\phi_3}{\partial x_3} = \frac{\partial\phi_i}{\partial x_i}$$

The double i indicates a summation. This term is shortened further by the use of a comma to indicate a derivative with respect to the i direction:

$$\frac{\partial\phi_i}{\partial x_i} = \phi_{i,i} \tag{2.81}$$

This notation will be used in the discussion that follows.

De Wet's relation for moisture movement in an isotropic porous medium is

$$\rho\dot{c}_m = \left[kT \left(\frac{\mu_1}{T_e} \right)_{,j} \right]_{,k} - \left[c \, \frac{T_{e,j}}{T_e} \right]_{,k} \tag{2.82}$$

where

ρ = mass density of the mixture,

$\dot{c}_m$ = the time rate of change of the mass density concentration of moisture,

μ_1 = energy of water phase,

$\left(\frac{\mu_1}{T_e} \right)_{,j}$ = gradient of $\frac{\mu_1}{T_e}$,

T_e = absolute temperature,

$T_{e,j}$ = temperature gradient,

k = a permeability,

c = a thermo-osmotic diffusivity.

The mass density of water is related to the mixture mass density by the equation

$$\frac{\rho_1}{\rho} = c_1 \qquad (2.83)$$

where

ρ_1 = the mass density of the water,

c_1 = the mass concentration of water.

This work may be extended to the case of an anisotropic porous medium by considering the following phenomenological equations:

$${}_a\rho\,{}_a u_k = -{}_a b_{kj} T_e \left(\frac{{}_a\mu}{T_e}\right)_{,j} + {}_a d_{kj}\left(q_j \pm \Sigma\,{}_a\rho\,{}_a\epsilon\,{}_a u_j\right) \qquad (2.84)$$

$$\frac{T_{e,k}}{T_e} = {}_a f_{kj} T_e \left(\frac{{}_a\mu}{T_e}\right)_{,j} + {}_a g_{kj}\left(q_j \pm \Sigma\,{}_a\rho\,{}_a\epsilon\,{}_a u_j\right) \qquad (2.85)$$

where

q_j = the heat flux at the surface,

${}_a\epsilon$ = the internal energy of the a^{th} component of pore fluid,

${}_a u_k, {}_a u_j$ = the velocity component of the a^{th} component of pore fluid relative to the mineral structure in the k or the j direction,

${}_a\mu$ = a total energy concentration in the a^{th} component of pore fluid, e.g., the sum of hydrostatic pressure head and chemical concentration in equivalent units.

In these equations the phenomenological tensors ${}_a b_{kj}$, ${}_a d_{kj}$, ${}_a f_{kj}$, and ${}_a g_{kj}$ relate the forces (energy gradients, heat flux, and internal energy flux) to the induced fluxes of temperature and fluid flow.

These equations may be manipulated to give the formula for mass flow of the a^{th} component fluid, a vector quantity.

$$\underset{a}{\rho}\underset{a}{u}_k = -\underset{a}{k}_{kj} T_e \left(\frac{\underset{a}{\mu}}{T_e} \right)_{,j} + \underset{a}{c}_{kj} \frac{T_{e,j}}{T_e} \tag{2.86}$$

The derivative of each velocity component in its own direction gives an equation of the following form:

$$-\left(\underset{a}{\rho}\underset{a}{u}_k \right)_{,k} = +\left[\underset{a}{k}_{kj} T_e \left(\frac{\underset{a}{\mu}}{T_e} \right)_{,j} \right]_{,k} - \left[\underset{a}{c}_{kj} \frac{T_{e,j}}{T_e} \right]_{,k} \tag{2.87}$$

The continuity equation for a mixture which reacts is

$$\rho \left(\underset{a}{\dot{c}} - \underset{a}{c} \right) = -\left(\underset{a}{\rho}\underset{a}{u}_k \right)_{,k} \tag{2.88}$$

where

$\underset{a}{\dot{c}}$ = the rate of change of mass density concentration of the a^{th} component of pore fluid,

$\underset{a}{c}$ = the rate of reaction of the a^{th} component of pore fluid.

Adding all a equations together and assuming that the sum of all reaction rates is zero gives

$$\rho \dot{c}_m = \left[K_{kj} T_e \left(\frac{\underset{1}{\mu}}{T_e} \right)_{,j} \right]_{,k} - \left[C_{kj} \frac{T_{e,j}}{T_e} \right]_{,k} + \left[\underset{1}{\mu} \sum_{\substack{a \\ \neq 1}} \underset{a}{K}_{kj} \underset{a}{\phi}_{,j} \right] \tag{2.89}$$

where

$\dot{c}_m$ = the time rate of change of the mass concentration of moisture,

$\underset{1}{\mu}$ = energy of water phase, and the following assumptions are made:

$$K_{kj} = K_{1kj} + \phi_2 K_{2kj} + \phi_3 K_{3kj} \tag{2.90}$$

$$\mu_a = \phi_a \mu_1 \tag{2.91}$$

The subscripts 1, 2, and 3 refer to water, air (including water vapor), and adsorbed pore fluid. The quantities ϕ are taken as functions; they may be constants in which case their gradients will be zero. This is not normally the case, however.

Equations such as these are based on physical fundamentals and serve to point out the assumptions that are made in assuming a certain form of permeability relation. For example, in Eq 2.89, it is apparent that in the absence of a temperature gradient, the total energy gradient in the water phase may not be sufficient to determine the rate of moisture change in soil. The functions ϕ_2 and ϕ_3 must also be constant for this condition to be valid. If they are incorrectly assumed to be constant, then some experimental error may be ascribed to the inadequacy of that assumption.

Investigation of these relations will be helpful in interpreting the results of experiments as well as in guiding the planning of experimental programs.

Summary of Permeability

Permeability of clay is determined by the properties of pore fluids and the geometric properties of the porous medium. Pore fluids include water in an adsorbed phase, a liquid phase, and a vapor phase, as well as air. The transfer of moisture is a rate process which depends upon the level of suction, the current geometrical arrangement of solids, and the size of the energy gradients causing the moisture movement.

Geometrical arrangement of the solids is likely to be anisotropic, requiring the macroscopic geometric properties of a porous medium to be expressed as a second-order tensor which is symmetric if there are three orthogonal principal permeabilities. These permeabilities may be better related to the properties of the medium by using the factors k , c , and μ which are functions of the three invariants of the permeability tensor. The factor k would be a function of porosity, c would correlate with a particle shape factor, and μ would be a function of directional tortuosity.

Unsaturated permeability is a function of many factors, including temperature gradients, and gradients of functions relating water phase energy to the energy contained in other water phases.

Interpretation of Gardner's empirical permeability formula in terms of changing geometry, level of suction, and activation energy reveals the fundamental reason for its success in modeling permeability-suction relations. Furthermore, Gardner's relation implies a permeability formula which includes as a special case an expression used by the Portland Cement Association to describe the permeability of hardened cement paste. The result is general enough to include many of the important factors determining the permeability of clays.

CHAPTER 3. SOIL SUCTION AND TOTAL POTENTIAL OF SOIL WATER

Soil suction is a macroscopic property of soil which indicates the intensity with which a soil sample will attract water. Suction is normally defined as a negative gage pressure, is generally expressed in inches or centimeters of water, and is not to be confused with pore water pressure which is a component of suction. Pore water pressure is normally associated with the density of liquid, distance from a free surface, and surface tension forces.

Suction results from the interplay of attraction and repulsion forces of charged clay particles and polar water molecules, together with surface tension forces in water, solution potentials due to dissolved ions, and gravity potential. The representation of suction, the sum of all of these forces, as an equivalent height of water has been called the capillary model. This model was a controversial subject until 1960 at the London Conference on Pore Pressures and Suction in Soils where substantial agreement was finally reached. At this conference, Aitchison (Ref 1) carefully defined the range of validity of the model and concluded that it is a useful concept over a very wide range of suction pressures. Since that time, there has been much less hesitation to use the capillary model. Aitchison's reservations should still be noted in any derivation using the capillary model since they define in physical terms the local conditions under which tension in pore water cannot exist. Briefly, these conditions are as follows:

(1) when tension in the soil water is sufficient to drain the pore,

(2) when the tension in the soil water does not exceed the pressure of the soil air, and

(3) when the pore is of such small dimensions that the attraction of the water molecules to the soil surface produces a pressure that is greater than the tension in the soil water.

Terminology is very important in this discussion. There is a difference between tension in pore water and suction in the water. Tension applies to the actual pressure state of the pore water; suction is a total head term which includes pore water pressure, osmotic pressure, and adsorptive pressure as components.

The International Society of Soil Science has given definitions of soil suction, its components, and the different potentials which make up the total potential of soil water. These definitions are quoted on page 9 of Ref 47. Basically, soil suction is considered to be composed of matrix suction and osmotic or solute suction. Matrix suction is a negative gage pressure which will hold soil water in equilibrium through a porous membrane with the same soil water within a sample of soil. This is also known as capillary suction. Osmotic or solute suction is a negative gage pressure which will hold pure water in equilibrium with soil water through a membrane which only allows water molecules to pass.

There is a close relation between these suction components and their corresponding potentials in the soil water. The total potential of soil water at a certain position is the amount of isothermal work per unit volume that must be done on a small quantity of water to move it from a pool of pure water at atmospheric pressure and a specified elevation to the soil water at the point under consideration. At least five components of this total potential can be identified in most problems:

(1) osmotic or solute potential,
(2) gravitational potential,
(3) matrix or so-called "capillary" potential,
(4) gas pressure potential, and
(5) structural or overburden pressure potential.

In many engineering problems, some of these potentials may be neglected. For example, soils containing small quantities of soluble salts which are rather uniformly dispersed will not be greatly affected by solute potentials. The gas pressure potential should be considered only when the gas pressure is greatly different from the atmospheric pressure. Structural or overburden pressure may need to be considered in most problems. Richards (Ref 48) recognized this fact but neglected the effect of overburden pressure in making his computer predictions of suction beneath a roadway pavement.

There are several practical reasons for not considering the effect of overburden pressure. In the first place, the depth within which suction was changing significantly in Richard's example problems was about 90 centimeters. This is not sufficient depth to cause a great difference in computed suction values. Secondly, inclusion of unsaturated effective stress parameters in the analysis requires an additional degree of complication in the computer program developed. Thirdly, since no laboratory measurements of effective stress parameters were reported by Richards, it is assumed that no correlation with field volume change was made.

Several more practical reasons could be given for not considering the effect of overburden pressure on soil suction. However, as Philip (Ref 41) points out, in deep soils of high colloid content the moisture-suction relation is influenced by overburden pressures and a complete solution of the problem must consider this factor.

These comments do not deprecate the excellent work done by Richards. He has produced the first example in the literature of two-dimensional suction distribution using discontinuous, suction-dependent diffusion constants that were measured in the laboratory. He has attempted to correlate this computer-predicted behavior with values he measured in the field. His work gives hope that the diffusion approach, just as in consolidation theory, may ultimately give engineers useful results. Increasing size and speed of computers will make it practical to include more of the pertinent variables.

Although it may seem to be an imposing task, this chapter will present a summary of current technical knowledge regarding soil suction. This does not mean that this chapter will be a comprehensive treatment of the subject. However, it will raise and partially answer questions in practically every endeavor of the field concerning suction.

Suction at a point within a body is a dependent variable. It depends for its value on the temperature, state of strain, water content, geometrical position with respect to boundaries, chemical concentration, and water pressure to mention a few. In order to discuss suction from a fundamental point of view it is first necessary to consider all of the properties of material which it depends upon.

It is necessary to know something about water and its properties. It is essential to consider the total soil water potential and how it relates to suction. The components of suction and methods of measuring them may then be discussed followed by methods of measuring the total suction. This chapter will follow the procedure outlined above. The discussion of measurement methods will be limited to a brief discussion of typical methods because an exhaustive account of measurement techniques would require several volumes.

Experimental results with artificial and real soils will be presented followed by a careful definition of the terms "saturated" and "unsaturated." A single dividing line is proposed based on the continuum properties of the clay-water-air mixture. This dividing line is implicitly recognized in the literature by the two fundamental methods of computing suction: (1) using an ideal incompressible liquid as a model for water and (2) using an ideal gas as a model for water vapor. The two are never combined into a mixture of compressible and incompressible fluids for the purpose of computing a theoretical value of suction. These notions are discussed and are followed by a purely speculative discussion on the shape of the suction-moisture curve using the idea of "probability of aggregation."

Water and Its Properties

A comprehensive treatment of the properties of water is given in N. E. Dorsey's book on Properties of Ordinary Water Substance (Ref 18) and it is recommended to all who wish a detailed knowledge of the subject.

Water is a unique substance containing the basic units of both acid (hydrogen ion) and base (hydroxl ion). The two hydrogen molecules cluster on one end of the larger oxygen molecule with the angle subtended by their centers about 105°. The distance from the center of the oxygen to the center of a hydrogen atom appears to be about one Angstrom unit or 10^{-8} cm. Water is held on the surface of clay particles by hydrogen bonding, water molecules being more oriented near the surface of the clay mineral than farther away where the molecular arrangement of water is more disorderly.

A property of ice that water molecules are strongly organized by hydrogen bonding between the electronegative oxygen atoms into a tetrahedral building block. At the vertices of the tetrahedron, picture hydrogen molecules, and at the body center, imagine an oxygen atom. Two hydrogen atoms are bound tightly and the other two are bound less intensely. Temperature and pressure change the properties of the macroscopic material we know as ice and water. Higher temperature and lower pressure permit the hydrogen bonds to dissociate, explaining the decrease of viscosity in water with an increase of temperature and giving a theoretical basis to the phase diagram of water such as shown in Fig 11.

The chart is taken from Winterkorn's paper (Ref 60) and shows the strange fact that Ice I with which we are familiar expands on freezing while the other

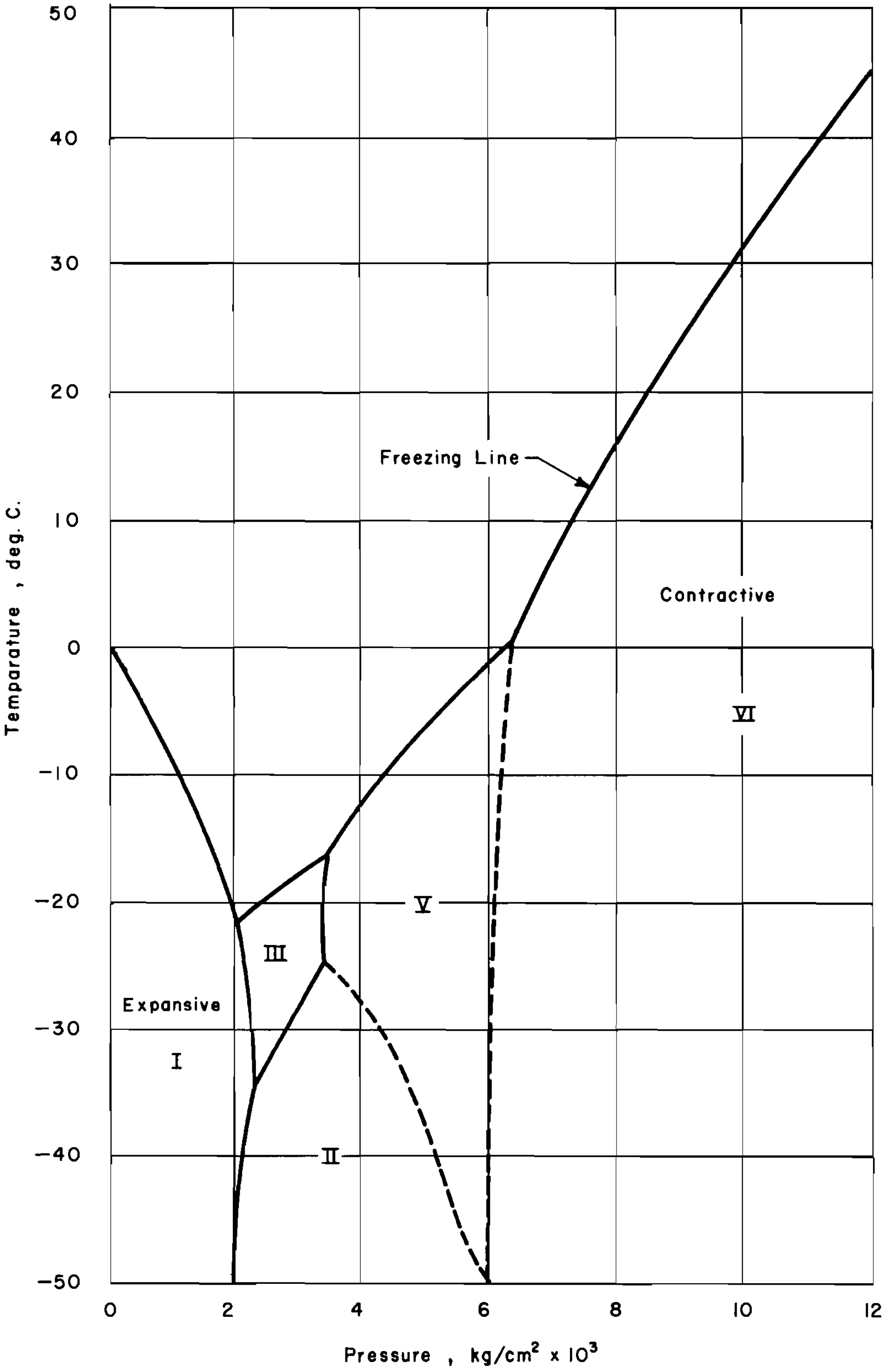


Fig 11. Pressure-temperature relation for ice.

phases of ice contract on freezing. It is worthy of note that with a high enough pressure, it is possible to have ice at room temperature.

The greater density of contractive ice coupled with the knowledge that water is denser and more viscous near the surface of a clay mineral led to the conclusion that water near a mineral surface is in a state much like ice but not quite so thoroughly organized or immobile.

Total Soil Water Potential

The strength of hydrogen bonding has an effect much like that of pressure in its organizing and compressing effect. In fact, it is possible to speak of the total energy per unit weight of water required to remove the water from its position near a clay mineral to a place where no more energy is required to move it. This is the approach taken by Silvan Andrei (Ref 4).

Before discussing the idea of Andrei, it is useful to examine the same idea as it refers to a spring. The total energy E required to displace the spring in Fig 12 a distance x_1 is $\frac{1}{2} kx_1^2$ where k is the spring constant. The force in the spring at any point x_1 is the derivative of E with respect to x at point x_1 , kx_1 . The energy in the spring is analogous to the specific energy L proposed by Andrei. The force in the spring is analogous to suction τ or to the pressure-like quantities used to express the strength of hydrogen bonding at different distances from the surface of a clay mineral. Figure 13 shows Andrei's idea. Obviously, additional shells of water around the spherical particle shown in the figure represent greater and greater water content. The derivative of the specific energy with respect to water content is the suction.

An explanation given by Aitchison (Ref 1) is the basis for the following approximate figure (Fig 14) which illustrates the size of the "suction" quantity at different distances from the surface of a clay mineral.

In concluding this section on the total water potential it should be noted that the soil water is rarely, if ever, in tension. Although its tensile strength is considerable (2800 psi - Ref 1), the cause of water moving into soil as fine-grained as clay will normally have little to do with capillary action. Rather, the cause will normally be the attractive and repulsive forces within the clay as a result of the electrostatic character of the minerals and water molecules themselves.

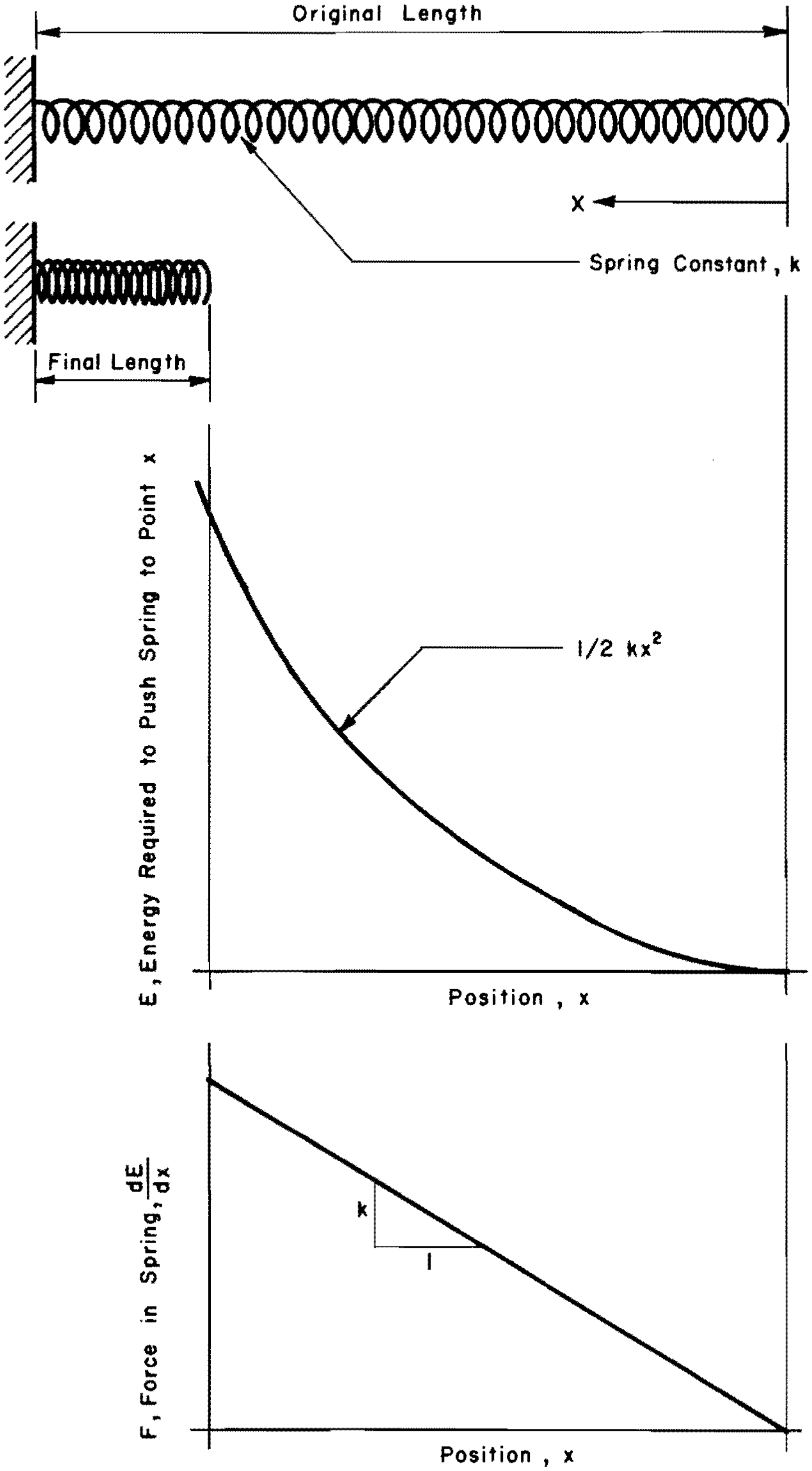


Fig 12. Spring analogy to soil water energy potential.

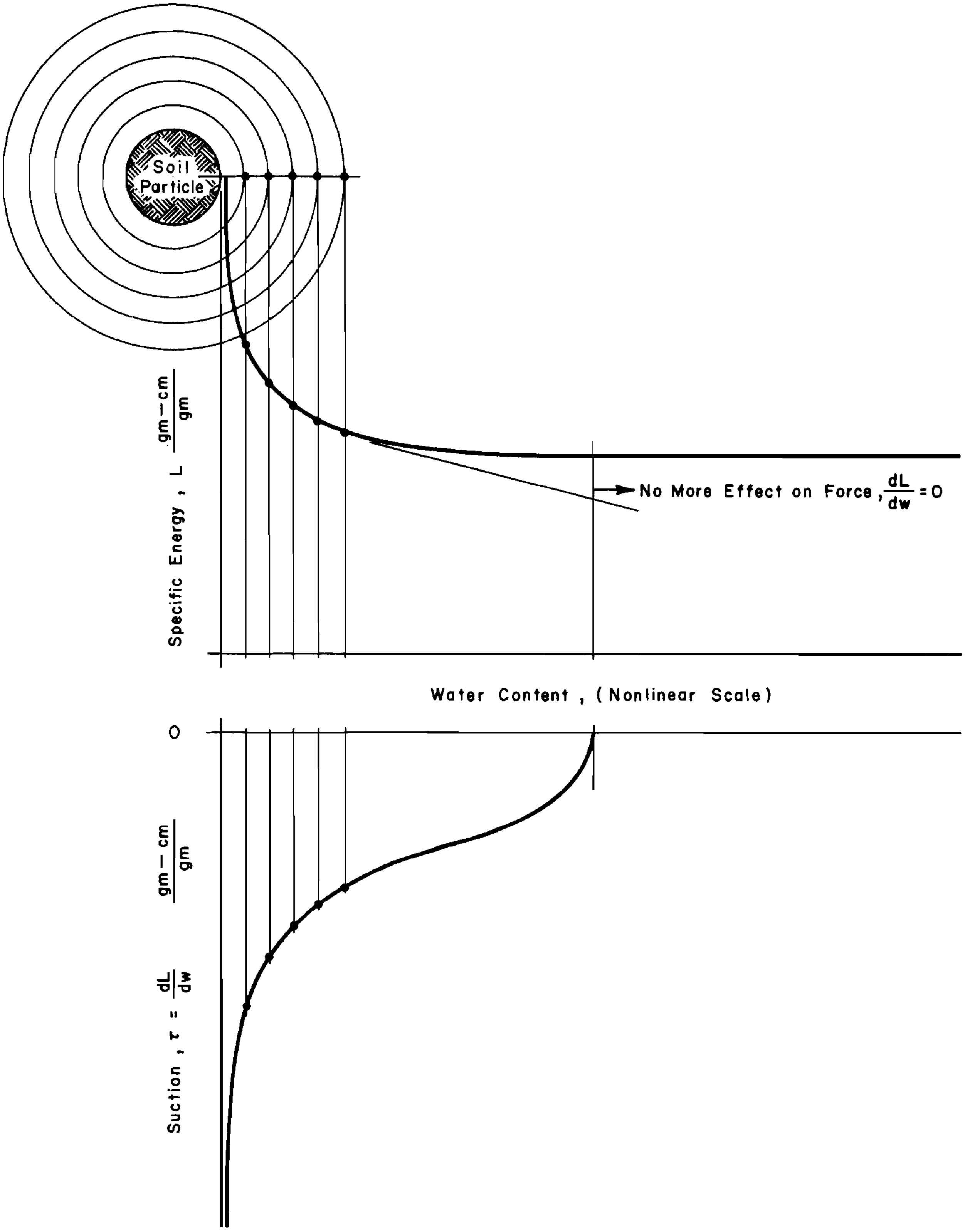


Fig 13. Specific energy and suction of water.

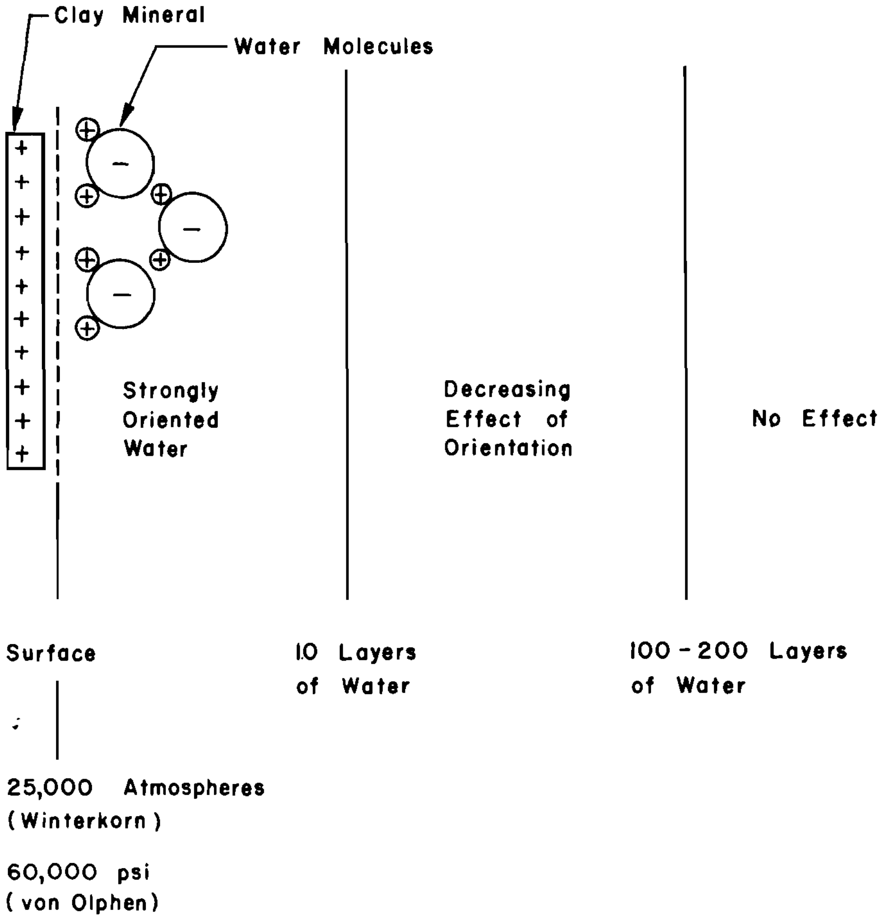


Fig 14. Strength of bonding with distance.

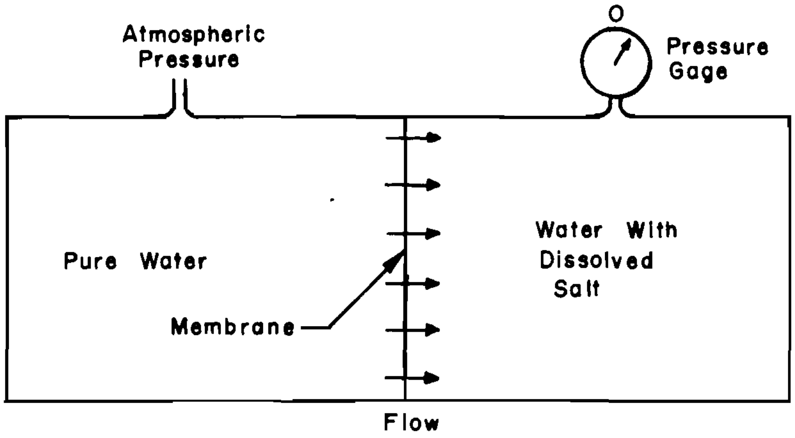
Components of Suction Head

An excellent discussion of the components of soil suction was given in a paper by Olson and Langfelder (Ref 39) in which is recognized that moisture movement takes place in soils as a result of a gradient in total head rather than in pressure head alone. Thus, an engineer who investigates pore water pressures in soils in an attempt to explain all moisture movement will be disappointed in most cases. The total head τ as defined by Olson and Langfelder is

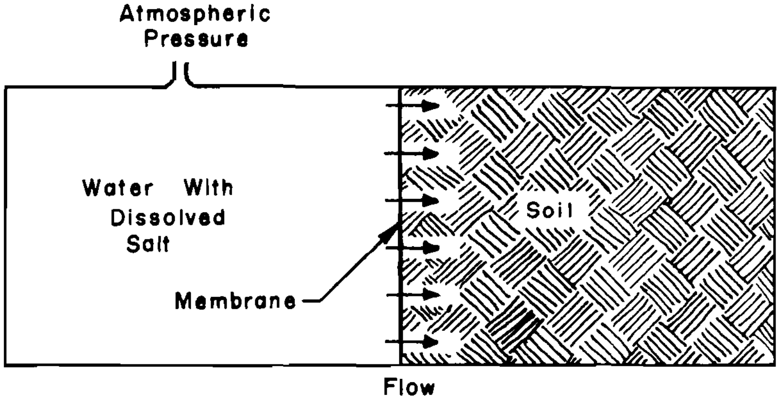
$$\tau = \frac{u}{\gamma_w} + z + \frac{v^2}{2g} + \omega + A \tag{3.1}$$

in which u is the pore water pressure, z is the gravity head referred to some datum, γ_w is the unit weight of water, v is the actual flow velocity of water, ω is the osmotic pressure head, and A is the adsorptive head, the equivalent of matrix suction. Because the velocity term is negligible and does not actually cause water movement, it will not be included in the total head term in this development. The pore water pressure term includes the effects of overburden pressure. The adsorption term includes surface tension and curvature of liquid-air interfaces.

Total suctions measured in the laboratory have been as high as -142,000 psi which greatly exceed the tensile strength of de-aired water, -2800 psi. This naturally raises doubt as to whether such suction actually can occur. The answer is that while pore water pressures cannot go below the tensile strength of water without causing cavitation, the other components of total suction head combine to give the effect on the water of the much greater suction. These components are negative because of the datum from which they are measured, that of pure water at atmospheric pressure. If flow occurs away from the pure water, then a negative gradient exists away from the pure water. If suction is defined as zero in pure water, then it must be negative in the region toward which flow occurs. A simple experiment will illustrate the point. In Fig 15(a), pure water is placed on one side of a semipermeable membrane and a salt solution of concentration C is placed on the other side. The pure water is open to atmospheric pressure and the pressure gage measures the pressure in the salt solution. Flow will occur in the direction shown until equilibrium is reached at time flow stops and the total heads on each



(a) Solute suction experiment.



(b) Matrix suction experiment.

Fig 15. Two types of suction experiments.

side of the membrane are equal. By definition, the total head in the pure water is zero, but the pressure gage indicates a gain in water pressure. The apparent anomaly is explained by writing the total head equation for each side of the membrane.

The increase in water pressure is an osmotic effect which can be predicted by the equation

$$u = CRT_e \tag{3.2}$$

where

C = the concentration of the salt solution,

R = the universal gas constant,

T_e = the absolute temperature.

The osmotic head is ω in the equation for total head:

$$\tau = 0 = \frac{CRT_e}{\gamma_w} + \omega \tag{3.3}$$

Thus, it is seen that the osmotic head is

$$\omega = -\frac{CRT_e}{\gamma_w} \tag{3.4}$$

and this is despite the fact that the water pressure to the right of the membrane is greater than atmospheric pressure.

Similarly in Fig 15(b) the same salt solution is placed on the left and soil containing the salt solution is placed on the right of a permeable membrane. Flow occurs into the soil until equilibrium is reached. A higher pore water pressure would be measured on the right. Equating total head terms gives

$$\frac{CRT_e}{\gamma_w} - \omega = o = \left(\frac{\Delta u}{\gamma_w} + \frac{CRT_e}{\gamma_w} \right) - \omega + A \tag{3.5}$$

in which case the adsorptive head is

$$A = -\frac{\Delta u}{\gamma_w} \tag{3.6}$$

a negative quantity again.

Similar demonstrations can be given for surface tension forces to indicate their effect on the total head. All head components must be referred to pure water at atmospheric pressure. It should be carefully noted that in each case shown in Fig 15 the equilibrium water pressure is greater (more compressive) than the pressure in the pure water. The movement of water is always in the direction of a negative gradient, and flow occurs until the gradient becomes zero. At this equilibrium condition, the initial strength of the gradient can be determined from the final difference in pressure.

This principle is used in devices which measure pore water pressure. The water in the device will have pressure u_p and solution concentration C_p. At equilibrium the total heads are equated

$$\frac{u_p}{\gamma_w} - C_p RT_e = \frac{u_s}{\gamma_w} - C_s RT_e + A_s + \omega_s \tag{3.7}$$

where the s subscript indicates the sample. Recall that A_s and ω_s are negative numbers. Solving for u_p gives

$$u_p = u_s - \gamma_w RT_e (C_s - C_p) + A_s\gamma_w + \omega_s\gamma_w \tag{3.8}$$

Measurement cannot be made if the water in the device cavitates. Thus, the experimental technique used increases u_s to a point where u_p is less than the tensile strength of water. Aitchison and Richards (Ref 3) note that a pressure measuring technique is used to measure suctions of nearly -1,420 psi. Beyond this point a vacuum desiccator technique is used which draws moisture from a sample with sulfuric acid of known concentration. This technique has been used to measure total suctions of -142,000 psi. Since pressure techniques use water in the measuring apparatus, they normally determine matrix suction, whereas the vacuum desiccator can determine total suction.

There will be no difference between the two readings if the sample water has no dissolved salts and the pressure device uses pure water. A brief discussion of some typical methods used to measure suction follows.

Some Suction Measuring Devices

Although there are a large number of methods of measuring, it is the purpose of this section to discuss the principle of operation of three of them. These principles will be useful in a subsequent section which distinguishes between effectively saturated and effectively unsaturated soil.

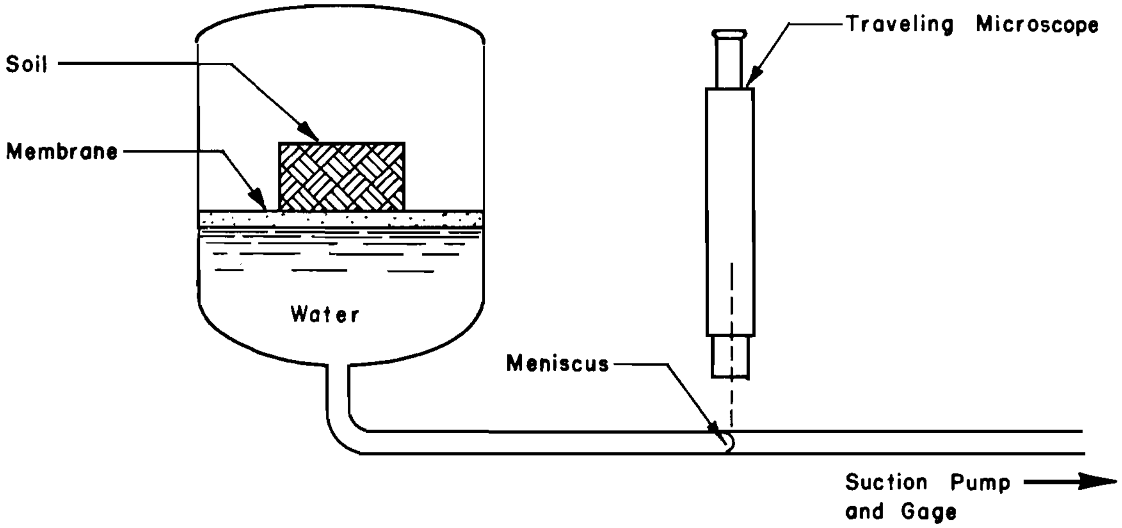
Two of these methods, the Suction Plate and the Pressure Membrane, are for lower values of suction, up to a pF (def p 72) of about 6.0. The third is for high suction or unsaturated soils, and is called the vacuum desiccator method.

Suction Plate. A sample of soil is placed on the membrane, the lid is put firmly in position, and suction from the pump is adjusted to pull on the water that wishes to enter the clay through the saturated membrane. A meniscus is watched through a traveling microscope to make sure that the meniscus remains stationary. Because water will cavitate around one atmosphere, the limiting suction for this device is 3.0 pF or 14.2 pounds per square inch. This apparatus is shown in Fig 16(a).

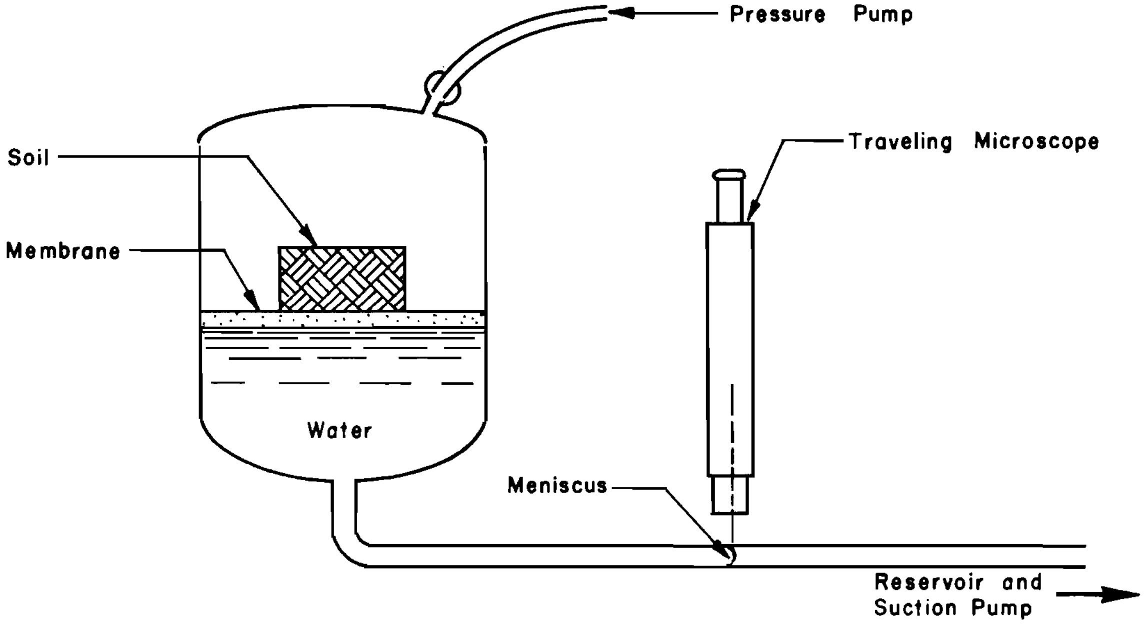
Pressure Membrane. This device shown in Fig 16(b) is identical in operation with the suction except that it uses a "translation" technique to measure suctions greater in magnitude than cavitation pressure. Again, pressure is applied to keep water from entering the soil sample through the saturated membrane. But in this case, the pressure is a back pressure which changes air solubility and bubble size some but does not affect overall results greatly (Ref 42). The equilibrium back pressure that maintains a stationary meniscus is said to be the suction value of the sample. Suction range with this method is 0 to 6.2 pF .

Both of the pressure devices use a low suction measuring method suitable for effectively saturated soils. The pressure membrane does measure suctions in what would be termed effectively unsaturated soils and gives results which overlap data obtained from the high suction device, the vacuum desiccator.

Vacuum Desiccator. The apparatus shown schematically in Fig 17 uses sulfuric acid to generate a relative vapor pressure. The vacuum jar is



(a) Schematic diagram of suction plate device.



(b) Diagram of pressure membrane device.

Fig 16. Diagrams of two types of suction devices.

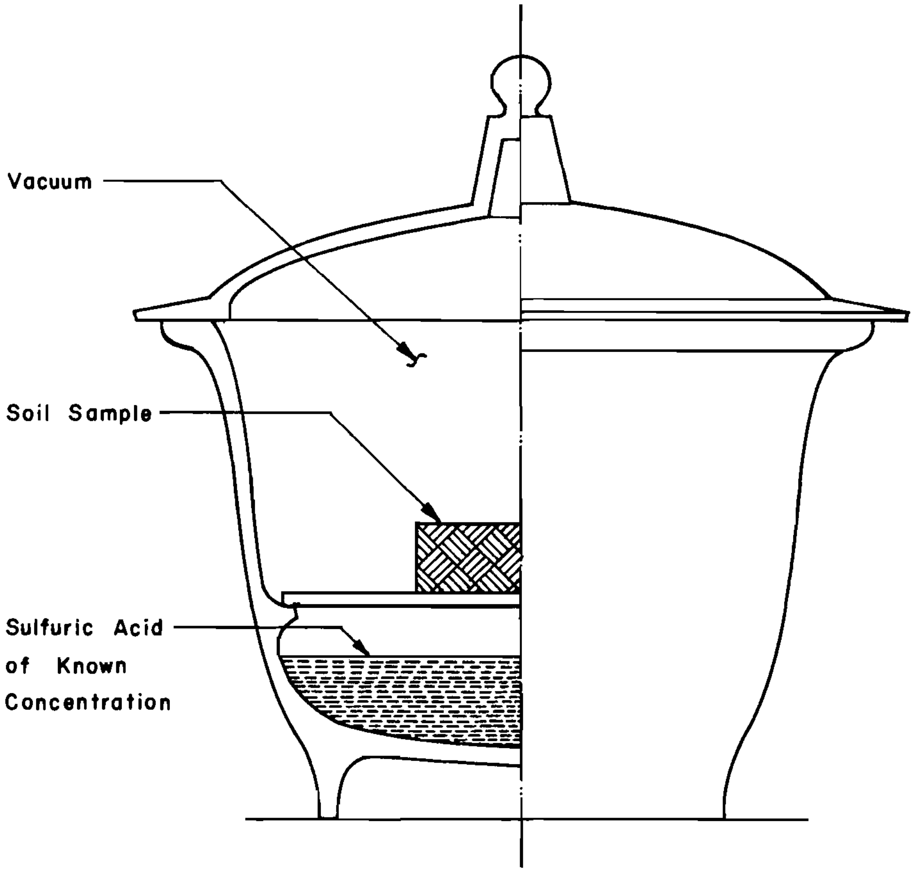


Fig 17. Vacuum desiccator.

evacuated to promote more rapid transfer of moisture between the soil and the acid solution. When equilibrium is reached, the soil moisture content and the acid density are determined. Typical values of relative vapor pressures and actual vapor pressures at 20° C are given in Table 3. Range of suction values for this method is 5.0 to 7.0 pF.

It is notable that although the suctions computed from these relative vapor pressures are negative and become increasingly larger in magnitude, the actual vapor pressures are small and approach zero. These unsaturated suction values are computed by assuming water vapor to be an ideal gas and by using an expression involving a natural logarithm of relative vapor pressure, the characteristic of which is negative. The large suction corresponding to a small value of vapor pressure emphasizes the fact that suction is a measure of a tendency to take on water or a hesitancy to give the water up. The greater the suction value in a soil, the more intensely will it dry up the air in its vicinity in its frantic search for water.

Moisture-Suction Relations for Artificial Soils

Some of the first work done on investigating the variation of suction with water content was reported in 1925 by W. B. Haines (Refs 24 and 25) who made numerous tests filling and draining a porous medium made up of regularly packed spheres of uniform size. He found that even these pure cases exhibit hysteresis. While a certain amount of suction was required to drain water from a void, the space refilled at a lower level of suction. Aitchison gave a very clear discussion of these data at the 1960 conference of Pore Pressures and Suction in Soils (Ref 1). Some of the pertinent points of his discussion are presented here.

An idealized soil system composed of uniform spheres in some stable packing arrangement is initially saturated with water and open to atmospheric pressure. The water is then drained from the system and the pressure in the water is measured. The pressure in the water below a meniscus is less than atmospheric pressure by the amount of surface tension in the water surface divided by the effective radius of the surface. If atmospheric pressure is taken as zero, then the water pressure is negative. This negative pressure may be decreased to a certain critical point at which the entire void space at a certain level will drain. At this point, the saturation of that void

TABLE 3. VAPOR PRESSURE DEVELOPED BY SULFURIC ACID OF DIFFERING SPECIFIC GRAVITY

Specific Gravity of Acid	Relative Vapor Pressure	Saturated Water Vapor Pressure at 20° C	Water Vapor Pressure at 20° C	Suction cm
1.050	0.977	--	.345 psi	-3.23×10^4
1.110	0.930	--	.328	-1.01×10^5
1.205	0.795	0.353 psi	.281	-3.09×10^5
1.344	0.484	--	.171	-1.01×10^6
1.578	0.101	--	.0357	-3.09×10^6
1.840	0.001	--	.000353	-9.60×10^6

suddenly plunges from 100 percent to very nearly zero percent. If the pressure in the remaining water film is decreased further, the saturation will not change perceptibly.

The process is now reversed and the voids are refilled while water pressure is monitored. When the critical drainage pressure is reached, saturation starts to increase but at a much slower rate than its previous decrease. Finally, saturation is reached at a pressure that is much higher than the critical drainage pressure.

The entire process is shown graphically in Fig 18. Obviously, with the closer packing of spheres, the critical drainage pressure is more negative.

It should be carefully noted that the packing structure of the spheres does not change in these tests and thus the total pore volume does not change with changing water pressure. This is not the case with real soils in which the structure and orientation of particles change with variations in soil water pressure. It is necessary to realize that even in the ideal soil composed of uniform spheres, hysteresis is observed between draining and filling cycles and thus hysteresis should be expected in real soils.

An investigation of hysteresis on a porous body with constant geometry and a variation of pore sizes was conducted by Poulovassilis (Ref 43). Using glass beads of several sizes for the porous medium, he cycled between completely saturated and completely unsaturated and found that the equilibrium points of suction and water content were reproducible on both the wetting and drying curves. Because of the different pore sizes, loss of water was observed at every point along the draining curve and gain of water was noted all along the refilling curve. Once these outer bounds were reliably established, partial saturation was reached and then full saturation recovered along scanning curves which follow paths inside the bounding curves. Typical experimental results are shown in Fig 19. In analyzing the data, Poulovassilis introduced the concept of independent domains used in other hysteresis considerations. An independent domain is a fictitious pore space containing a quantity of water that is gained in a certain suction range in the filling process and is lost in another suction range in the draining process. The concept is probably not a completely valid model of the porous body, but it is a useful idea much like that of permeability. Viewed microscopically neither can be justified. But on a macroscopic basis, the empirical results of Poulovassilis' experiments seem to correlate well.

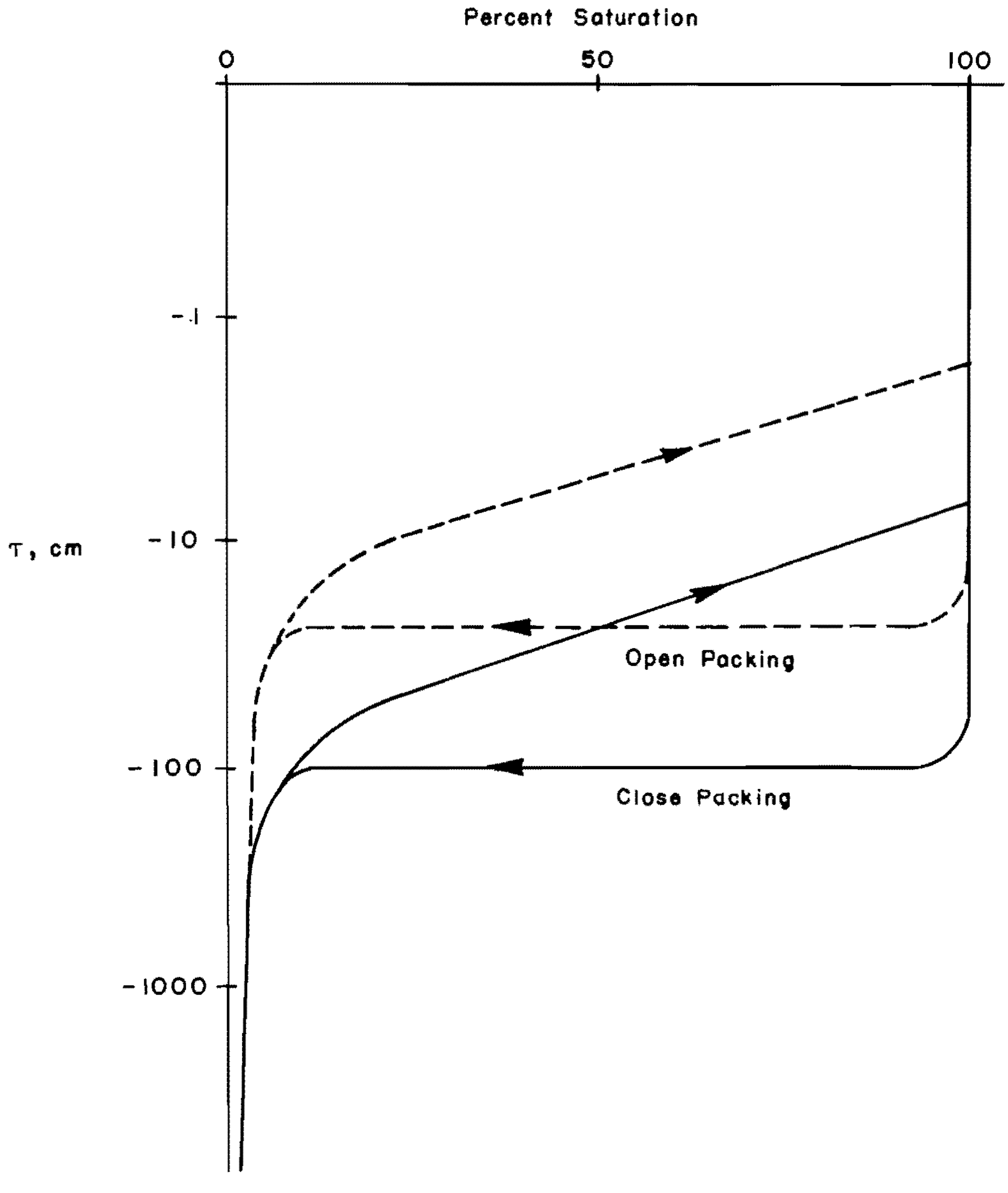


Fig 18. Suction versus saturation for porous medium composed of spheres of uniform size.

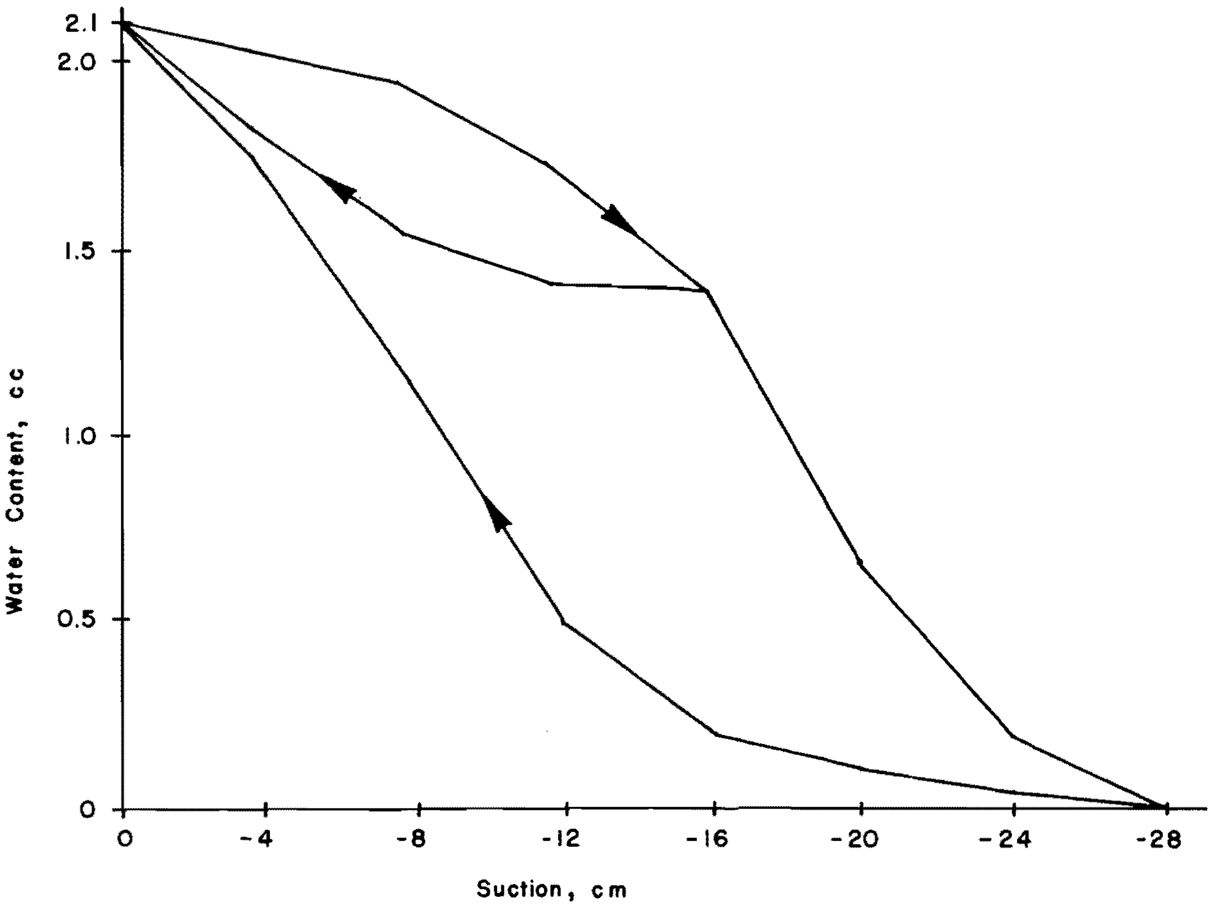
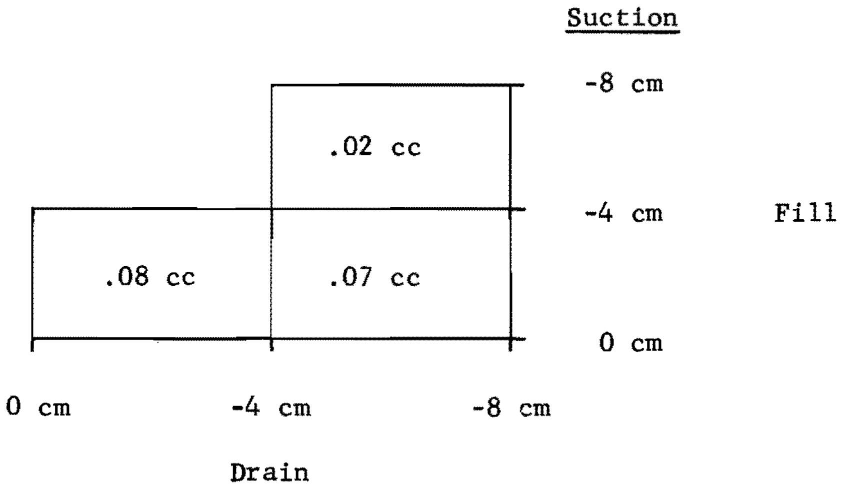


Fig 19. Suction versus water content for an ideal incompressible porous medium.

As is seen in Fig 19, an increase of suction from 0 to -4 cm results in a total loss of .08 cc of water. Further suction increase to -8 cm drains an additional .09 cc of water from the specimen. If the process were now reversed, and suction decreased to -4 cm, only .02 cc would be regained. A further decrease to zero suction would return the remaining .15 cc of water. These results could be charted as shown below



Thus, no water is lost in the process. A series of such experiments could fill up a table such as shown in Fig 20 and the same process can be used to predict scanning curves.

Moisture-Suction Relations for Real Soils

The curve shown in Fig 21 is taken from experimental data reported by Croney, Coleman, and Black (Ref 14) and is the suction-moisture content curve for a heavy clay. Table 4 shows an approximate analysis of this curve. The moisture contents were changed to percent saturation and saturation increments were used in a scanning study of this curve. The term pF , frequently used in reporting suction-moisture data, the Briggs logarithm (base 10) of the suction expressed in centimeters. A pF of 7 corresponds to a total suction of 142,000 psi.

The chart shown in Fig 22 is an approximate analysis of the Croney, Coleman, and Black data by independent domains. It is evident that the wetting and drying curves are closer together. This is shown in the domain study by numbers clustered close to the diagonal of the chart. A scanning

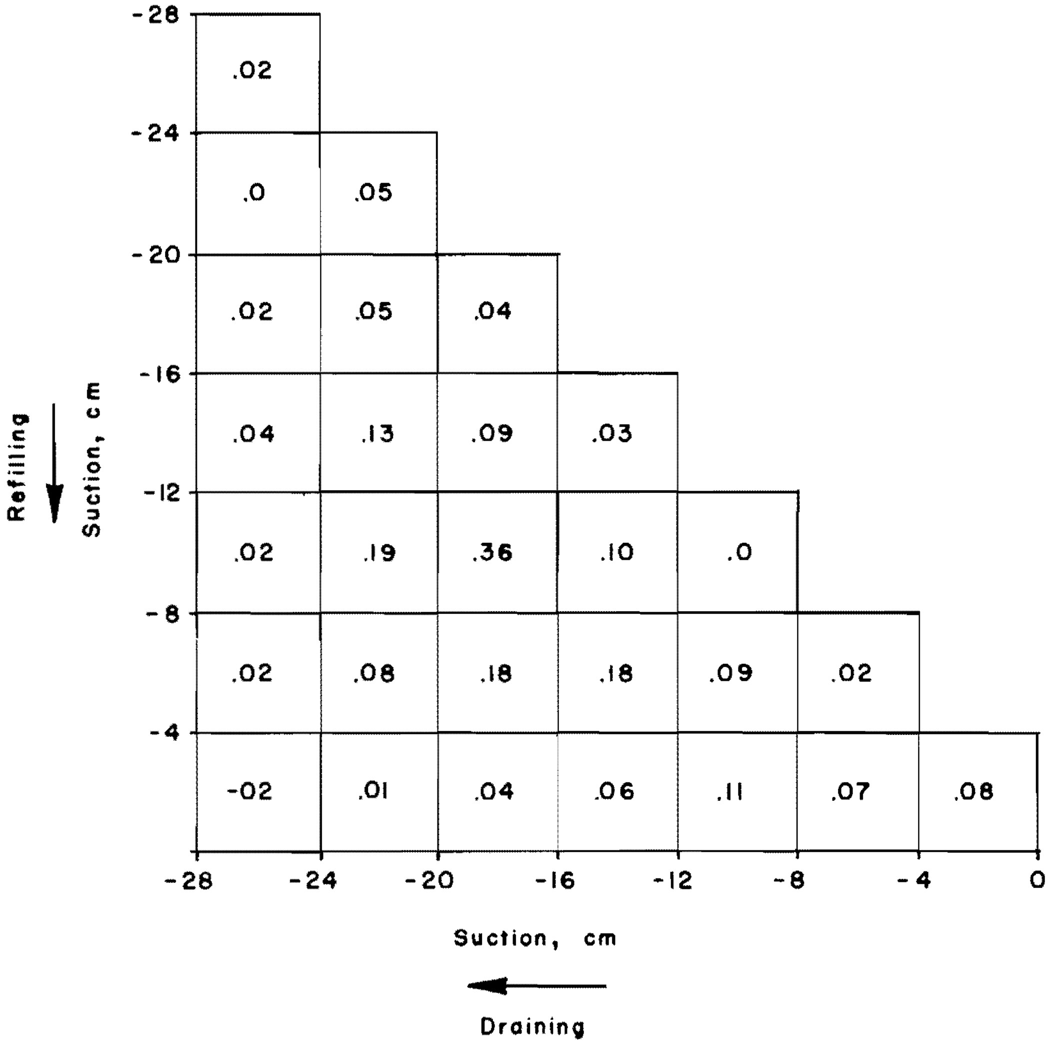


Fig 20. Independent domain analysis of suction-water content relation for an ideal incompressible porous medium.

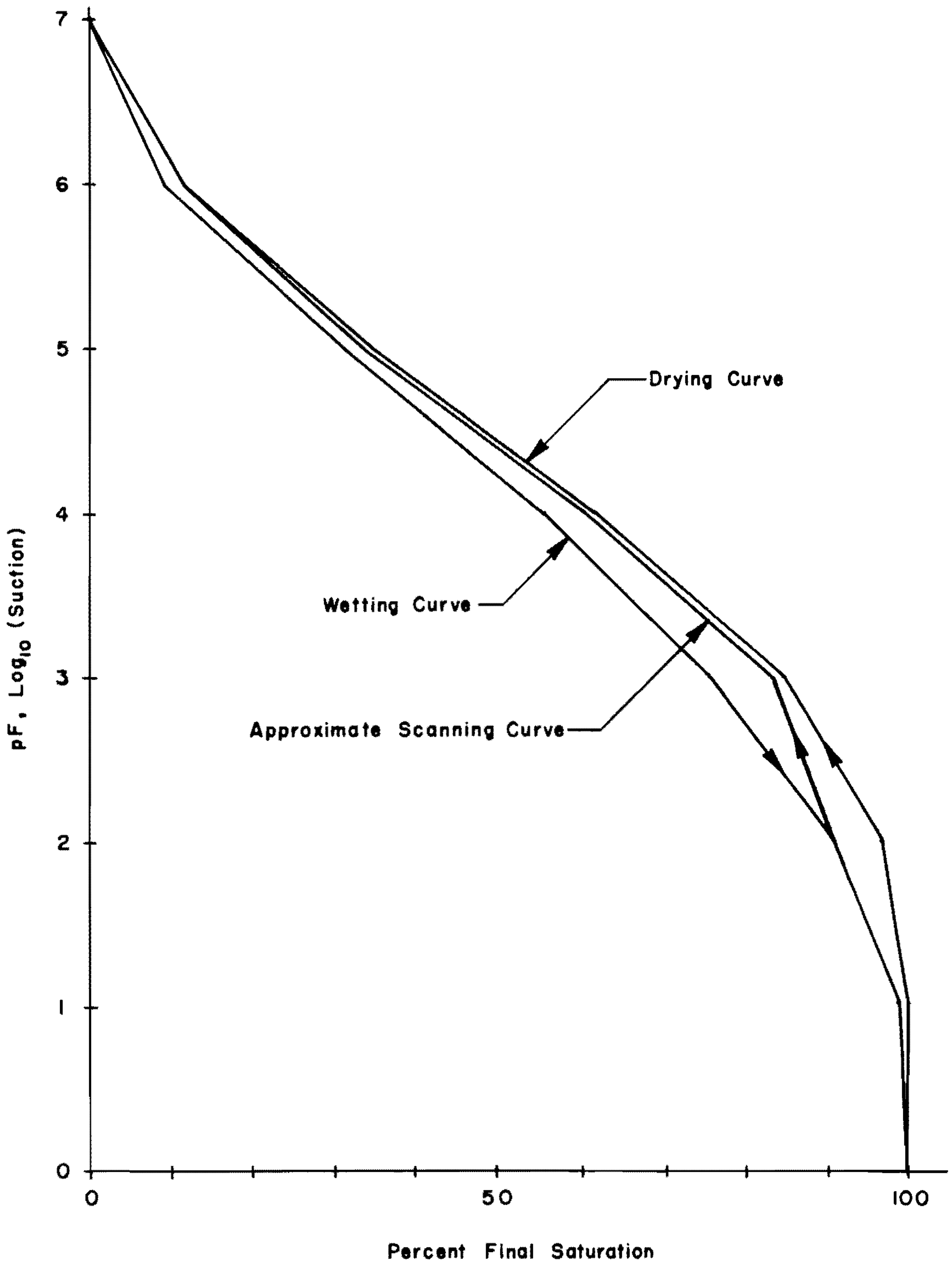


Fig 21. pF-Moisture relation for a heavy clay (from Croney, Coleman, and Black, Ref 14).

TABLE 4. WETTING AND DRYING CURVE DATA
(From Croney, Coleman, and Black, Ref 14)

	WETTING			DRYING		
pF	Water Content, %	Percent Final Saturation	Saturation Increment	Water Content, %	Percent Final Saturation	Saturation Increment
0	30.5	100		30.5	100	
			1			0
1	30.0	99		30.5	100	
			8			3
2	27.8	91		29.5	97	
			15			12
3	23.0	76		25.8	85	
			20			23
4	17.0	56		19.0	62	
			24			27
5	9.8	32		10.8	35	
			22			23
6	3.0	10		3.8	12	
			10			12
7	0	0		0	0	

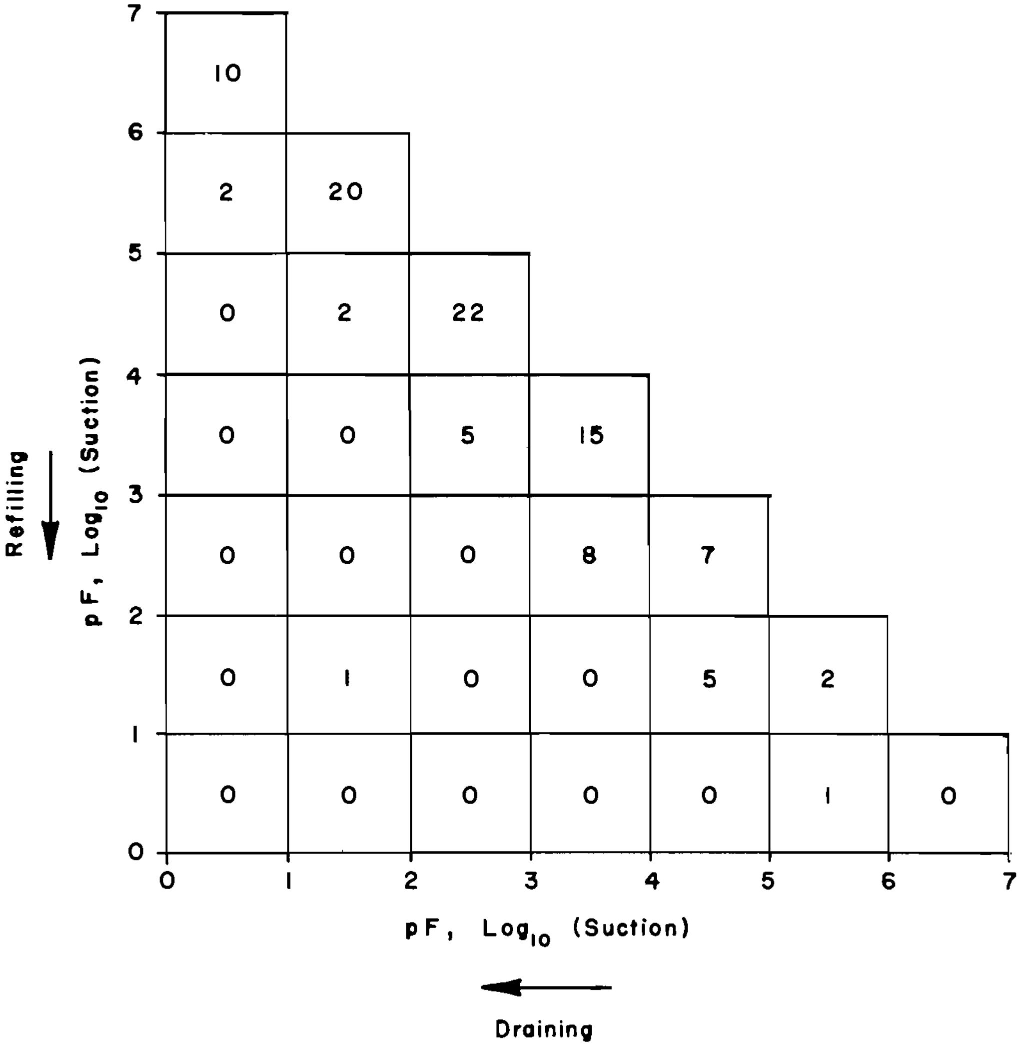


Fig 22. Independent domain analysis of Croney, Coleman, and Black data for heavy clay.

curve shown in Fig 21 starting from a pF of 2.0 on the wetting curve and drying from that point shows only slight deviation from the boundary drying curve.

Consideration of hysteresis is certainly important in accurately predicting values of suction. But as seen in this approximate study of a real soil curve, water content is not greatly affected by a large change in suction in the high suction ranges. Thus, if soil is dry there is considerable latitude in estimating the suction in the soil when moisture conditions are to be predicted.

Experimental scanning curves such as reported by Croney, Coleman, and Black (Ref 14) move from one bounding curve to another within a relatively small range of suction. This indicates that there is some important difference between the idealized medium made of glass beads studied by Poulovassilis and real soil. A real soil is compressible. The spheres used by Haines and the glass beads used by Poulovassilis were not compressible and this is one other aspect in which real soils were not modeled. The soil is compressed by a transfer of water pressure to the soil grains. Of course, the less saturated the soil is, the less pressure can be transferred to soil grains because of the smaller wetted surface. As Aitchison (Ref 1) notes, a more complete relation would include degree of saturation, suction, and porosity. A microscopic view of the problem would indicate the importance of particle orientation, dissolved salts, and other such properties.

The characteristic S-shape of the suction-moisture curve can be understood qualitatively as a direct result of the instantaneous pore-size distribution. If it can be assumed that suction at a point on the curve is that required to drain all pores of the same size in a given sample, then computation of the capillary size that can be drained at that suction will give an effective pore-size distribution. Whether this effective pore-size distribution has any relation to the actual distribution is problematical, but Childs and Collis-George (Ref 12) applied a statistical model of connected pores of different sizes to data of this sort and found fair correlation with measured permeability. It is certain that at the inflection point of the S-shape, the greatest amount of water leaves or enters for a set change of suction. The suction at which this occurs does depend on how small the predominant pore space is, and the corresponding moisture content depends on the mineral type. Montmorrilonite clays have a much higher water content at the inflection point.

In a general way, there is some qualitative correlation between the characteristic suction-moisture curve and the mechanical analysis curve for the same soil. The connection seems far more difficult to form, however. Rao and Ramacharlu (Ref 45) have stated that "predominant particle size determines predominant pore-size," but it seems that the particle-size distribution is also very important in determining the pore-size distribution with which the suction-moisture curve correlates more directly.

At final saturation or maximum water content, substantial changes in pF (though not suction) cause little change in moisture content. Except for the volume-change effect of the water pressure on the soil, it would be tempting to assume the slope of the curve at final saturation to be infinite.

Similarly, at the point of virtually no moisture a very large change in pF (and also suction) changes water content very little. At a pF higher than about 4.8, water can be removed only by a vacuum desiccator or an oven. These high suctions are required to remove the adsorbed moisture bonded to the clay particles. Croney, Coleman, and Black (Ref 14) have found other interesting features of the suction-moisture curve in what they term heavy clays. The plastic limit occurs at a pF of about 3.4 for several such soils. In a shrinkage (drying) test of the soil with suction-moisture characteristics shown in Fig 21, it was found that air first entered the soil at pF 4 . This statement requires explanation.

Heavy clays as defined by Croney, Coleman, and Black are compressible. An increase of suction in the soil water induces a compression of the soil skeleton. Thus, a soil sample may start at some moisture content, say 30 percent, and be effectively saturated. Suction is applied to the pore water and the soil sample is compressed so that at a moisture content of around 20 percent, the voids in the sample may still be effectively saturated. In this process, the decrease of sample volume is exactly equal to the volume of water lost and this occurs until a point is reached at which air enters the sample. At this point and beyond, sample volume changes at a much slower rate than the change in moisture volume. More detailed discussion of the factors involved in this process is given in the next chapter.

The general shape of the pF-θ curve is known and some check points have been established by experiment. As more knowledge is gained of these relations, it may become possible to construct a useful, though not exact, moisture-suction curve from simple experimental data.

The terms "final saturation" and "effective saturation" have been used in the foregoing discussion and these terms need more precise definition before the discussion is continued.

Definition of Saturation

The term "saturation" must be carefully defined because soil exists in several states of saturation or near saturation and the term "unsaturation" remains nebulous. Aitchison (Ref 2) gave some precise and fairly well-defined distinctions between the types of soil saturation. These distinctions are based on the degree of saturation, the states of pressure in the pore water and pore air, and the relation to the negative pore water pressure required for air entry into a specimen. Table 5 indicates Aitchison's four states of saturation and the distinctions between them.

None of the soils which will be discussed in the succeeding pages will be taken to exist in States 1 or 2. The discussion will be centered around soils with negative pore water pressure which contain a certain amount of trapped air.

For the purpose of abbreviation, State 3, "partially saturated," will be called "final saturation" or simply "saturation." State 4a will be called "effective saturation" and States 4b and 4c, which are beyond the air entry points, will be called "effectively unsaturated" or more simply "unsaturated."

The point now becomes more clear that once air enters a sample to the extent that its total volume begins to change less than its water volume then a point of different macroscopic behavior has been reached. This point of air entry is near the shrinkage limit of soil and is normally lower by several percent than the moisture content of the plastic limit. The air entry point may admittedly be more difficult to obtain experimentally on a production basis, but it seems to mark the first fundamental point of departure between a solid and a semisolid state of soil. It also marks a very convenient and perhaps microscopically valid point of differentiation between "effective" saturation and "effective" unsaturation.

Although he chooses the shrinkage limit to comment about, instead of the air entry point as explained above, the thinking of Raymond J. Krizek (Ref 31) appears to be along the same lines. The small difference in moisture content between air entry point and shrinkage limit makes his remarks applicable to this discussion.

> A clay soil at its shrinkage limit manifests a phenomenon very similar to a polymer at its glass temperature. The shrinkage limit of a soil is defined as that moisture content associated with the minimum volume attained by a saturated sample; in other words, it is that moisture content at which a plot of volume versus moisture content exhibits a sharp break. (Writer's emphasis) Above the shrinkage limit, a saturated clay changes volume as it changes moisture content; below the shrinkage limit, changes in moisture content are accompanied by negligible volume changes, thus yielding a discontinuous rate of volume change. The shrinkage limit also designates qualitatively the division between the solid and semisolid states of soil in a manner similar to the glossy and rubbery states of a polymer. Although from a behavioral point of view, the plastic limit of a clay may seem to bear a closer resemblance to the glass temperature of a polymer, the shrinkage limit has greater fundamental significance and appears to offer the more appropriate and interesting parallel. It is conceivable that a change in the bonding mechanisms of the clay particles occurs at the shrinkage limit similar to the alteration of the mobility of the molecular segments of a polymer.

Discussion will be presented in subsequent sections further substantiating the idea that air entry point, rather than shrinkage limit, is more closely the fundamental property described by Krizek. His analogy of soil-polymer behavior is compelling and especially so is the notion of the change of bonding mechanisms when air entry point is reached.

A speculative model of such a bonding mechanism is presented in the section which discusses the mechanical model of aggregation.

Computation of Saturated and Unsaturated Suction

With the dividing line established between unsaturated and saturated behavior of clay, it is now possible to consider more clearly the difference between the two methods of computing suction. Although terms involving chemical concentration and other factors could be included, they will not be for the sake of clarity.

Suction is a change in force potential energy from some datum state which has been chosen to be pure water at standard temperature and pressure. It is expressed as the equivalent height of water of a pressure that is normally negative. In equation form, suction is basically

$$\Delta F = \frac{p}{\gamma} = \int_{p_o}^{p} \frac{1}{\gamma}\, dp$$

$$= \int_{p_o}^{p} v\, dp \qquad (3.9)$$

where

ΔF = change in free energy from the datum condition,

p = pressure,

γ = weight per unit volume,

v = volume per unit weight,

p_o = reference pressure.

Now the two categories of suction may be formed. One category, saturated suction, applies to the moisture condition in which liquid water is important. The second category, unsaturated suction, applies where water vapor is more important in terms of total water movement. The two categories are best compared in parallel columns.

These equations imply that flow is of one type or the other - either liquid water or water vapor. No provision is made for mixed flow conditions. Nor is provision ordinarily made for considering the effect of attraction and repulsion forces in the water vapor.

At a suction pressure of around -1 atmosphere (pF 3.0) free water will cavitate. In the more organized and bonded state near the surface of clay minerals, the suction pressure can be brought by capillary surface tension nearer to the tensile strength of water which occurs at around -200 atmospheres or at a pF of about 5.3.

Inferring from the experimental evidence of Croney and Coleman (Ref 13) that the air entry point occurs at about pF 4.0 , it would seem that capillary surface tension reaches about -142 psi when the soil begins to act as a solid. It is proposed here that the pF at air entry point should be the point of differentiation between the two suction formulas. It is a fairly safe assumption that liquid water will flow under pressure gradients below this point and

TABLE 5. SATURATION TABLE

State	Description	Degree of Saturation, Percent	Pore Water Pressure (+, 0, -)	Below or Above Air Entry Pressure	Pore Air Pressure
1	Fully Saturated	100	+, 0	Above	No Air
2	Quasi-Saturated	100	-	Above	No Air
3	Partially Saturated	< 100	+, 0	Above	+
4a	Unsaturated	< 100	-	Above	+
4b	Unsaturated	< 100	-	Below	Air Drained to Atmosphere, 0
4c	Unsaturated	< 100	-	Below	Trapped Air, +

TABLE 6. COMPUTATION OF SATURATED AND UNSATURATED SUCTION

Effectively Saturated	Quantity	Effectively Unsaturated
$= \int_{p_o}^{p} v \, dp$	Relative free energy, ΔF	$= \int_{p_o}^{p} v \, dp$
$p_o = o$, gage pressure, atmospheric	Datum pressure, p_o	= saturated vapor pressure at standard temperature
	Assumption	
water is an incompressible liquid $\gamma = \text{constant}$		water vapor is an ideal gas following the law $pv = \frac{RT_e}{mg}$
$= \frac{1}{\gamma} (p - o)$	Equation for ΔF	$= \int_{p_o}^{p} \frac{RT_e}{mg} \frac{dp}{p}$ $= \frac{RT_e}{mg} \ln \frac{p}{p_o}$ $\frac{p}{p_o}$ is the relative humidity p is the water vapor pressure p_o is the saturated water vapor pressure

that water vapor is the predominant phase of moisture transfer above this point. The following section pursues this speculative tone further to attempt some phenomenological explanation of the shape of the moisture-suction curve of real soils.

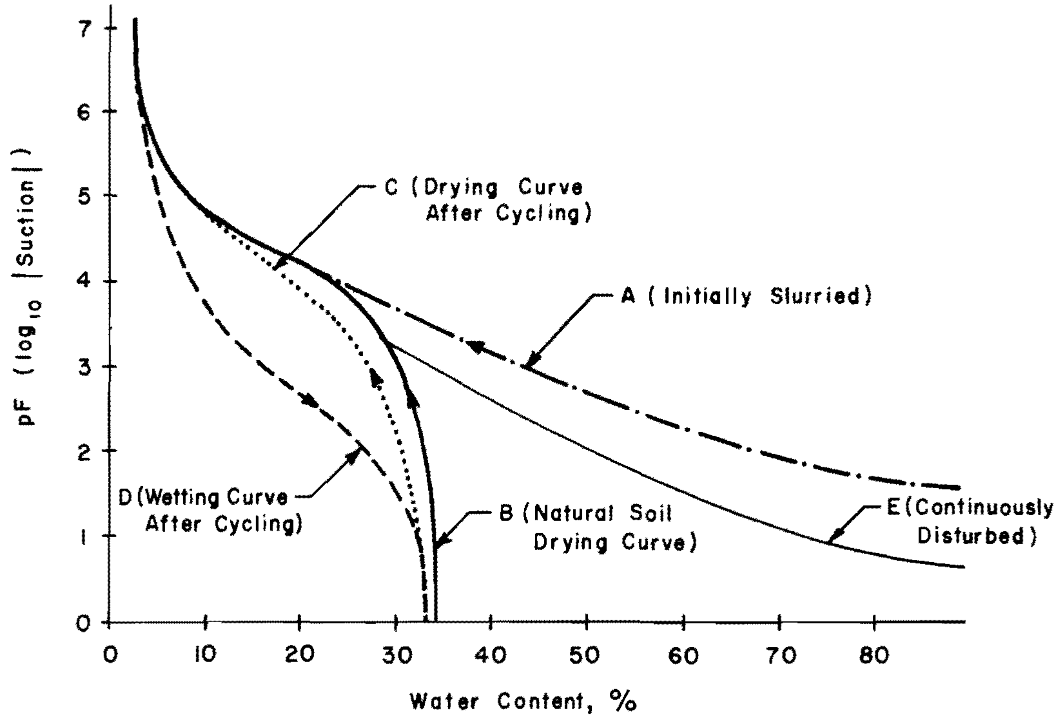
Speculation on the Shape of the Moisture-Suction Curve

This speculation is offered partially as theory and partially as a tool for visualizing the effects on clay behavior of changing water content. A model is proposed which could lead to phenomenological models of clay much like those developed for paint and polymers by rheologists (Ref 46). Arguments from the properties of this model can explain the general shape of the moisture-suction relation, and it is hoped that extensions of these notions along the lines suggested by Krizek (Ref 31) will bring a new level of scientific description of clay behavior.

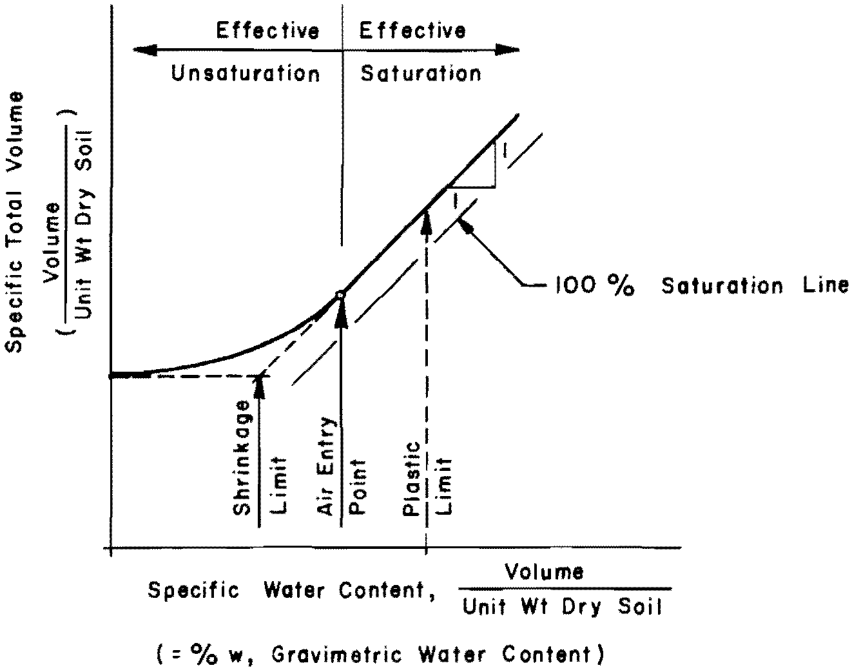
This discussion presumes that no external mechanical pressure will restrain the free volume change of the soil in any process described. The development of this discussion will use the following format. First, characteristic shapes of moisture-suction and shrinkage curves for real soils will be shown and will be followed by an idea on the equation for the simplest case of moisture-suction relation, that of the initially slurried soil. Next, a mechanical model of aggregation will be proposed and the probability of aggregation idea will be developed. Then the effect of aggregation on the moisture-suction curve will be presented for both unsaturated and saturated conditions.

Characteristic Shapes of Moisture-Suction and Shrinkage Curves.

The tests of Croney, Coleman, and Black (Ref 14) have revealed five characteristic shapes of the moisture-suction relation. All of these are shown in Fig 23(a). The curve marked A is for soil that is initially slurried. In this condition, individual clay particles are separated from each other and they have comparatively weak attractive and repulsive interaction with each other. Curve B is the drying curve for the same soil which has been through a number of wetting and drying cycles. Curve D is the wetting curve for the soil which has been wetted and dried a number of times. Curve E shows the variation of suction with water content if the soil is continuously disturbed as water is added. Because the vertical scale of Fig 23(a) is logarithmic, the fact is not as apparent as it would be on an arithmetic scale that Curve E approaches



(a) Five characteristic suction-moisture curves.



(b) The shrinkage curve.

Fig 23. Characteristic shapes of moisture-suction and shrinkage curves.

Curve A as a limiting value. Downward-sloping parallel lines on the logarithmic scale imply a logarithmically decreasing difference in the value of suction.

Figure 23(b) shows the shrinkage curve of the soil and locates the air entry point (sometimes called the point of unsaturation), the shrinkage limit as normally defined, and the approximate location of the plastic limit. The shrinkage limit is a theoretical moisture content computed by dividing the soil dry weight into the weight of water required to fill the dry sample voids to the same degree of saturation as at the air entry point. It is at the air entry point that the slope begins to change from its 1:1 value.

Moisture-Suction Relation for an Initially Slurried Soil. Olson (Ref 38) has pointed out that the shape of the swelling pressure-water content curve for some clays can be copied by the double layer repulsion theories (see van Olphen, Ref 57) but they fail to copy the magnitude at the proper water content.

First of all, it should be made clear that when clay is saturated, suction converted to pressure is the same as swelling pressure. The failure of the microscopic double layer repulsion theories to copy experimental saturated moisture-suction curves points up a weakness in all microscopic theories: that once the basic assumption of grain orientation is made, the natural consequences of this assumption may or may not match experimental data.

Thus, while the exact nature of the interaction among particles may be unknown, the assumption may still be valid that phenomenological quantities such as pressure and volume obey gas laws. Indeed, even those materials we think of as elastic obey gas laws. The range of behavior over which we observe their pressure-volume relation leads us to postulate a linear dependence of one on the other but these data can be considered to be part of an hyperbola of the form $pV^n = \text{constant}$.

The success achieved by Juarez-Badillo (Ref 30) in postulating a gas law for saturated clay in compression lends credence to the validity of the assumption that

$$\frac{dV}{V} = -\alpha \frac{dp}{p} \qquad (3.10)$$

where V is a volume and p is a pressure. When this is integrated, it gives the form of the gas law

$$p(V)^{\frac{1}{\alpha}} = p_o(V_o)^{\frac{1}{\alpha}} \tag{3.11}$$

Using the suction and specific volume of soil at the air entry point as the basis for developing an equation for suction as a function of volume in the effectively saturated range, we have

$$P_A(V_{T_{DRY}} - V_{W_{SL}} + V_{W_A})^{\frac{1}{\alpha}} = p(V_{T_{DRY}} - V_{W_{SL}} + V_W)^{\frac{1}{\alpha}} \tag{3.12}$$

where $V_{T_{DRY}}$ is the total specific volume of dry soil and $V_{W_{SL}}$ is the specific water volume at the shrinkage limit. P_A is the pressure (or suction converted to pressure) at the air entry point and V_{W_A} is the specific water volume at that point. The quantities p and V_W are the pressure and specific water volume at any lower pressure level. Obviously, this equation can be stated in terms of the water content at the plastic limit which can be stated as a function of the plasticity index. This function may be taken from the chart relating liquid limit to plasticity index. Because many clays of similar geologic origin plot in narrow, almost linear bands roughly parallel to the "A"-line, the function would often be of the form

$$\text{P.L.} = \alpha + (\beta - 1)\text{P.I.} \tag{3.13}$$

Returning now to the "gas law" for slurried soil, it is noticed that

$$\log_{10}\left(\frac{p}{\gamma_W}\right) = \log_{10}\left(\frac{P_A}{\gamma_W}\right) + \frac{1}{\alpha}\log_{10}(V_{T_{DRY}} - V_{W_{SL}} + V_{W_A})$$

$$- \frac{1}{\alpha} \log_{10} (V_{T_{DRY}} - V_{W_{SL}} + V_W) \qquad (3.14)$$

This equation is of the form

$$pF = a - \frac{1}{\alpha} \log_{10} (b + V_W) \qquad (3.15)$$

which has a slope of

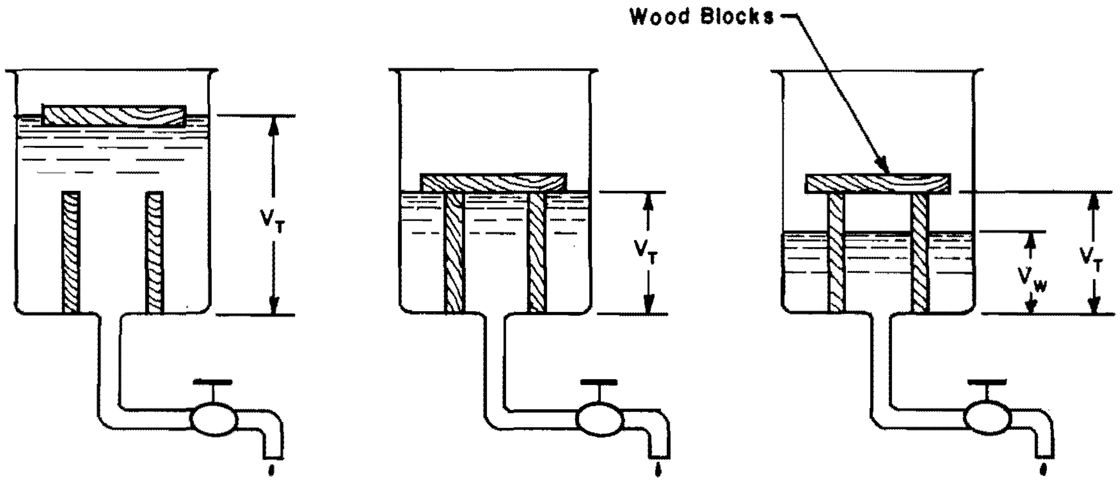
$$\frac{d}{dV_W} (pF) = - \left(\frac{1}{\alpha} \right) \frac{\log_{10} e}{b + V_W} \qquad (3.16)$$

which varies with water content. The larger the water content, the slighter is the slope. This describes a curve of the same shape as Curve A in Fig 23(a) for initially slurried soils. This equation contains enough constants to fit experimental data satisfactorily but is proposed only for effectively saturated soils where suction is related to pressure by a single number, the unit weight of an assumed incompressible water.

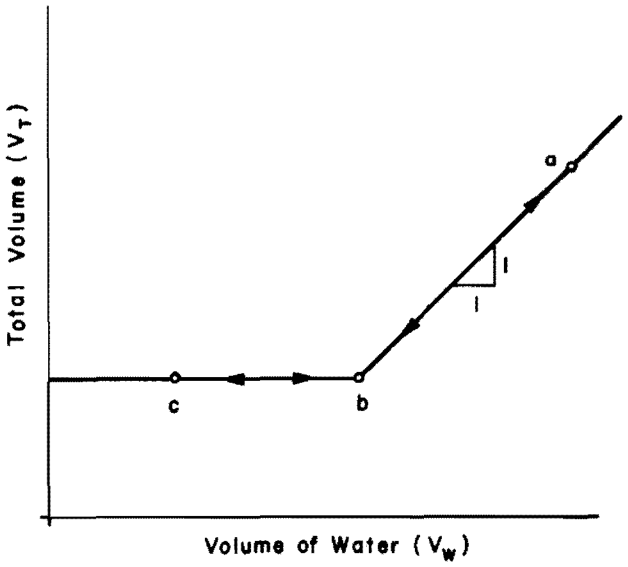
The remainder of the suction-moisture curve must use the value of suction derived from the assumption of an ideal gas. The fact that experimental curves of pF versus moisture appear continuous with slope everywhere indicate that, in reality, there is a gradual transition from one predominant phase of moisture transfer to the next.

The equations presented above can give phenomenological constants for moisture-suction relations such as Curve A in Fig 23(a). But this suction-moisture relation is characteristic of a soil the particles of which either have not formed any effective micro-structure or have for the first time hardened and dried to the point where water vapor is all that moves in the void spaces. Such is the moisture-suction relation to be expected of virginal clays in the process of deposition. Desiccated soils have a different characteristic moisture-suction shape. The higher suction part remains lower than but of essentially the same shape as the virgin curve. At water contents greater than the air entry point, however, the pF-water content curve has a falling characteristic. At higher water contents, the slope becomes steeper downward. It is this peculiarity that is visualized and at least partially explained by considering the probability of aggregation.

Aggregation Model. The first step in the development of this idea is an appeal to a model of the shrinkage process which gives a curve similar to the shrinkage curve. Consider the cup in Fig 24 with three wooden blocks inside of it. In Fig 24(a), the model soil formed by the wooden blocks and the water is in the semisolid stage. Total volume is measured from the top of the floating block to the bottom of the cup. Pressure on the floating block would cause a dashpot effect, decreasing the total volume to a point which could not be less than the total volume shown in Figs 24(b) or 24(c). The plot of total volume versus water volume in Fig 24(d) shows that the two change at the same rate as water is drained out of the cup from stage a to stage b. At point b, there is a sharp change in the behavior of the soil model. It reaches its air entry point and shrinkage limit simultaneously and becomes essentially a solid post-and-lintel type structural unit. Pressure on the top block of this unit will not cause any dashpot effect unless the supporting posts can be compressed enough to allow the lintel to contact the water again. Because no permanent bond is made between posts and lintel, the shrinkage curve developed with this model is reversible on adding water. The sharp break at point b is not characteristic of most soils, however. The gradual bending of the $V_T - V_W$ curve can be copied in a discrete manner by using a system of inverted post-and-lintel units such as shown in Fig 25(a). A ruse is used in this model. Hooks hold the post-and-lintel units down to keep them from floating up. When the top block contacts each unit in turn, it becomes a bouyant box which will lift the top block free of the water surface much as the iceberg lifts its crown above the water. More bouyancy is achieved when the next post-and-lintel unit is reached and so on. An approximation of the shrinkage curve from the model is shown in Fig 25(b). Points a, b, and c represent the points where contact between units is made. This model is admittedly not as satisfactory as the first but it does illustrate the one point that is to be made about aggregation of particles: the flatter the shrinkage curve, the greater the number of contacts or bonds that have probably been made. When clay particles come into contact, they develop bonds and the shrinkage and swell curves for clay should not be expected to be reversible. However, just as in the model, the flatness achieved on the shrinkage curve is a measure of the structural integrity achieved by the soil at the appropriate water content.

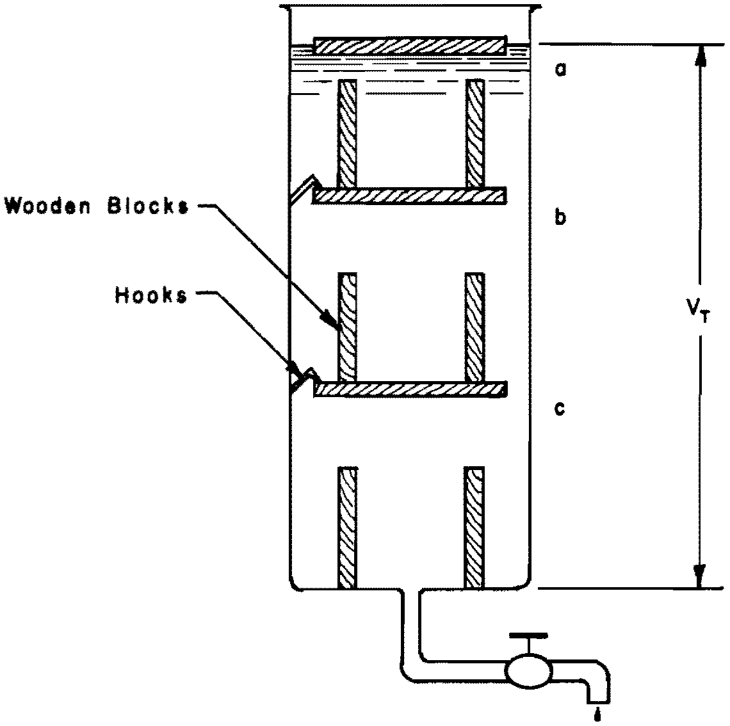


(a) Semisolid. (b) Transition. (c) Solid.

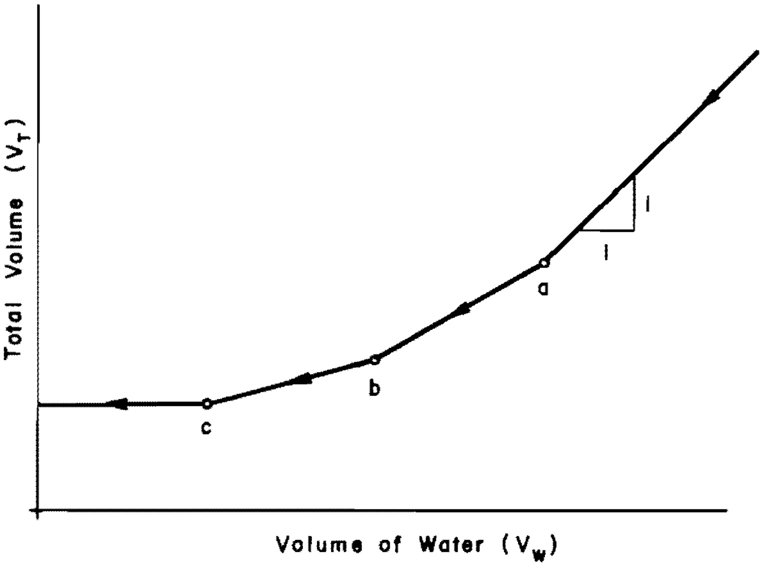


(d) Reversible shrinkage curve.

Fig 24. Aggregation models and shrinkage curve.



(a) Aggregation model.



(b) Shrinkage curve.

Fig 25. Discrete aggregation model and shrinkage curve.

When the soil is wetted again, some bonds will undoubtedly be broken and some structural integrity destroyed. But the structural integrity that remains and the bonding locations excluded to water molecules by the bond between particles is what can account for the different shape of the pF minus water content curve for soil that has been through several wetting and drying cycles.

Probability of Aggregation. This section is the most speculative of all and the ideas presented here are purely theoretical. Their applicability to real soils can and rightly should be challenged with experimental data for the purpose of proof or disproof.

Let N_T be the total number of bonds possible in a given macroscopic body of soil. Then let N be the number of bonds actually made and N_W be the number of bonds possible at a given water content. Not all of these bonds will be effective in building a structural unit but all permanent bonds will be effective in excluding "locations" for hydrogen bonding of water molecules. And if our present picture of the surface chemistry of clay minerals is correct, these hydrogen bonds must lie at the basis of the phenomenon of suction.

The total number of possible bonds N_T is assumed to be impossible to reach until all water has been dried from the soil. It will be assumed that the ratio of N_W to N_T is probably related to the slope of the shrinkage curve α as follows:

$$\frac{N_W}{N_T} = 1 - \alpha \tag{3.17}$$

and the term $(1 - \alpha)$ is the probability of aggregation. Not all of these N_W bonds will be made nor will all that are made N be retained on rewetting. It will be presumed that the total number will probably be proportional to $(1 - \alpha)$.

For drying

$$\frac{N}{N_W} = (1 - \alpha)^m \tag{3.18}$$

For rewetting, with M bonds broken of N actually made on drying the relative retained bonding is

$$R = \frac{N - M}{N} = 1 - \frac{M}{N} \tag{3.19}$$

It would seem reasonable to suppose that the smaller α became, the smaller would be the relative number of bonds broken on rewetting. For this reason, it can be assumed that

$$\frac{M}{N} = k\alpha^n \tag{3.20}$$

In order to find the relative bonding per unit volume we must find

$$\frac{1}{V}\left(\frac{N - M}{N_T}\right) = \frac{1}{V}\left(\frac{N_W}{N_T}\right)\left(\frac{N}{N_W}\right)\left(\frac{N - M}{N}\right)$$

$$= \frac{1}{V}(1 - \alpha)(1 - \alpha)^m(1 - k\alpha^n) \tag{3.21}$$

The reduction of suction below that theoretically determined from the phenomenological equation for a virgin soil is presumed to be of the same form as the attractive reduction of pressure in Van der Waal's equation of state:

$$\left[p + a\left(\frac{n}{V}\right)^2\right]\left[V - nb\right] = nRT_e \tag{3.22}$$

where

$a\left(\frac{n}{V}\right)^2$ = an effective reduction of pressure,

p = the pressure,

a = a constant involving geometry, attractive energy per bond, and number of molecules,

$\frac{n}{V}$ = a molar density, that is, the number of moles per volume,

n = the number of moles of gas,

nb = an excluded volume due to the fact that one molecule cannot penetrate another,

R = the universal gas constant,

T_e = the absolute temperature.

The formula for the attractive "force" is based on principles of statistical mechanics and use of a square-well potential for bonding strength with distance. For a discussion of the derivation of Van der Waal's equation of state, see Jackson's book, Equilibrium Statistical Mechanics (Ref 29).

In an analogous manner, the suction reduction term is proposed to have the same form:

$$\text{Reduction} = g\epsilon_b \left(\frac{N - M}{V} \right)^2 \tag{3.23}$$

where

g = a geometry term,

ϵ_b = the strength of bond between particles,

$\frac{N - M}{V}$ = bonds per unit volume.

This reduction can be cast in different terms using the formulas involving α and N_T such that

$$\text{Reduction} = g\epsilon_b \left(\frac{N - M}{N_T V} \right)^2 N_T^2 \tag{3.24}$$

$$\text{Reduction} = \left(\frac{N_T}{V} \right)^2 g\epsilon_b (1 - \alpha)^{2m+2} (1 - k\alpha^n)^2 \tag{3.25}$$

which is of the general form

$$\text{Reduction} = \beta(1 - \alpha)^q (1 - k\alpha^n)^2 \tag{3.26}$$

which can be combined with the pressure formula for the virgin suction condition to give

$$p = p_A \left(\frac{V_{T_{DRY}} - V_{W_{SL}} + V_{W_A}}{V_{T_{DRY}} - V_{W_{SL}} + V_W} \right)^{\frac{1}{\alpha}} - \beta(1 - \alpha)^q (1 - k\alpha^n)^2 \qquad (3.27)$$

In this derivation, β , k , and n are composed of terms that may vary with water content: β containing a geometry term and k and n referring to broken bond density. The expression for pF is greatly complicated now but it is seen that if the pressure correction term is constant, then eventually, with higher V_W the difference between the virgin pressure term and the pressure correction term will be $\frac{1}{\gamma_W}$ unit. In that case, pF = 0 , and the curve shown in Fig 26 can be successfully described. The constant difference causes the lower curve to be offset from the virgin curve logarithmically with higher water content. A brief study of data reported by Croney, Coleman, and Black (Ref 14) indicates that this correction term may not be quite constant and it is around 70 to 100 psi for the real soil drying curve.

The shrinkage curve referred to in these derivations is that determined from a soil in a stress-free condition. If stress is applied, obviously the effective α could eventually be any value between its initial value and zero due to the displacement of water. The number of bonds made in such a case would be proportional to the eventual value of α .

More could be said of the effects of aggregation on the moisture-suction relation. For example, the hysteresis between wetting and drying curves and the effects of repeated drying and wetting could be discussed. The latter effect will almost certainly be reflected in the changing shrinkage curve. However, the ideas presented in this section have been speculative and it is better to prove the larger notions before proposing phenomenological explanations of the more detailed effects.

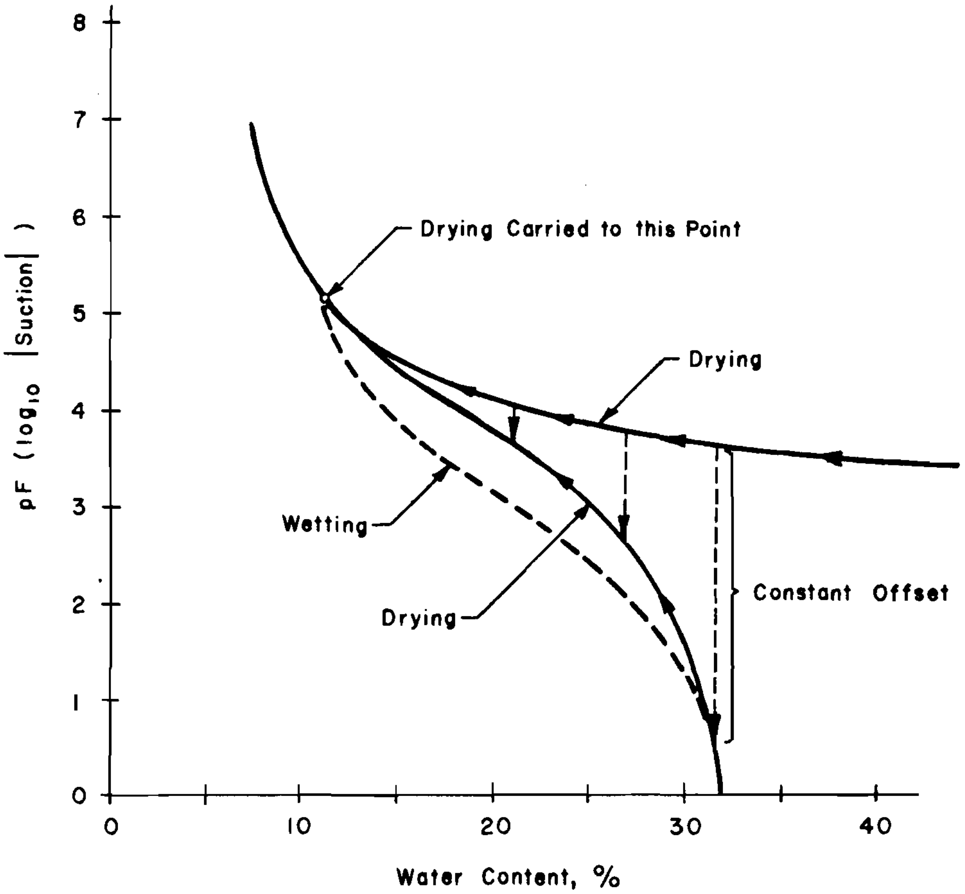


Fig 26. Drying suction-moisture relation described by offsets from the virgin drying curve.

CHAPTER 4. THE UNSATURATED SOIL STRESS PARAMETERS α AND χ

Much technical ingenuity has been used in conducting experiments to determine unsaturated soil stress parameters. An extension of the effective stress concept used in saturated soil seems to permit explanation of most of the findings, but the phenomenon of collapsing soil is an anomaly which indicates that more factors must be included before satisfactory agreement is reached. The discussion in this chapter will be limited to topics related to the factors α and χ which will be defined shortly. Both of these factors have one fundamental property in common: they are empirical constants which must be determined for each soil of interest. In addition, they will both exhibit a hysteresis characteristic based on whether the soil is being dried or wetted. Both are dependent on the way a soil changes volume and shape when subjected to stress. Neither can be viewed as a fundamental soil property, but each can be used as a convenient concept.

In Research Report No. 118-2, a theory of clay-water-air mixtures is presented which treats the total stress, the pore water pressure, and the stress in the solids as functions of volume strain TRE and mass-density water content ρ_L . Because the α and χ factors may be clearly interpreted from the mixture theory point of view throughout this chapter, the basic form of the isothermal constitutive equation is given at the outset of the discussion:

$$T_S = (\phi_2 \mathrm{TRE} + \phi_3 \rho_L) I + \phi_4 E^* \tag{4.1}$$

$$P = (\phi_5 \mathrm{TRE} + \phi_6 \rho_L) I \tag{4.2}$$

$$\rho_L = \frac{\gamma_L}{g} \frac{V_L}{V} \tag{4.3}$$

where

T_S = the partial stress tensor in the solid,

P = the mixture partial pressure in the water and the functions,

ϕ_i, i = 2 to 6 relate these measurable quantities of strain and water density to the partial stresses,

$E^* = E - 1/3\ \text{ITRE}$ = the deviatoric strain tensor,

I = the unit matrix,

γ_L = unit weight of water,

g = acceleration due to gravity,

V_L = volume of liquid,

V = total volume,

$\text{TRE} = e_{11} + e_{22} + e_{33}$ = trace of strain tensor.

In terms that are more familiar to soil mechanics, the effective stress is

$$\sigma' = \frac{1}{3}\,\text{TRT}_S = (\phi_2 \text{TRE} + \phi_3 \rho_L) \tag{4.4}$$

where TRT_S is the trace of the partial stress tensor for the solid. Also, the pore water pressure u is

$$u = \frac{V}{V_L}\,P = \frac{V}{V_L}\,(\phi_5 \text{TRE} + \phi_6 \rho_L) \tag{4.5}$$

It is significant to note that if the mixture partial pressure in water P remains constant, a decrease of water volume causes an increase of pore water "pressure." This "pressure" can approach very large numbers as the volume of liquid approaches zero. The total pressure on soil is the sum of the σ' and the P and it amounts to

$$\sigma = (\phi_2 + \phi_5)\text{TRE} + (\phi_3 + \phi_6)\rho_L \tag{4.6}$$

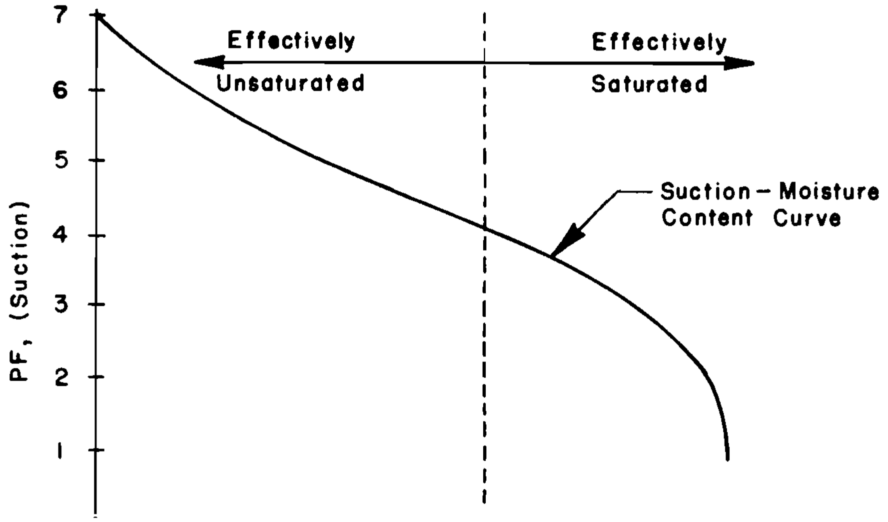
All of the equations given on the preceding page carry the assumption that air pressure does not change greatly in any process and that the momentum transferred to water or soil by the air is negligible. These equations are derived on the basis of a simplification that regards unsaturated soil as a mixture of soil solids, water, and void space with no compressibility properties. For a complete treatment of these equations, see Research Report No. 118-2.

The Coefficient α

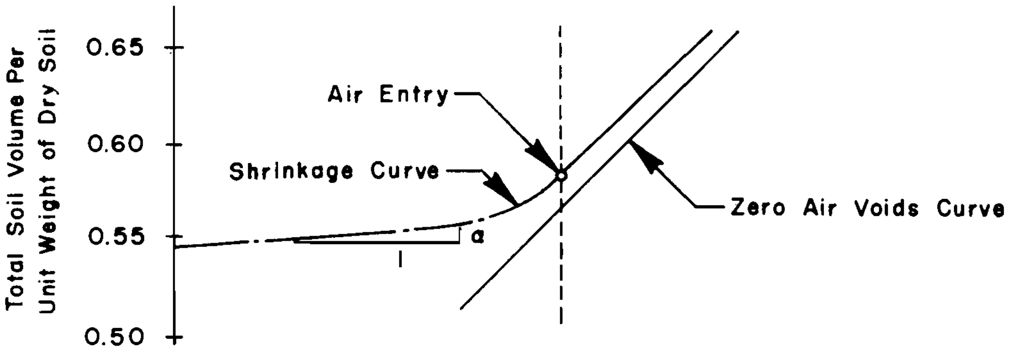
Croney, Coleman, and Black (Ref 14) presented the curves shown in Figs 27(a) and 27(b), describing them as typical of a heavy clay-soil. Figure 27(c) has been computed from these curves and is appended as helpful information only. The data describe a drying test on a sample of natural soil. The pF is increased and the total sample volume decreases linearly as moisture is lost. In fact, until the moisture content reaches 20 percent, the change in sample volume is equal to the change in water volume. The quantity α , which is the slope of this curve at each point, is equal to 1.0 along this portion of the curve. The quantity called the shrinkage limit occurs somewhere between 14 and 20 percent gravimetric moisture content. In this portion of the curve, α is undergoing a rapid change from 1.0 down to very nearly zero. From 0 to 14 percent, the curve is relatively straight with α being virtually zero.

It is recognized that in any natural soil, a certain amount of air will be trapped in the water and this volume makes up the discrepancy between the actual shrinkage curve and the zero air voids curves labeled in Fig 27(b). The soil is effectively saturated, although computation would reveal that the volume of voids exceeds the water volume by the amount of trapped air. At the 20 percent moisture content, external air begins to enter the soil and it is no longer effectively saturated. The computed saturation in Fig 27(c) is based on assuming a specific gravity of solids of 2.73. This may seem a fine distinction, but in reality it is at the 20 percent moisture-content point that the behavior of this compressible soil changes from effectively saturated to effectively unsaturated behavior. The rather high pF of about 4.25 corresponding to a total suction of -250 psi should be carefully noted.

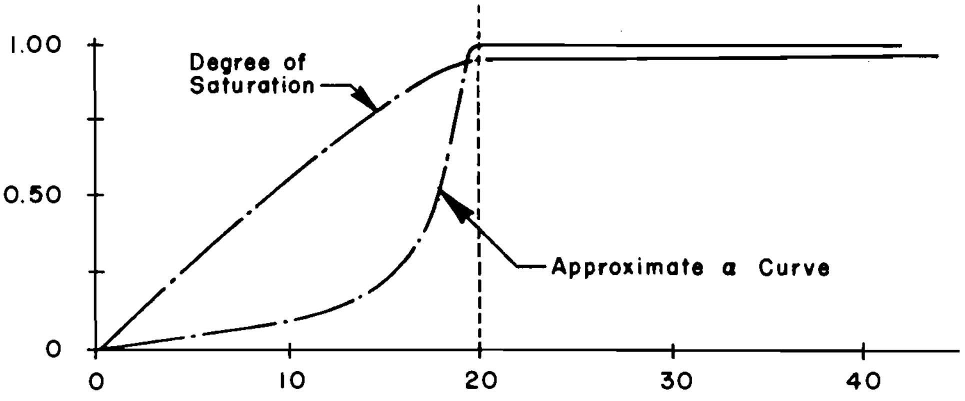
The α-Factor in Saturated Soil. If the total volume of a fixed mass of soil is V_T , the moisture included in that volume is V_w , the external air pressure on the volume is p , and the total suction in the soil water is τ ;



(a) Typical pF versus water content curve.



(b) Typical volumetric shrinkage versus water content curve.



(c) Slope (α) of shrinkage curve versus water content.

Fig 27. Shrinkage test data for a heavy clay (from curves of Croney, Coleman, and Black, Ref 14).

then the total free energy contained in that volume is given by the following expression:

$$F = pV_T - \tau V_w \gamma_w \tag{4.7}$$

If the pressure is increased by Δp , then V_T acquires an increment $-\Delta V_T$. In the same process, τ is changed by $+\Delta\tau$ and V_w is increased by ΔV_w . The new free energy is

$$F + \Delta F = (p + \Delta p)(V_T - \Delta V_T) - (\tau + \Delta\tau)(V_w + \Delta V_w)\gamma_w \tag{4.8}$$

If p and τ are constant and the volumes change, then

$$F + \Delta F = p(V_T - \Delta V_T) - \tau(V_w + \Delta V_w)\gamma_w \tag{4.9}$$

The change in free energy is

$$\Delta F = -p\Delta V_T - \tau \Delta V_w \gamma_w \tag{4.10}$$

and the following equation results:

$$\left(\frac{\Delta F}{\Delta V_w} \right) = -p \left(\frac{\Delta V_T}{\Delta V_w} \right) - \tau \gamma_w \tag{4.11}$$

Since p and τ are constant, the partial derivative of Eq 4.11 with respect to p gives the expression for α as previously defined.

$$\frac{\partial^2 F}{\partial V_w \partial p} = - \left(\frac{\partial V_T}{\partial V_w} \right)_{p\text{-const.}} = -\alpha \tag{4.12}$$

In a similar manner, V_T and V_w may be held constant and it is found that

$$\Delta F = \Delta p V_T - \Delta\tau V_w \gamma_w \tag{4.13}$$

which gives the following result:

$$\frac{\Delta F}{\Delta p} = V_T - \left(\frac{\Delta \tau}{\Delta p} \right) V_w \gamma_w \tag{4.14}$$

Since V_w is held constant, a partial derivative with respect to V_w will give another expression for α

$$\frac{\partial^2 F}{\partial p \partial V_w} = - \gamma_w \left(\frac{\partial \tau}{\partial p} \right)_{V_w\text{-const.}} = -\alpha \tag{4.15}$$

Thus, it is shown theoretically that

$$\left(\frac{\partial V_T}{\partial V_w} \right)_p = +\gamma_w \left(\frac{\partial \tau}{\partial p} \right)_{V_w} = \alpha \tag{4.16}$$

It remains to demonstrate whether Eq 4.15 has been found to be valid experimentally. Croney, Coleman, and Black (Ref 14) made both tests: a shrinkage test and a constant water-volume test to determine the change of τ with respect to p . The latter tests made on heavy clay were conducted in the effectively saturated range in which α should be 1.0. The experimental results verified this identity.

The derivation just completed carries the assumption that the unit weight of water does not change. The derivation is somewhat inconsistent for a change of volume is assumed with change of suction and this implies that water density must change. For this reason, the derivation given above is considered useful only when soil is saturated and water is virtually incompressible.

A more general approach to the problem of defining α is to use the results from the theory of mixtures given in the first section of this chapter.

The α-Factor for a Soil-Water Mixture

Two α-factors can be derived. One of these is based on results of the free swell test in which total volume and water volume change under no restraint from external pressure. The other is determined from a constant water content test in which the change of water pressure with respect to total pressure is measured. The mixture theory formulas for the two show that they are related but not necessarily the same.

Free Swell Test α-Factor. The relative mass density water content ρ_L must first be defined in terms of the specific water volume and the unit weight of water.

$$\rho_L = \frac{\gamma_W}{g}\left(\frac{V_W}{V_T}\right) \tag{4.17}$$

where

V_W = the volume of water,

V_T = the total volume,

γ_W = the unit weight of water,

g = the acceleration due to gravity.

A change of ρ_L which accompanies a change of water volume and total volume is approximately

$$\Delta\rho_L = \frac{\gamma_W}{g} \cdot \frac{1}{V_T}\left[1 - \frac{V_W}{V_T}\alpha_{FS}\right]\Delta V_W \tag{4.18}$$

where α_{FS} is the free swell test α-factor. A change in the specific total volume of soil is

$$\frac{\Delta V_T}{V_S\gamma_S} \tag{4.19}$$

and a change in the specific volume of water in a soil is

$$\frac{\Delta V_W}{V_S\gamma_S} \tag{4.20}$$

The α-factor for a free swell test is given as the ratio of change of specific total volume to specific water volume, i.e.,

$$\alpha_{FS} = \frac{\dfrac{\Delta V_T}{V_S \gamma_S}}{\dfrac{\Delta V_W}{V_S \gamma_S}} \tag{4.21}$$

Making use of the fact that

$$\frac{\Delta V_T}{V_T} = \mathrm{TRE} \tag{4.22}$$

the following equation is found:

$$\alpha_{FS} = \frac{1}{\left[\dfrac{V_w}{V_T} + \dfrac{\Delta \rho_L}{\mathrm{TRE}} \dfrac{g}{\gamma_w} \right]} \tag{4.23}$$

But because P and σ are zero, σ' is zero in this case and we find that

$$\sigma' = \phi_2 \mathrm{TRE} + \phi_3 \rho_L = 0 \tag{4.24}$$

$$P = \phi_5 \mathrm{TRE} + \phi_6 \rho_L = 0 \tag{4.25}$$

and thus

$$\mathrm{TRE} = - \frac{\phi_3}{\phi_2} \rho_L = - \frac{\phi_6}{\phi_5} \rho_L \tag{4.26}$$

which can be substituted into Eq 4.23 to give

$$\alpha_{FS} = \frac{1}{\left[\frac{V_w}{mV_T} - \left(\frac{\phi_5}{\phi_6}\right)\frac{\Delta\rho_L}{\rho_L}\frac{g}{\gamma_w}\right]} \tag{4.27}$$

This equation will be compared with that for α_P which is computed from results of a constant water content pressure test.

It should be noted that for unsaturated soil, the unit weight of water vapor is

$$\gamma_w = \frac{pmg}{RT_e} \tag{4.28}$$

All terms in this equation have been defined in Chapter 3. It is recalled that p is the vapor pressure. If this equation is substituted into Eq 4.29, an expression for unsaturated α_{FS} results:

$$\alpha_{FS} = \frac{1}{\left[\frac{V_w}{V_T} - \left(\frac{\phi_5}{\phi_6}\right)\frac{\Delta\rho_L}{\rho_L}\frac{RT_e}{mp}\right]} \tag{4.29}$$

Notice that as the vapor pressure decreases, α_{FS} also decreases.

As soil becomes more saturated, α_{FS} approaches 1.0 and the collection of terms in the expression for α_{FS} approaches a limit as follows:

$$-\frac{\phi_5}{\phi_6}\frac{\Delta\rho_L}{\rho_L}\frac{g}{\gamma_w} \rightarrow \frac{V_T - V_w}{V_T} \tag{4.30}$$

The last two relations show that the limits of α_{FS} are zero at zero water content and 1.0 when soil is effectively saturated.

<u>Constant Water Content Pressure Test α-Factor</u>. In this test ρ_L remains constant and the functions ϕ_2 , ϕ_3 , ϕ_5 , and ϕ_6 have specific values

based on the value of ρ_L . The ratio of water partial pressure P to total isotropic pressure is

$$\frac{P}{\sigma} = \frac{1}{\left(1 + \frac{\phi_2}{\phi_5}\right)} \frac{\mathrm{TRE} + \left(\frac{\phi_6}{\phi_5}\right) \rho_L}{\mathrm{TRE} + \frac{(\phi_3 + \phi_6)}{(\phi_2 + \phi_5)} \rho_L} \tag{4.31}$$

As has been stated previously, the pore water pressure change from some equilibrium condition u is the equivalent of $\frac{PV}{V_L}$. Thus, the factor relating the change of pore water pressure to the total isotropic pressure will be

$$\alpha_P = \frac{u}{\sigma} = \frac{V}{V_L} \frac{1}{\left(1 + \frac{\phi_2}{\phi_5}\right)} \frac{\mathrm{TRE} + \left(\frac{\phi_6}{\phi_5}\right) \rho_L}{\mathrm{TRE} + \frac{(\phi_3 + \phi_6)}{(\phi_2 + \phi_5)} \rho_L} \tag{4.32}$$

It seems somewhat unlikely that α_{FS} and α_P are necessarily equal. Proofs of the equality of α_{FS} and α_P may be derived for the unsaturated case in which a change in total pressure P causes changes in water vapor volume and pressure in a manner similar to that shown for the effectively saturated condition. The assumption that is made in this case is that the adsorbed water does not change its free energy under imposed total pressure. This does not seem to be a safe assumption and consequently no conclusion is drawn on the equality of α_{FS} and α_P for all conditions of soil moisture.

The Coefficient χ

As suction increases, the volume of a clay soil decreases and in turn the reverse is true. The question arises whether suction is the equivalent of pressure in changing volume. Blight (Ref 8) conducted tests on an effectively saturated, remolded clay over a range of pressures from 0 to 90 psi. He subjected the sample to mechanical pressure in a triaxial cell, to pore water-pressure change due to expelling by raised air pressure, and to centrifuging. In each case, he measured volume changes that plotted along an identical line.

The interpretation is that each of these processes cause identical volume change in saturated soil, and the equivalence of suction and pressure is thus experimentally demonstrated. The maximum pressure to which the sample was subjected was the equivalent of a suction of about pF 3.8 .

Aitchison (Ref 2) tested samples of Waite loam, kaolin, red clay, and black clay in consolidometers and in various suction devices and found that void-ratio change was reasonably close for the latter three soils regardless of process. However, the Waite loam was compressed in the consolidometer but was unaffected by suction change although pF's as high as 4.5 were imposed on the soil. Aitchison rated Waite loam as incompressible and the other three as highly compressible. He based his terminology on the results of the suction test. These tests illustrate an important consideration of the equivalence of pressure and suction.

The Waite loam had a clay content of 17 percent. The other soils had at least 65 percent clay. The Waite loam was composed primarily of a chemically and electrically inert material which, though compressible to mechanical pressure, has no significant internal adsorption or hydration head or double layer attraction to aid in the volume-change process. In addition, the capillary interparticle forces generated by draining water apparently were not sufficient to compress the soil. The larger grain size and consequently larger pore size permits drainage at lower suction values. The grain structure of the loam was undoubtedly more stable than that of the clays.

It becomes qualitatively clear that at moisture contents below the air entry moisture content, the pore water pressure becomes much less significant in causing volume change and that in this range, clays must rely on other components of suction to perform some of the work of volume reduction or increase.

The fraction of the net pore water and pore air pressure which causes volume change may be defined as the coefficient χ . Above the point where air enters the soil, χ is essentially equal to 1.0. Below this critical moisture content, χ falls rapidly to a value close to zero. Some historical and theoretical considerations are needed to properly explain the development of χ .

In a lecture in Oslo, Norway, in 1955, A. W. Bishop proposed a tentative expression for effective stress

$$\sigma' = \sigma - u_a + \chi(u_a - u_w) \tag{4.33}$$

The lecture was later published in Teknisk Ukeblad, No. 39, October 22, 1959. The symbols are defined as follows:

σ' = effective stress,

σ = total applied stress,

u_a = pressure in pore air,

u_w = pore water pressure,

χ = a proportionality factor which is a function of the degree of saturation and the state of stress itself.

Normally, any definition of χ states that it must be measured experimentally. This presumes that all quantities σ', σ, u_a, and u_w are measured and χ is computed from Eq 4.33. In some cases it is possible to measure all but the effective stress. This must be obtained from some other formula based on the particular process of interest. For example, if effective stress is supposed to contribute to shear strength, experimentally measured shear strength is used to calculate σ' according to a proposed Mohr-Coulomb formula. This calculated value of σ' is then used in Eq 4.33 to compute χ. Another example is more closely related to the present discussion. Effective stress change is assumed to cause volume change. A formula that appears to satisfactorily predict volume change in saturated soils is presumed to be valid in the unsaturated range.

For example, Blight (Ref 8) works with the following form:

$$\left(\frac{\Delta V}{V} \right) = c(\Delta\sigma') \tag{4.34}$$

The soil compressibility c is assumed the same in both saturated and unsaturated ranges. Intuitively, this seems valid. Drained compression tests measure volume change after all excess pore-water pressure has been relieved, and thus the entire load is carried in the mineral skeleton. The volume change measured should be simply a function of the strength of the soil structure itself. The amount of pore water contained in the soil should be irrelevant. This is the assumption: Blight gives some substantiation for the notion in an expansive clay but also indicates test data for sand that tend to destroy faith in the assumption.

Values of χ calculated from volume change and from shear strength data for the same soil do not tend to agree. It is apparent from the preceding discussion that at least three reasons for this can be given:

(1) The effective stress equation is incorrectly written.

(2) Either the shear strength formula or the volume change formula or both are based on incorrect assumptions.

(3) There has been an experimental error.

Possibility (3) is always present, of course, but possibilities (1) and (2) are certainly worth investigation. Possibility (2) will be discussed first.

Newmark's opening address to the 1960 Research Conference on Shear Strength of Cohesive Soils (Ref 36) contains the suggestions that both volume strain and shearing strength are functions of isotropic stress, octahedral shear stress, and a factor ϕ which contains the third stress invariant and the octahedral shear stress. Of course, other factors such as stress rate could be included and a thoroughly comprehensive theory should include such factors. However, in view of Newmark's suggestion alone, neither the shear stress formulas nor the volume strain formulas used by experimenters in computing effective stress are completely valid. The form of the equation for volume strain used by experimenters is in question even for saturated clays. Juarez-Badillo (Ref 30) proposed the following form which has been able to predict D. J. Henkel's experimental data (Ref 26) on drained tests quite well:

$$\left(\frac{\Delta V}{V} \right) = -\gamma \left(\frac{\Delta\sigma}{\sigma} \right) \tag{4.35}$$

Of course, in drained tests on saturated soils it is assumed that

$$\frac{\Delta\sigma}{\sigma} = \frac{\Delta\sigma'}{\sigma'} \tag{4.36}$$

where

γ = compressibility coefficient,

$\Delta\sigma$ = change in total stress,

$\Delta\sigma'$ = change in effective stress.

If the computed value of effective stress is substantially in error, then the value of χ computed in turn from σ' should be suspect. The coefficient χ , a constant, is made to take the place of a function. Even plots showing variation of χ with saturation are probably not completely valid since χ probably depends on stress level, shearing stress, and shearing stress distribution as well as saturation. Blight (Ref 8) remarks that in the case of a collapsing sand a continuous curve of χ versus suction cannot be developed experimentally because as suction is reduced, sudden volume decrease is caused by sensitivity to "concentrations of secondary shear stress which are unavoidable in any body of soil, even under laboratory conditions."

There is much controversy over the validity of Eq 4.33 as applied to unsaturated soils because of the misuse or misunderstanding of the factor χ . Basically speaking, the total stress applied to a three phase system composed of soil, water, and air will be carried by stress in each of the components of the system. The amount of stress carried by each is the complex problem which the coefficient χ is supposed to treat. That it has not been completely successful to date may be due to the ways experimenters measure it. This much is certain: when soil is saturated, χ is one; when soil is dry, χ is zero. It is actually possible to compute negative values of χ in the intermediate range when a soil collapses with a positive increase of effective stress. As an example of this, Eq 4.34 is used to compute effective stress for a hypothetical experiment. Compression is assumed positive and air pressure is assumed at atmospheric. Constant stress is maintained and suction is decreased by $+\Delta u_w$. The soil sample collapses, changing in volume by $+\Delta V$. Equation 4.33 gives the following relation:

$$\sigma' = \sigma - \chi u_w \tag{4.37}$$

$$\Delta\sigma' = -\chi \Delta u_w \tag{4.38}$$

and Eq 4.10 gives the additional information that

$$+\frac{\Delta V}{V} = c\Delta\sigma' \tag{4.39}$$

Obviously, $\Delta\sigma'$ is positive; and because Δu_w is also positive, χ must be negative if it is a constant. If χ varies with stress then the change

$$\Delta(\chi u_w) = \chi\Delta u_w + u_w\Delta\chi \qquad (4.40)$$

must be less than zero. If χ is sensitive to shearing stress concentrations as has been suggested, $\Delta\chi$ can be of such negative magnitude to cause the change term to be negative. In any case, unusual values of χ can be predicted from experimental data, and it appears that the coefficient need not be positive and between zero and one.

A very thorough study of the χ-factor in incompressible spherical media has been presented by A. B. Walsh-Sparks (Ref 58) who noted that surface tension should be included in the expression for χ . If it is not included, the χ-factor can be found to be infinite at the point at which isolated bubbles change to open pores and pendular water. In this condition, water pressure and air pressure theoretically become equal and the expression for χ

$$\chi = \frac{\sigma - u_a - \sigma'}{u_W - u_a} \qquad (4.41)$$

becomes undefined. Walsh-Sparks presents a formula for computing χ based on the geometry, changes of water content and pressure, and presence of occluded air bubbles. He has suggested formulas to use for the χ-factor for all ranges of saturation. The formulas are restricted to incompressible soils, however, and do not include the effects of particle bonding.

In the next section, a derivation of the χ-factor is given which follows the development of J. D. de Wet (Ref 16).

Derivation of the χ-Factor for High Water Content

It must be made explicit that the χ-factor is an empirical function which depends for its value on the type of stress applied to the soil. For example, different authors have proposed that the χ-factor be used in processes involving shearing strength alone. Another such factor η with the same form of relation to effective stress has been proposed for isotropic pressure and volume change tests. The volume change process is of interest in this report and no other process will be considered. The development to be presented in this section is due mainly to de Wet (Ref 17).

When some positive air pressure p is applied to an element of soil surrounded by a membrane, that pressure is felt throughout the element. Before any drainage takes place, this applied pressure will increase the presence in the soil solids, the water, the air, and the absorbed water. When drainage occurs, the fluid stresses are gradually transferred to the soil skeleton.

The initial pressure condition can be described by the following identity:

$$p = p \frac{V_S}{V_T} + p \frac{V_W}{V_T} + p \frac{V_A}{V_T} + p \frac{V_B}{V_T} \tag{4.42}$$

where

V_S = the volume of solids,

V_W = the volume of water,

V_A = the volume of air,

V_B = the volume of absorbed water,

V_T = the total volume.

To effect a differentiation later on, the overall pressure p will be replaced by the symbols for pressure in each of the soil constituents. For this reason, Eq 4.42 is rewritten as

$$p = p_S \frac{V_S}{V_T} + p_W \frac{V_W}{V_T} + p_A \frac{V_A}{V_T} + p_B \frac{V_B}{V_T} \tag{4.43}$$

where

p_S = a mean pressure in the solids,

p_W = the mean pressure in water,

p_A = the mean pressure in the air,

p_B = the mean pressure in the absorbed water.

This is the equilibrium equation for pressure in all phases of the soil system. In dynamic processes, momentum terms should be considered but here they are assumed to be small. It is at this point where two different χ-factors must be defined

(1) χ_E - from the equilibrium equation, and
(2) χ_T - from the time derivative of the equilibrium equation.

The first gives the dependence of effective stress on air, water, and absorbed water pressure for a small change of total pressure. The second gives the rate of change of the relation between effective stress and water pressure, a relation which obviously changes with time.

Equilibrium χ-Factor, χ_E. This χ-factor applies at any equilibrium situation or in any state of impending volume change. It is applicable when stress is applied and before drainage has taken place to any appreciable degree.

The soil pressure p_S is not the actual effective stress because soil grains are "buoyed" up by fluid pressure acting on them. The proportion of soil surface acted upon by each fluid would ideally be determined by the stereogonic angle subtended by each phase. These proportions will be assumed to be:

(1) k_W for water,
(2) k_A for air, and
(3) $1 - k_A - k_W$ for absorbed water.

The effective stress is given by the following relation:

$$\sigma' = p_S \frac{V_S}{V_T} - \left[k_W p_W \frac{V_S}{V_T} + k_A p_A \frac{V_S}{V_T} + (1 - k_W - k_A) p_B \frac{V_S}{V_T} \right] \qquad (4.44)$$

But from Eq 4.43 it is apparent that

$$p_S \frac{V_S}{V_T} = p - p_W \frac{V_W}{V_T} - p_A \frac{V_A}{V_T} - p_B \frac{V_B}{V_T} \qquad (4.45)$$

which when substituted into Eq 4.44 gives

$$\sigma' = p - p_W \left[\frac{V_W}{V_T} + k_W \frac{V_S}{V_T} \right] - p_A \left[\frac{V_A}{V_T} + k_A \frac{V_S}{V_T} \right]$$

$$- p_B \left[\frac{V_B}{V_T} + (1 - k_A - k_W) \frac{V_S}{V_T} \right] \qquad (4.46)$$

Recall that at the instant the pressure p is imposed all of the other pressure changes are equal to p. Thus, Eq 4.46 may be rewritten substituting p_A for p and p_W for p_B to obtain

$$\sigma' = (p_A - p_W) \left[\frac{V_W + V_B}{V_T} + (1 - k_A) \frac{V_S}{V_T} \right] \qquad (4.47)$$

and because the degree of saturation S is

$$S = \frac{V_W + V_B}{V_T - V_S} \qquad (4.48)$$

the expression for effective stress is found to be

$$\sigma' = (p_A - p_W) \left[S + (1 - k_A - S) \frac{V_S}{V_T} \right] \qquad (4.49)$$

Thus, for this equilibrium condition,

$$\chi_E = S + (1 - k_A - S) \frac{V_S}{V_T} \tag{4.50}$$

When the soil is saturated, $k_A = 0$, $S = 1$, and $\chi_E = 1$.

The factor $(1 - k_A)$ depends upon the values of k_W and k_B which in turn depend upon temperature, suction level, surface tension, and electrical and geometrical character of mineral surfaces.

It must be emphasized that χ_E is the factor that governs sudden processess such as quick shear tests where no drainage is permitted and where final condition is reached in a relatively short period of time.

Time-Rate χ-Factor, χ_T . In a process in which drainage is permitted, the ratio of water pressure to effective stress may change continuously with time. As a general rule, the value of χ_T may start at χ_E and decay to a value of zero as drainage proceeds. Before beginning with the derivation of χ_T it will be assumed that the excess pressure in the water and in the absorbed water are the same, i.e.,

$$p_W = p_B \tag{4.51}$$

Based on this assumption, Eq 4.46 will read as follows:

$$\sigma' = p - p_A + (p_A - p_W) \left[\frac{V_W + V_B}{V_T} + (1 - k_A) \frac{V_S}{V_T} \right] \tag{4.52}$$

Now the ratio of effective stress to the difference in air and water pressure is formed and the time rate of change of this quantity is essentially x_T .

$$\frac{\sigma'}{-p_W + p_A} = R = \frac{p - p_A}{-p_W + p_A} + \left[\frac{V_W + V_B}{V_T} + (1 - k_A) \frac{V_S}{V_T} \right] \tag{4.53}$$

The time derivative of R gives an expression for the instantaneous value of χ .

$$\frac{dR}{dt} = \chi_T = \frac{p(\dot{p}_W + \dot{p}_A) + p_W\dot{p}_A - p_A\dot{p}_W}{(p_A - p_W)^2}$$

$$- \frac{1}{V_T} (\dot{V}_W + \dot{V}_B - V_S\dot{k}_A) + \frac{\dot{V}_T}{V_T^2} [V_T - V_A - k_AV_S] \qquad (4.54)$$

where the dot over a symbol indicates a time derivative of that quantity. This is a complicated expression which is capable of evaluation only if all of the volumes, pressures, and rates of all constituents can be measured or neglected. The equation is valid for all conditions except one: the case in which water pressure and air pressure are equal. In this case, the assumption in Eq 4.51 is probably invalid.

A very careful distinction must be made between the χ_E and χ_T because of their conditions of validity. The value of χ_E may be determined experimentally only by using one additional assumption: that the pressure-volume change ratio for a material is unique whether it is saturated or not. Then consolidation or expansion test data on a saturated sample may be compared with the suction-shrinkage volume change data for a sample with the same initial void ratio. Identical volume changes may be assumed to be caused by equivalent pressure systems. The ratio of equilibrium mechanical pressure to equilibrium suction converted to water pressure gives an experimental value of χ_E from which k_A may be determined.

Derivation of the χ-Factor for Low Water Content

All of the derivations presented in the previous chapter are valid for low water contents. However, certain simplifications are possible in these unsaturated ranges.

Unsaturated χ_E . At low water contents where the pF is 4.0 or above, k_A approaches 1 and the equilibrium χ_E approaches

$$\chi_E = \frac{V_W + V_B}{V_T} = nS \tag{4.55}$$

where n is the porosity of the soil and S is the degree of saturation. This means that for a soil with $n = 0.40$ the slope of the χ_E - S curve will be lower than 1.0 until k_A begins to drop toward zero. A typical χ - S curve to be expected from these predictions is shown in Fig 28.

The general shape of these curves is given by A. D. Walsh-Sparks (Refs 58 and 59) and by J. A. de Wet (Ref 17).

Unsaturated χ_T. In the region of fairly dry soil k_A is nearly 1 and will not change rapidly under imposed load. Neither air nor water pressure will be greatly altered by imposition of mechanical pressure p and their rates are negligible. The rate of change of water and absorbed water volume will be small. Under these conditions, the value of unsaturated χ_T approaches

$$\chi_T = \frac{\dot{V}_T}{V_T^2} [V_T - V_S - V_A] \tag{4.56}$$

A further assumption that at these low water contents

$$V_T \cong V_S + V_A \tag{4.57}$$

shows that

$$\chi_T = 0 \tag{4.58}$$

or the time rate of strain of the soil is nearly zero. This means that the soil is nearly elastic or nearly rigid.

The Relation of α to χ

In the literature, some attempts have been made to equate the α and χ factors. Based on observations in the previous sections, it appears that this

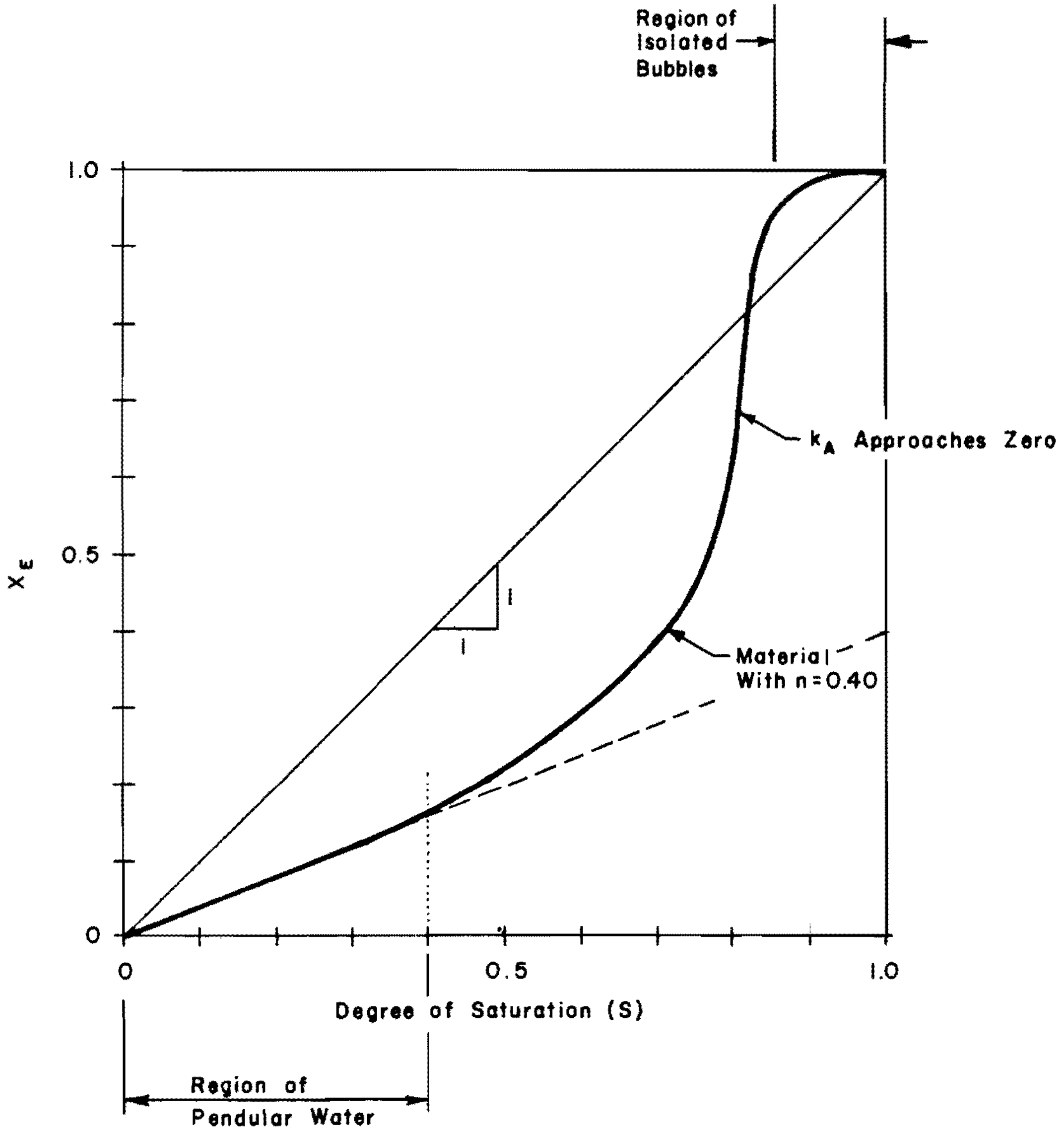


Fig 28. A typical χ - S curve.

need not be the case at all. To review briefly, two α-factors and two χ-factors have been delineated: α_{FS} , α_P , χ_E , and χ_T . All of these except χ_E have instantaneous values, i.e., values that change with time. The χ_E factor alone is determined at some equilibrium state of impending change. The following table summarizes definitions and testing techniques required to obtain these values.

Possibly no other ratio is more pertinent to a volume change process than the current ratio of effective stress to total stress. As the value of excess pore water pressure approaches zero, the ratio $\frac{\sigma'}{p}$ approaches 1.0. The initial value of this ratio is obtained from the effective stress equation. Air pressure is considered to be atmospheric.

$$\sigma' = p - \chi_E p_W \tag{4.59}$$

But from previous equations it was found that initially

$$p_W = \alpha_{PO} p \tag{4.60}$$

where α_{PO} is the initial value of α_p . With this substitution it is obvious that

$$\left(\frac{\sigma'}{p} \right)_{t = o} = 1 - \chi_E \alpha_{PO} \tag{4.61}$$

and for all subsequent times

$$\frac{\sigma'}{p} = 1 - \chi_E \alpha_p \tag{4.62}$$

These points are covered more completely in the sections to follow.

It is significant that the product of χ_E and α_p in the unsaturated range is precisely $\frac{P}{\sigma}$, the ratio of the mixture partial water pressure P to the total imposed pressure σ . This observation leads to the conclusion that the product $\chi_E \alpha_p$ represents the ratio $\frac{P}{\sigma}$ for both saturated and unsaturated

TABLE 7. DEFINITIONS OF THE α AND χ-FACTORS

Factor	Formula Definition	Definition	Tests and Quantities to be Measured in the Tests
α_{FS}	$\dfrac{\frac{\Delta V_T}{V_S \gamma_S}}{\frac{\Delta V_W}{V_S \gamma_S}}$	Ratio of change of total specific volume to change of specific water volume. Changes with water content.	Free swell test. <u>Measure</u> Total volume V_T . Total water volume V_W . Dry weight of soil $V_S \gamma_S$.
α_P	$\dfrac{u}{\sigma}$	Ratio of change of pore water pressure to change of total pressure. Has initial value and changes with time to zero.	Undrained compression test for initial value conducted at atmospheric air pressure. Drained compression test for subsequent values conducted at atmospheric air pressure. <u>Measure</u> Constant total pressure σ . Pore water pressure u .

(continued)

TABLE 7 (continued)

Factor	Formula Definition	Definition	Tests and Quantities to be Measured in the Tests
χ_E	$\frac{\sigma'}{p_W - p_A}$	Ratio of equilibrium effective stress to equilibrium difference in pore air and pore water pressure. Has one value for each porosity and degree of saturation.	Companion tests. Shrinkage test. Consolidation test. <u>Measure</u> In shrinkage test: suction-convert to p_W water content. In consolidation test: total pressure and equilibrium void ratio.
χ_T	$\frac{d}{dt}\left(\frac{\sigma'}{p_A - p_W}\right)$	Rate of change of ratio of effective stress to difference in pore air and pore water pressure. Total pressure does not change.	Drained compression test with constant total pressure. <u>Measure</u> Pressure of air and water. Volume of air, water, and total. Rate of pressure change of air and water. Rate of volume change of air, water, and total.

ranges and lends credence to the notion that the important physical quantity in volume change processes is the mixture partial water pressure P .

Effective Stress in a Constant Water Content Test

Another hypothetical experiment will indicate the way in which α and χ enter the effective stress equation. A clay is initially unstressed by external load. The pressure in the pore water is u_o , the total suction is τ_o , the adsorptive head is A_o , salt concentration is c_o , and bubble effective radius is r_o . The expression for the initial pore water pressure is as follows:

$$u_o = \gamma_w \tau_o - A_o \gamma_w + C_o RT_e - \frac{2T_s}{r_o} \tag{4.63}$$

Now an exterior stress σ is applied in a constant water content test. The new pore water pressure will be u . The new total suction will be τ and the change in suction is $\Delta\tau$.

$$\tau = \tau_o + \Delta\tau = \tau_o + \left(\frac{\alpha_{po}}{\gamma_w} \right) \sigma \tag{4.64}$$

The same water is contained in the soil throughout the application of σ , and this leads to the assumption that there is no change in adsorptive head and salt concentration. The new pore water pressure is

$$u = \gamma_w \tau - A_o \gamma_w + C_o RT_e - \frac{2T_s}{r} \tag{4.65}$$

The change in pore water pressure is as follows:

$$u - u_o = \gamma_w (\tau - \tau_o) - 2T_s \left(\frac{1}{r} - \frac{1}{r_o} \right) \tag{4.66}$$

But $(\tau - \tau_o)$ is equal to

$$\Delta\tau = \frac{\alpha_{po}}{\gamma_w} \sigma \tag{4.67}$$

and the change in pore water pressure can be expressed as

$$\Delta u = \alpha_{po}\sigma - 2T_s \left(\frac{1}{r} - \frac{1}{r_o} \right) \tag{4.68}$$

The effective stress induced by application of the stress σ is given in the following equation:

$$\sigma' = \sigma - u_a + \chi_E \left[u_a - \alpha_{po}\sigma + 2T_s \left(\frac{1}{r} - \frac{1}{r_o} \right) \right] \tag{4.69}$$

The terms of this equation may be rearranged to indicate the contributions of stress components to effective stress.

$$\sigma' = \sigma(1 - \alpha_{po}\chi_E) - (1 - \chi_E)u_a + 2T_s\chi_E \left(\frac{1}{r} - \frac{1}{r_o} \right) \tag{4.70}$$

Since effective stress is assumed proportional to volume change, it appears that volume change is proportional to the product of α_{po} and χ_E and inversely proportional to the effective bubble radius.

The surface tension term is not important in water that is far enough away from the air-water interface of an isolated air bubble. The term can be of significant size when clay becomes effectively unsaturated. The effect of the surface tension is to increase the effective stress, and, consequently, to increase the volume change due to applied stress.

Effective Stress Increase with Drainage

Now if the water is allowed to drain from the soil, the pore water pressure tends to return to u_o, the suction to τ_o, and so on. This is a time-dependent consolidation process, and it has as its final objective that

$$\Delta u = 0 = \alpha_p \sigma - 2T_s \left(\frac{1}{r} - \frac{1}{r_o} \right) \tag{4.71}$$

The effective bubble radius will return to r_o. However, σ remains and this means α_p must approach zero with Δu. This kind of process is needed to explain the action of clay which is in the range of effective saturation. Reference to Eq 4.70 will indicate that when α_{po} and χ_E are both 1.0, then applied stress causes no effective stress. This is true when the soil is initially loaded, but it is no longer true once drainage has occurred. Thus, the coefficient α_p starts at the value dictated by the initial water content of the soil and decays with time to zero.

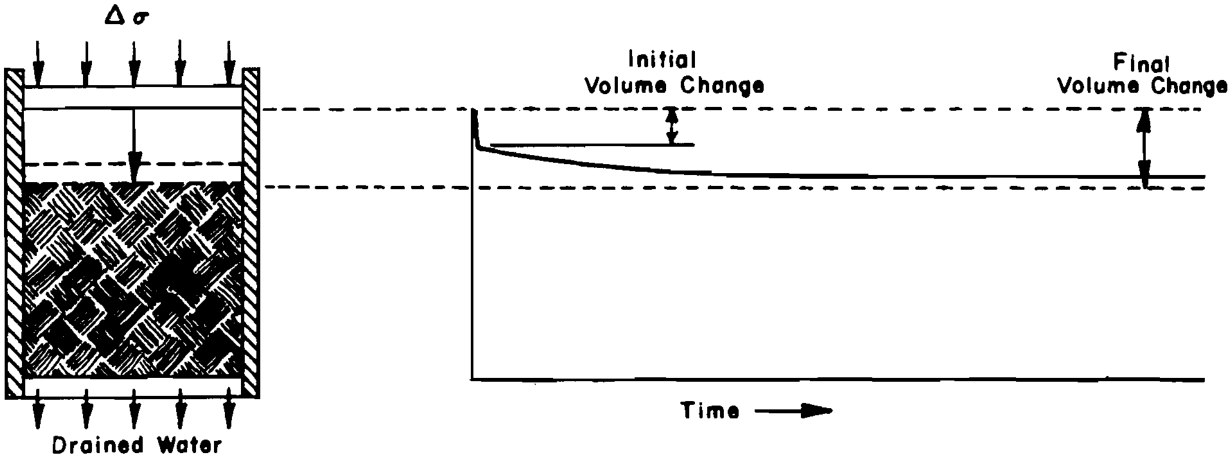
The coefficient χ_E could remain equal to 1.0 during the whole drainage process provided the soil remains effectively saturated. It is time-dependent only to the extent that it depends on the structural characteristics of the soil-air-water system which may change with time as a result of volume reduction or increase and shearing.

If a sample of soil that is effectively unsaturated is tested in compression and drainage is allowed, there is an immediate decrease in volume as the load is applied. After this initial volume decrease, there is a time-dependent volume decrease that takes place until the final volume is reached. An illustration of this process is given in Fig 29(a). This behavior can be related to the α_p and χ_E factors. An over-simplified version of this relation may be shown by considering less general forms of Eqs 4.34 and 4.70.

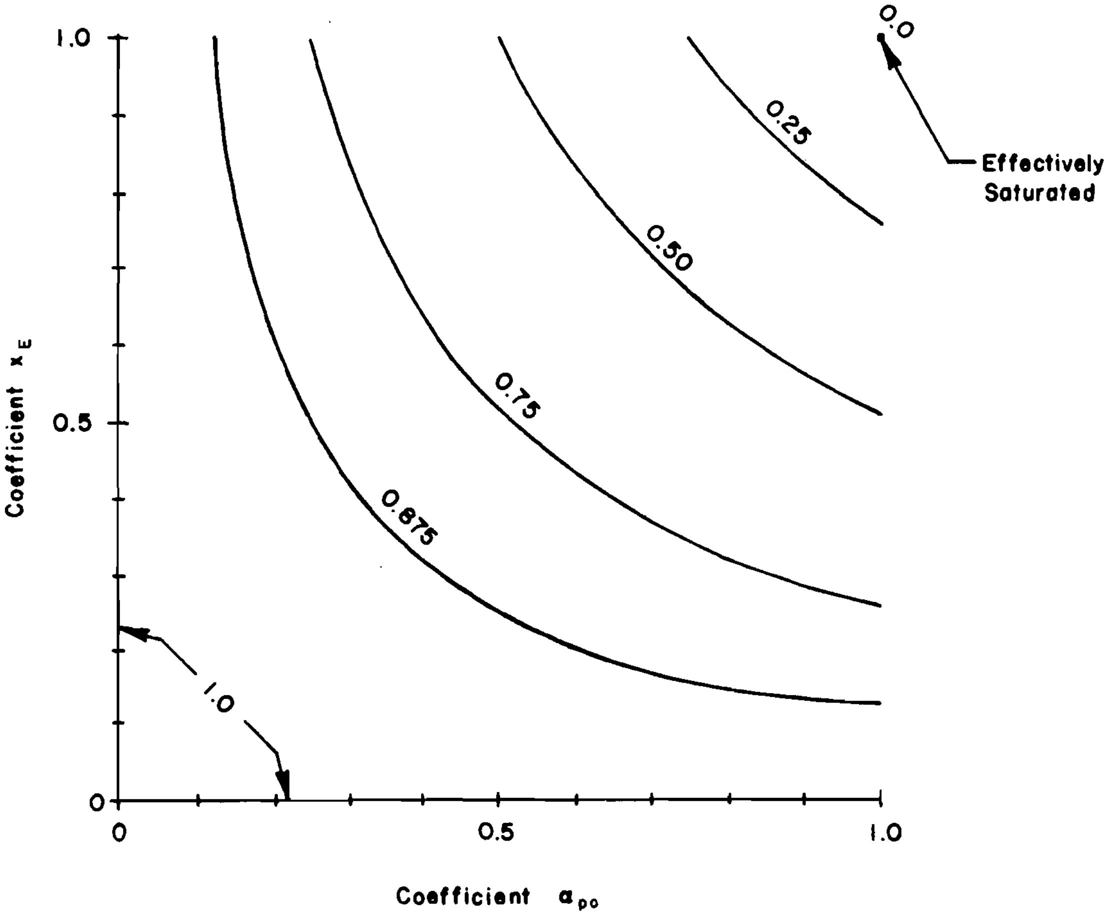
$$\left(\frac{\Delta V_T}{V_T} \right) = c\Delta\sigma' \tag{4.72}$$

$$\Delta\sigma' = \Delta\sigma(1 - \alpha_{po}\chi_E) \tag{4.73}$$

One other relation is necessary: that the compressibility c is taken from the final volume change which occurs after all excess pore water pressure has been relieved, and the entire stress change is carried by the soil structure.



(a) Volume change in unsaturated soils.



(b) A relation between χ_E and α_{po} for selected values of $\frac{\Delta V_{Ti}}{\Delta V_{Tf}}$.

Fig 29. Illustrated properties of partially saturated soils.

$$\left(\frac{\Delta V_T}{V_T} \right)_{final} = c\Delta\sigma \tag{4.74}$$

The initial volume change is equal to

$$\left(\frac{\Delta V_T}{V_T} \right)_{initial} = c\Delta\sigma(1 - \alpha_{po}\chi_E) \tag{4.75}$$

If the volume changes are related to the same total volume, the two equations can be manipulated to give an expression for χ_E in terms of α_{po} .

$$\chi_E = \frac{1 - \frac{\Delta V_{Ti}}{\Delta V_{Tf}}}{\alpha_{po}} \tag{4.76}$$

A graph of this function is shown in Fig 29(b). Thus, if α_{po} is known from Eq 4.76 and initial and final volume changes are known from a consolidation test, then an approximation of χ may be established.

While this discussion has considered volume change as a decrease, the same remarks can be made about a swelling test. Initial and final swell upon load reduction in a consolidometer or other such device can be used in this determination of the χ_E-factor.

If soil volume is assumed to change according to the Juarez-Badillo equation (Eq 4.35)

$$\left(\frac{\Delta V_T}{V_T} \right) = -\gamma \left(\frac{\Delta\sigma'}{\sigma'} \right) \tag{4.77}$$

then the equation relating α and χ becomes

$$\alpha_{po}\chi_E = 1 - \frac{\left[\left(1 - \frac{\Delta V_{Ti}}{V_{To}} \right)^{-\frac{1}{\gamma}} - 1 \right]}{\left[\left(1 - \frac{\Delta V_{Tf}}{V_{To}} \right)^{-\frac{1}{\gamma}} - 1 \right]} \tag{4.78}$$

where

V_{To} = the original total volume,

ΔV_{Ti} = the initial change in total volume,

ΔV_{Tf} = the final change in total volume,

γ = the compressibility coefficient.

Because this equation involves stress level and initial volume as well as initial and final changes in volume, it may be a better representation of the relation between α_{po} and χ_E.

The preceding illustration is not an attempt at being strictly rigorous in explaining unsaturated soil behavior. It is meant to note that certain fundamental relations do exist among the unsaturated soil parameters.

Change of Suction with Water Content in Expansible Soils

One final development will give an indication of the way suction changes with water content if drainage is allowed. This development will demonstrate some of the mechanisms which make overburden pressure worth considering in "heavy" or compressible clays.

The volumetric water content of an element of soil θ is given below:

$$\theta = Sn = \frac{SV_V}{V_T} \tag{4.79}$$

where

n = the porosity,

S = the degree of saturation,

V_V = the volume of voids,

V_T = the total volume.

A change of volumetric water content would be affected as follows:

$$\Delta\theta = \frac{S}{V_T}\Delta V_V + \frac{V_V}{V_T}\Delta S - \frac{V_V}{V_T^2} S\Delta V_T \qquad (4.80)$$

But since

$$V_T = V_S + V_V \qquad (4.81)$$

where

V_S = the volume of solids

then

$$\Delta\theta = \frac{S}{V_T}(1 - n)(\Delta V_V) + \frac{V_V}{V_T}(\Delta S) - \frac{V_V}{V_T^2} S(\Delta V_S) \qquad (4.82)$$

$$\Delta\theta = \underbrace{\theta(1 - n)\frac{\Delta V_V}{V_V}}_{\substack{\text{Term}\\ \text{(A)}}} + \underbrace{n(\Delta S)}_{\substack{\text{Term}\\ \text{(B)}}} - \underbrace{\frac{\theta}{V_T}(\Delta V_S)}_{\substack{\text{Term}\\ \text{(C)}}} \qquad (4.83)$$

In Eq 4.83, Term (A) has to do with the compressibility of the soil structure. Scott (Ref 53) has shown how Term (B) changes with pore water pressure, but Shuurman (Ref 54) has proven experimentally that such expressions are valid only for soil that is between 85 and 100 percent saturated. Term (C) is related to the compressibility of the soil solids. This term is negligible in the range of stresses that occur near the surface of the earth. In the rest of this discussion, Term (C) will be neglected. Since the degree of saturation is

$$S = \frac{V_w}{V_V} = \frac{V_V - V_A}{V_V} = 1 - \frac{V_A}{V_V} \qquad (4.84)$$

then the change in saturation is

$$\Delta S = -\frac{\Delta V_A}{V_V} + \frac{V_A}{V_V}\left(\frac{\Delta V_V}{V_V}\right) \tag{4.85}$$

which can be substituted into Eq 4.83 to give Eq 4.86.

$$\Delta\theta = \frac{\theta}{n}\left(1 - n - \frac{V_A}{V_w}\right)\frac{\Delta V_V}{V_T} - \frac{\Delta V_A}{V_w} \tag{4.86}$$

where V_a is the volume of air. Because the total volume change is equal to the change in the volume of voids, Blight's equation for volume change may be used (Eq 4.72)

$$\frac{\Delta V_T}{V_T} = c(\Delta\sigma') \tag{4.87}$$

where c is the compressibility of the soil. Inserting this expression in Eq 4.86 gives Eq 4.88

$$\Delta\theta = c\,\frac{\theta}{n}\left(1 - n - \frac{V_A}{V_w}\right)\Delta\sigma' - \frac{\Delta V_A}{V_w} \tag{4.88}$$

This equation will form the basis for the subsequent discussion on the change of suction with water content as a function of the compressibility of soils.

Saturated Suction Change. In the saturated range, we assume that suction is a pore water pressure head rather than a water vapor pressure head as in the unsaturated condition. For this reason, a change of suction will result automatically in a change of effective stress according to the relation

$$\Delta\sigma' = -\chi_E\gamma_W\Delta T \tag{4.89}$$

Thus, Eq 4.88 may be written as

$$\frac{\Delta\theta}{\Delta\tau} = -c\,\frac{\theta}{n}\,\gamma_W\chi_E\left(1 - n - \frac{V_A}{V_W}\right) - \frac{1}{V_W}\left(\frac{\Delta V_A}{\Delta\tau}\right) \qquad (4.90)$$

but because in the saturated range

$$\frac{1}{V_W}\,\frac{\Delta V_A}{\Delta\tau} = \frac{\Delta V_A}{\Delta V_W}\cdot\frac{\Delta V_W}{\Delta\tau}\cdot\frac{1}{V_W} \qquad (4.91)$$

and

$$\frac{\Delta V_A}{\Delta V_W} = (\alpha_{FS} - 1)F = 0 \qquad (4.92)$$

where F is some compressibility factor based on the pressure acting on the soil, then

$$\frac{\Delta\tau}{\Delta\theta} = -\,\frac{n}{c\theta\gamma_W\chi_E\left(1 - n - \frac{V_A}{V_W}\right)} \qquad (4.93)$$

which is to be added to the value of the slope of the pressure free suction-moisture curve. Thus, in the saturated range, the actual change of suction with water content is

$$\frac{\partial\tau}{\partial\theta} = \underset{\substack{\text{Term}\\ \text{(D)}}}{\frac{\partial\tau_o}{\partial\theta}} + \underset{\substack{\text{Term}\\ \text{(E)}}}{\frac{\partial\tau_p}{\partial\theta}} = \frac{\partial\tau_o}{\partial\theta} - \frac{n}{c\theta\gamma_W\chi_E\left(1 - n - \frac{V_A}{V_W}\right)} \qquad (4.94)$$

where Term (D) is the pressure free value and Term (E) is the contribution due to pressure and compressibility of the soil.

Unsaturated Suction Change. In the unsaturated range, change of effective stress is related to change of water vapor pressure. The derivation which follows applies to small changes of suction only. The specific volume of water vapor V is

$$V = \frac{RT_e}{mg} \cdot \frac{1}{p} \tag{4.95}$$

where R, T_e, m, and g have been defined and p is the water vapor pressure which is

$$p = p_o e^{\frac{\tau mg}{RT_e}} \tag{4.96}$$

where τ is the suction and p_o is the saturated vapor pressure. For some small change of suction $\Delta\tau$, one may assume a constant specific volume of water vapor which leads to an expression for the change of effective stress.

$$\Delta\sigma' = -\chi_E \frac{\Delta\tau}{V} = -\chi_E \Delta\tau \cdot \frac{mg}{RT_e} p_o e^{\frac{\tau mg}{RT_e}} \tag{4.97}$$

Substitution of this expression into Eq 4.88 yields the following:

$$\frac{\Delta\theta}{\Delta\tau} = -c \frac{\theta}{n} \chi_E \left(1 - n - \frac{V_A}{V_W} \right) \frac{mg}{RT_e} p_o e^{\frac{\tau mg}{RT_e}} - \frac{1}{V_W} \left(\frac{\Delta V_A}{\Delta\tau} \right) \tag{4.98}$$

The remaining task in this derivation is to describe the change of air volume with change of suction. To do this it is first recognized that

$$\Delta V_T = \Delta V_W + \Delta V_A \tag{4.99}$$

where

ΔV_T = the change in total volume,

ΔV_W = the change in water volume,

ΔV_A = the change in air volume.

Dividing this equation by ΔV_W gives

$$\alpha_{FS} = \frac{\Delta V_T}{\Delta V_W} = 1 + \frac{\Delta V_A}{\Delta V_W} \tag{4.100}$$

or

$$\alpha_{FS} - 1 = \frac{\Delta V_A}{\Delta V_W} \tag{4.101}$$

for a pressure-free swelling condition. If pressure is present, the factor must be modified by a solubility relation between the pressures and volumes of air and water. This relation will be indicated as a multiplying factor F. A detailed discussion of the form of F is presented by Teerawong (Ref 56). Thus, in general,

$$\frac{\Delta V_A}{\Delta V_W} = F(\alpha_{FS} - 1) \tag{4.102}$$

It is understood that F is 1.0 when the soil is in the free swell condition and becomes smaller with increasing pressure. The air compressibility term in Eq 4.102 becomes

$$\frac{\Delta V_A}{\Delta V} \cdot \frac{\Delta V_W}{\Delta \tau} \cdot \frac{1}{V_W} = \frac{\Delta V_A}{\Delta V_W} \cdot \frac{\Delta \theta}{\Delta \tau} \cdot \frac{1}{\theta} \tag{4.103}$$

since

$$\frac{\Delta V_W}{V_W} = \frac{\Delta \theta}{\theta} \tag{4.104}$$

and finally

$$\frac{1}{V_W} \frac{\Delta V_A}{\Delta \tau} = \frac{F}{\theta} (\alpha_{FS} - 1) \frac{\Delta \theta}{\Delta \tau} \tag{4.105}$$

Substitution of this expression into Eq 4.106 leads to the equation sought.

$$\frac{\Delta T}{\Delta \theta} = - \frac{n}{c\theta \chi_E} \frac{\left[1 + \frac{F}{\theta} (\alpha_{FS} - 1) \right]}{\left[1 - n + \frac{V_A}{V_W} \right]} \left(\frac{1}{p_o} \right) \frac{RT_e}{mg} e^{- \frac{\tau mg}{RT_e}} \qquad (4.106)$$

which is added to the slope of the pressure-free suction moisture curve to obtain the actual rate of change of suction with respect to water content considering the compressibility of the soil.

Swelling Pressure: Saturated and Unsaturated

Swelling pressure has been measured in the laboratory in the following manner. A sample of soil is confined and allowed free access to water. Over a period of time increasing pressure is applied to the soil to keep its total volume strain equal to zero. When an equilibrium pressure is reached, the soil normally has a 90 percent or higher degree of saturation.

This is an effectively saturated equilibrium swelling pressure. The factors which affect these pressures are discussed in Chapter 3 in the section entitled Speculation on the Shape of the Suction-Moisture Curve. Obviously, the previous degree of drying, shrinkage, and cycling will lower the amount of swelling pressure. Disturbance of the natural soil or shearing of it loosens the bonding between particles and makes available a greater amount of suction to draw in moisture to be converted eventually into an equilibrium swelling pressure.

The swelling pressure reported in the literature is an end point of a potential expansion process made possible by the entry of water. In undisturbed natural soils, increase of water content will occur from a total volume-water volume condition much like that of the shrinkage curve. For a compacted material, water content increases into the effectively saturated range from a curve which is an inverted moisture density graph. These processes are shown in Fig 30. Data for the compaction and swelling pressure curves are taken from tests made by Eyuboglu (Ref 19) on Taylor clay.

It is virtually certain that dry soil that has been compacted is different in its fundamental makeup from soil that has been permitted to dry, compress, and cement over a long period of time. Although greater compactive effort

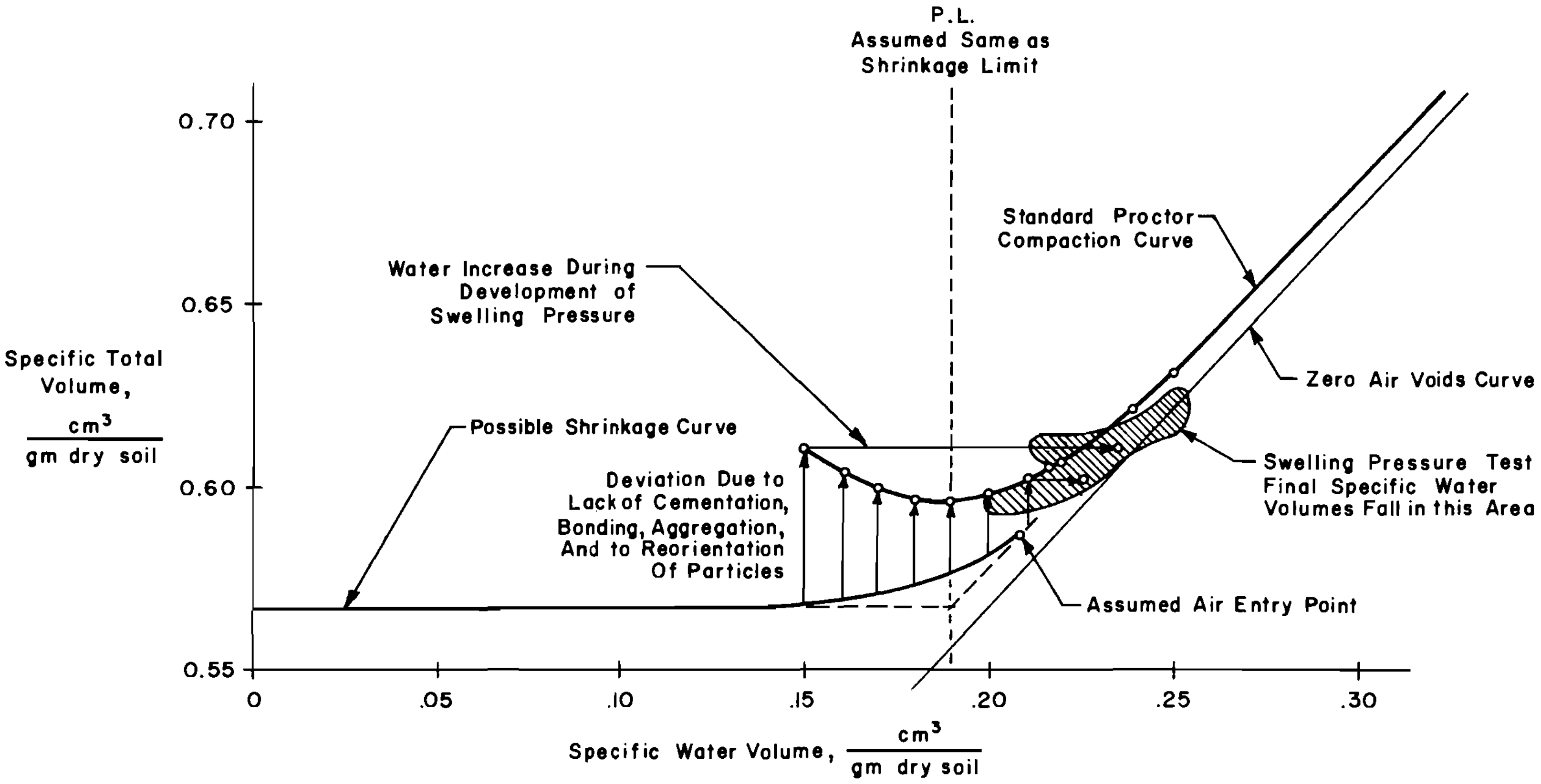


Fig 30. Compaction curve and subsequent swell.

seems to orient particles, it appears certain that the particles of a soil that has been compacted dry of optimum will be arranged in a more or less random fashion, called a flocculated structure. Many internal bonds will be broken from the shearing that takes place in the compaction process. Swell pressure that is generated can be expected to be higher than that measured for an undisturbed natural soil that swells from the same initial moisture content.

It is apparent from Fig 30 that any equilibrium swelling pressure measured in an immersed test will be that developed when the soil is in an effectively saturated condition. It can be asserted from the size of saturated moisture vapor pressure (0.353 psi at 68° F) that as long as soil is effectively unsaturated, virtually no swelling pressure can be generated. A test of this assertion would be to conduct swell pressure tests in 20, 50, and 80 percent relative humidity rooms on soils that are initially drier than the air entry water content. The currently used immersion tests supply the soil with all of the water it needs to become effectively saturated and generate maximum swelling pressure.

It is also evident from Fig 30 that the drying or consolidating process can produce a more dense material with greater cementation and bonding between particles. This statement may be incorrect for higher compactive efforts, however.

In Fig 31, swelling pressure measured by Eyuboglu is plotted against the specific total volume. This graph indicates that swell pressure is not a single-valued function of specific total volume (a reciprocal of dry density). These data are plotted in Fig 32 on a three-dimensional graph of the relation of specific water volume, specific total volume, and swelling pressure. The shape of the pressure build-up curves is purely speculative. The final swelling pressure is plotted along the zero air voids curve; and although this is an inaccuracy, it is not an extremely serious one. These data are not consistent with some of the findings of Roy E. Olson (Ref 38) who determined that a certain density gives a certain swell pressure for an illitic clay. However, Olson did find that two swelling pressures at the same dry density are possible for a silty clay provided that there are different initial particle arrangements. A dispersed particle structure produced the lower swelling pressure as does the Taylor clay reported here. The dispersed structure is characteristic of particle arrangements for soils compacted wet of optimum.

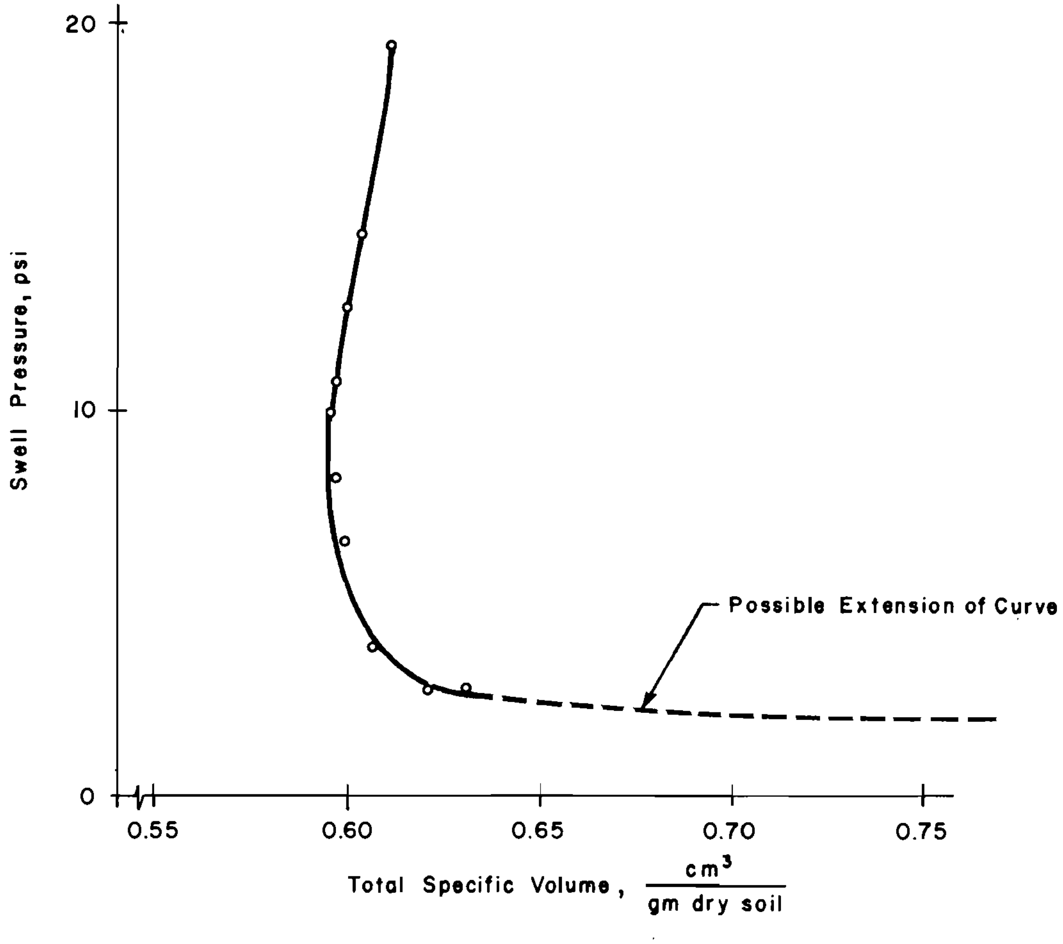


Fig 31. Swelling pressure as a function of compacted specific volume.

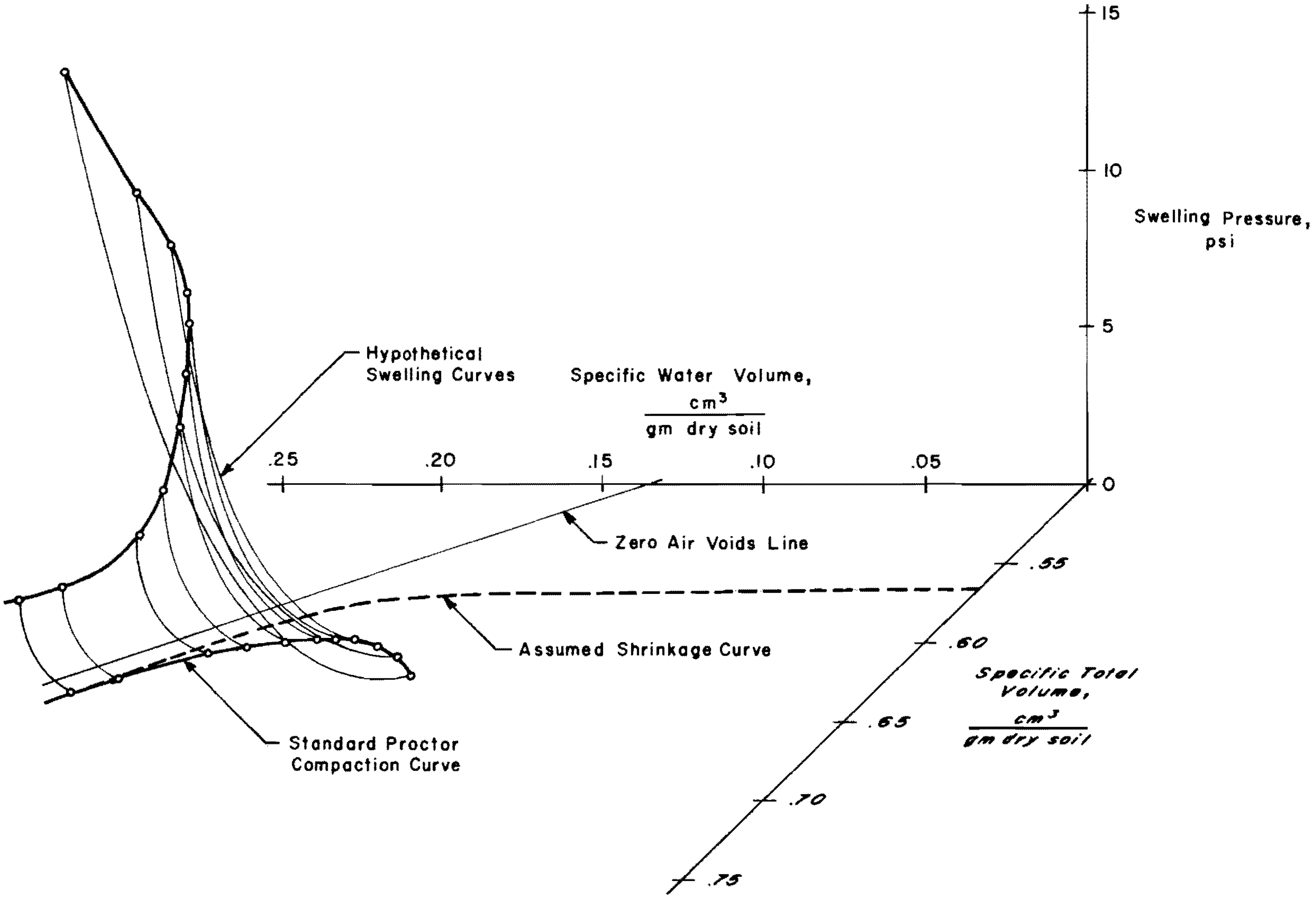


Fig 32. The $P\text{-}V_T\text{-}V_W$ surface for a Taylor clay.

Speculation on the Pressure-Total Volume-Water Volume Relation

The shape of the swelling pressure-total volume-water volume relation is largely unknown for real soils which have been dried beyond the air entry point. A practical reason for this is that samples of such materials are hard to trim and surface irregularities are difficult to deal with.

Much of the shape of this surface could be surmised if answers were known to the following questions:

(1) How large is equilibrium swelling pressure when the soil is permitted access to a limited amount of water?

(2) Does oven dry soil have swelling pressure?

(3) Is the swelling pressure of soil dried below the air entry point duplicated by the swelling pressure of soil compacted on the wet side of optimum to an identical density?

Tentative answers which can only be verified by experiment are as follows:

(1) Effectively unsaturated swelling pressure is very small and rises sharply to its peak equilibrium value when soil becomes 90 to 95 percent saturated.

(2) Oven dry soil should have swelling pressure. On the other hand, a fired clay brick may not have significant swelling pressure because of internal fusion of particles.

(3) The curves may not be identical. Compaction on the wet side produces a dispersed structure and drying may produce a flocculated soil structure which, in Olson's silty clay experiments, produces higher swelling pressures. In neither case is swelling pressure as high as it could be if there were no bonding between particles.

Given these tentative answers, the swell pressure-total volume-water volume relation for natural uncompacted soil may resemble the surface in Fig 33.

It is assumed that compaction breaks particle bonds and frees areas close to mineral surfaces for the adsorption of water. Because of this assumption, swelling pressure is assumed to rise when a dry soil is compacted.

As noted before, these assumptions must be proved by experiment before a comprehensive picture of clay behavior under load can emerge.

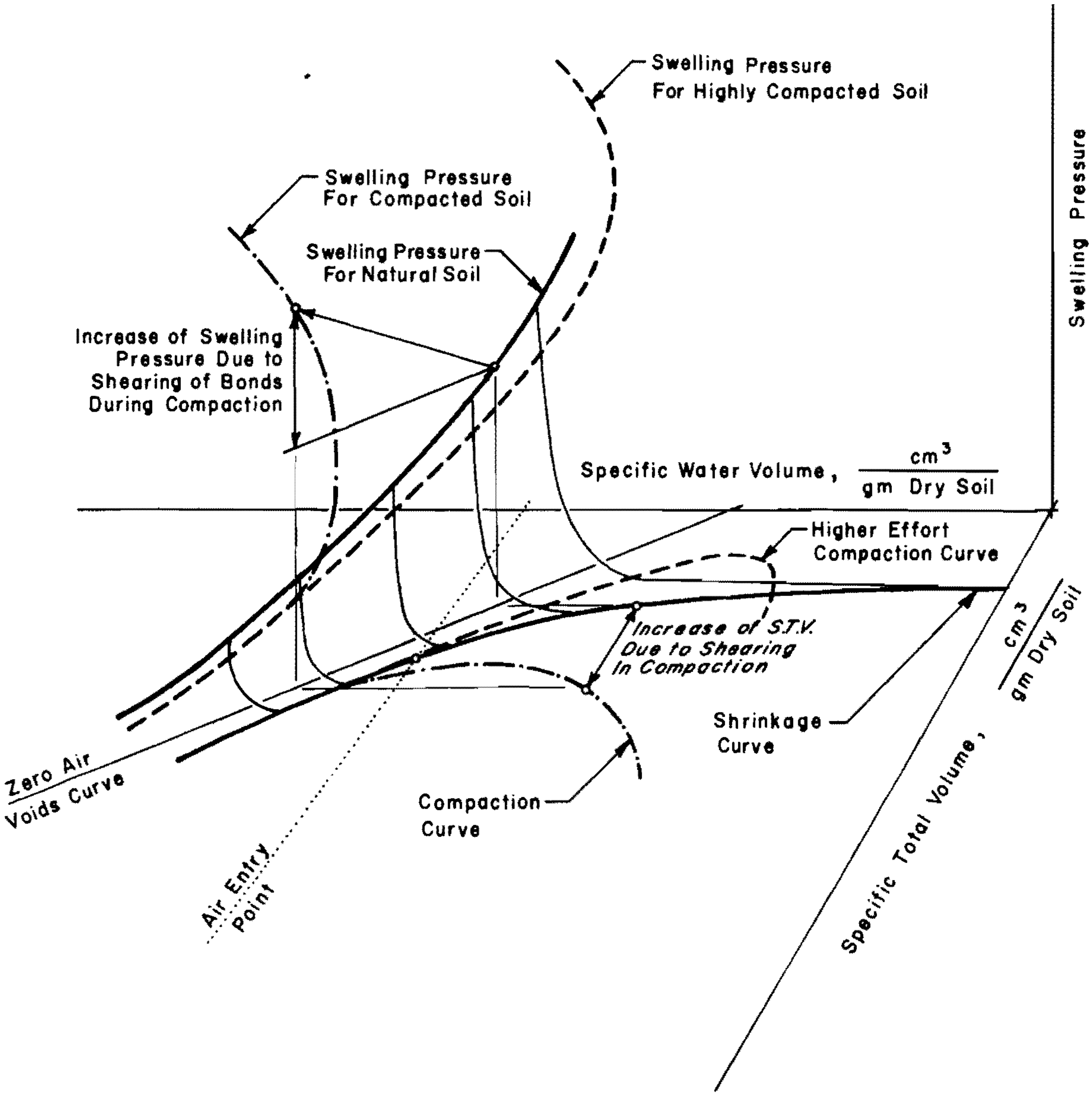


Fig 33. Hypothetical $P\text{-}V_T\text{-}V_W$ surface for a natural soil.

CHAPTER 5. FLOW THROUGH UNSATURATED SOIL

It is evident in Chapter 2 that permeability is a useful, though not always reliable, concept. It will be necessary in this chapter to assume the validity of the tensor form of permeability derived in Chapter 2 to develop the equations of flow through unsaturated soils. Needless to say, all of the limitations of that development will be inherent in the equations to be derived and methods of solution proposed.

There is one valid physical principle which applies to the movement of fluids, whether liquid or gas, regardless of the assumed relation between flow velocity and total head gradient. This is the continuity principle which states that the net mass of fluid entering or leaving a certain volume in a certain amount of time is equal to the amount of fluid stored or lost from that volume in that period of time.

Equations of Flow in Rectangular Coordinates

In more simple terms, the continuity principle states

$$(\text{Inflow Rate} - \text{Outflow Rate}) \times \Delta\text{Time} = \text{Storage}$$

Figure 34 shows a rectangular element of volume of a porous medium. The continuity condition is met when

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = - \frac{\partial(\rho\theta)}{\partial t} \tag{5.1}$$

where

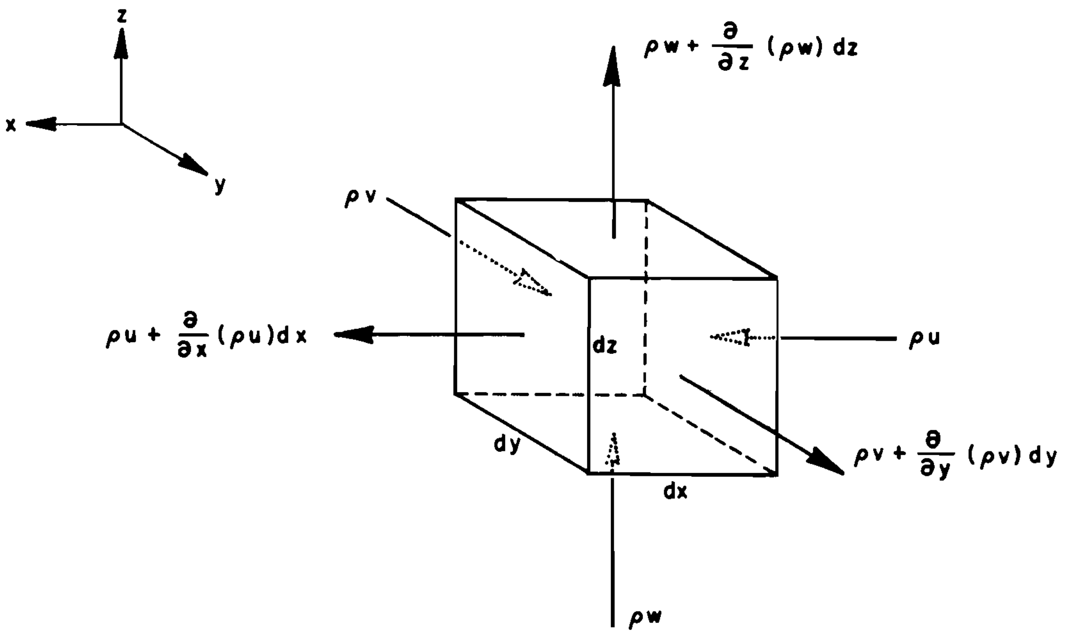
ρ = density of the fluid,

u = velocity in x-direction,

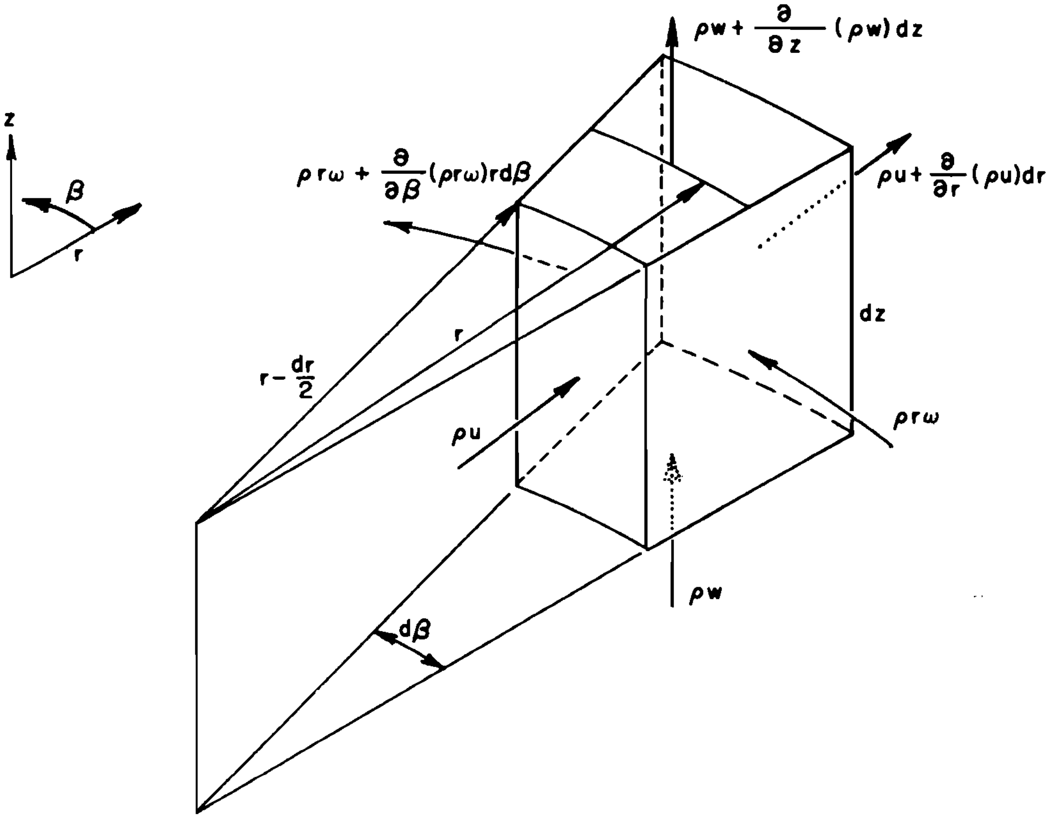
w = velocity in z-direction,

θ = volumetric water content.

t = time



(a) Rectangular element.



(b) Cylindrical element.

Fig 34. Movement of water through infinitesimal elements of soil.

The essential elements of the derivation are given in Rouse's book (Ref 49). The volume element should not change size appreciably with an incremental change in volumetric water content if this equation is to remain valid. At this point, assumptions are made. One common assumption which is microscopically invalid in soils is that the fluid density is constant. Large enough variations in temperature can easily cause this assumption to be incorrect. Certainly, in the vicinity of a clay particle surface, the density may change markedly in a very short distance. If x is noted as x_1, y as x_2, and z as x_3, the continuity equations becomes

$$\frac{\partial(\rho\theta)}{\partial t} = -\frac{\partial(\rho v_i)}{\partial x_i} \qquad i = 1,2,3 \qquad (5.2)$$

where v_i is the velocity in the i^{th} direction. Chapter 2 produced the result that

$$\rho v_i = -\rho k_{ij} \frac{\partial H}{\partial x_j} \qquad i,j = 1,2,3 \qquad (5.3)$$

and thus

$$\frac{\partial(\rho\theta)}{\partial t} = \frac{\partial}{\partial x_i}\left(\rho k_{ij} \frac{\partial H}{\partial x_j} \right) \qquad (5.4)$$

where

H = the total head,

k_{ij} = the permeability tensor.

Water can move through soil in several phases: liquid, vapor, and in the film flow of the adsorbed state. Several natural forces combine to drive water through: pressure, temperature, ion concentration, molecular attraction, and electromotive force to name a few. Permeability is affected by each of these forces and the total potential head must necessarily include each. One of the best discussions of the many factors that are involved and how they are interrelated is given by Philip (Ref 41). It is, of course, impossible at the present time to consider all possible factors in any practical application. The present derivation will consider only those components of total head which are suction, gravity, and temperature dependent.

Because the term τ has been defined in Chapter 4 to include suction and pore pressure terms, it is apparent that the most general functional relation that is to be considered here is as follows:

$$H = H\left(\tau , T_e, + \frac{\gamma x_3}{\gamma}\right) \tag{5.5}$$

where

τ = the total suction,

T_e = the absolute temperature,

$+\frac{\gamma x_3}{\gamma}$ = the gravity head.

The x_3 coordinate is taken as vertical and thus

$$\frac{\partial H}{\partial x_j} = \frac{\partial H}{\partial \tau}\frac{\partial \tau}{\partial x_j} + \frac{\partial H}{\partial T_e}\frac{\partial T_e}{\partial x_j} + 1 \cdot i_3 \tag{5.6}$$

where i_3 = a unit vector in the vertical direction . Equation 5.4 becomes

$$\frac{\partial(\rho\theta)}{\partial t} = \frac{\partial}{\partial x_i}\left(\rho k_{ij}\frac{\partial H}{\partial \tau}\frac{\partial \tau}{\partial x_j} + \rho k_{i3}\right) + \frac{\partial}{\partial x_i}\left(\rho k_{ij}\frac{\partial H}{\partial T_e}\frac{\partial T_e}{\partial x_j}\right) \tag{5.7}$$

Two more functional relations must be considered: the relation of the volumetric water content to the total suction τ and the temperature T_e and the relation of suction to temperature. The first of these two relations is written in functional form as

$$(\rho\theta) = \rho\theta(\tau, T_e) \tag{5.8}$$

so that the time-derivative of $(\rho\theta)$ is as follows:

$$\frac{\partial(\rho\theta)}{\partial t} = \underbrace{\frac{\partial(\rho\theta)}{\partial\tau}\frac{\partial\tau}{\partial t}}_{\text{Term(A)}} + \underbrace{\frac{\partial(\rho\theta)}{\partial T_e}\frac{\partial T_e}{\partial t}}_{\text{Term(B)}} \tag{5.9}$$

The expression given in Term (A) has been discussed in Chapter 4 with density considered constant. Term (B) includes the change of temperature with time and the change of volumetric water content with temperature. This last item is a coefficient of thermal expansion but is not necessarily a simple relation because water may exist in all three phases, each of which has a different thermal expansion coefficient.

The total suction τ is a function of pressure, gravity, adsorption, and concentration potentials, each of which is a function of temperature with the exception of gravity. The functional relation which will be considered here is

$$\tau = \tau\left(T_e, p, u, \omega, A, +\frac{\gamma x_3}{\gamma}\right) \tag{5.10}$$

where, as in Chapter 3,

T_e = the absolute temperature,

p = the pressure from overburden and imposed loads,

u = the pore water pressure,

ω = the concentration head,

A = the adsorptive head,

$+\frac{\gamma x_3}{\gamma}$ = the gravity head.

The time derivative of τ is as follows:

$$\frac{\partial\tau}{\partial t} = \frac{\partial\tau}{\partial T_e}\frac{\partial T_e}{\partial t} + \frac{\partial\tau}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial\tau}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial\tau}{\partial\omega}\frac{\partial\omega}{\partial t} + \frac{\partial\tau}{\partial A}\frac{\partial A}{\partial t} \tag{5.11}$$

The gravity head is assumed not to change with time. If the pressure, concentration, and adsorption terms could be considered as functions of temperature alone, the equation would simplify greatly. In many cases, this is not a correct assumption.

If over some time increment Δt the pressure, ion concentration, and adsorption terms were changed by the temperature alone, then for that special case

$$\frac{\partial \tau}{\partial t} = \frac{\partial \tau}{\partial T_e} \frac{\partial T_e}{\partial t} \tag{5.12}$$

which gives the special result that

$$\frac{\partial T_e}{\partial t} = \frac{\partial T_e}{\partial \tau} \frac{\partial \tau}{\partial t} \tag{5.13}$$

If this relation is substituted into Eq 5.9, it becomes Eq 5.14.

$$\frac{\partial(\rho\theta)}{\partial t} = \left(\frac{\partial(\rho\theta)}{\partial \tau} + \frac{\partial(\rho\theta)}{\partial T_e} \frac{\partial T_e}{\partial \tau} \right) \frac{\partial \tau}{\partial t} \tag{5.14}$$

Equation 5.14 may in turn be substituted into Eq 5.7 to give an expression for the change of suction with time.

$$\frac{\partial \tau}{\partial t} = \frac{1}{\left(\frac{\partial(\rho\theta)}{\partial \tau} + \frac{\partial(\rho\theta)}{\partial T_e} \frac{\partial T_e}{\partial \tau} \right)} \frac{\partial}{\partial x_i} \left(\rho k_{ij} \frac{\partial H}{\partial \tau} \frac{\partial \tau}{\partial x_j} + \rho k_{i3} \right.$$

$$+ \frac{1}{\left(\frac{\partial(\rho\theta)}{\partial \tau} + \frac{\partial(\rho\theta)}{\partial T_e} \frac{\partial T_e}{\partial \tau} \right)} \frac{\partial}{\partial x_i} \left(\rho k_{ij} \frac{\partial H}{\partial T_e} \frac{\partial T_e}{\partial x_j} \right) \tag{5.15}$$

This differential equation appears to be rather intractable. Certainly any attempt to solve a practical problem by classical mathematics would prove to be formidable if not impossible. Idealized versions of the relations among

τ , T_e , H , ρ , and θ would have to be used, and these may be so far removed from a real condition that the task would be useless to undertake. Numerical methods offer an attractive alternative, but even here the limitations of memory, size, and speed of computers will dictate the extent to which the solution of Eq 5.15 may be carried.

Another limitation is that imbedded in Eq 5.15 are the assumptions that the permeability tensor adequately describes the flow properties at a point, that deformations in the soil mass are small, and that suction varies with temperature and time alone. If all of these assumptions are accepted as valid, then Eq 5.15 will do an adequate job describing soil suction change with time.

In addition to the previous assumptions, it can be assumed that

(1) The average density of water ρ does not change with time or distance.

(2) The temperature does not change during a flow process.

(3) The total head equals the total suction head τ , plus the gravity head.

Then Eq 5.15 becomes the relatively simple differential equation written as Eq 5.16.

$$\frac{\partial \tau}{\partial t} = \frac{\partial \tau}{\partial \theta} \frac{\partial}{\partial x_i} \left(k_{ij} \frac{\partial \tau}{\partial x_j} + k_{i3} \right) \tag{5.16}$$

This differential equation has been used in the computer programs to be considered in Research Report Nos. 118-3 and 118-5.

Another approach that has been used to predict moisture movement is concerned with moisture gradients rather than suction gradients as shown above. There are advantages to each point of view and in order to consider them, a brief derivation of the moisture gradient, a differential equation, will be presented.

Moisture Gradient Differential Equation. Beginning with Eq 5.4

$$\frac{\partial(\rho\theta)}{\partial t} = \frac{\partial}{\partial x_i} \left(\rho k_{ij} \frac{\partial H}{\partial x_j} \right)$$

the assumption is made that the total force potential head is independent of temperature and is given completely by suction τ . No assumption is made regarding the mass density of water, but the term $\rho\theta$ is assumed to be a function of suction. This is expressed in function form as

$$\rho\theta = \rho\theta(\tau) \tag{5.17}$$

The total force potential head gradient is then considered to be

$$\frac{\partial\tau}{\partial x_j} = \frac{\partial\tau}{\partial(\rho\theta)} \cdot \frac{\partial(\rho\theta)}{\partial x_j} \tag{5.18}$$

and Eq 5.4 may be written as

$$\frac{\partial(\rho\theta)}{\partial t} = \frac{\partial}{\partial x_i}\left[\rho k_{ij} \frac{\partial\tau}{\partial(\rho\theta)} \cdot \frac{\partial(\rho\theta)}{\partial x_j} \right] \tag{5.19}$$

The coefficient of the moisture gradient given in Eq 5.19 is called diffusivity D_{ij} .

$$D_{ij} = \rho k_{ij} \frac{\partial\tau}{\partial(\rho\theta)} \tag{5.20}$$

The diffusivity equation used is given in Eq 5.21.

$$\frac{\partial(\rho\theta)}{\partial t} = \frac{\partial}{\partial x_i}\left[D_{ij} \frac{\partial(\rho\theta)}{\partial x_j} \right] \tag{5.21}$$

In this equation, the total mass of water that changes with time, regardless of phase, may be computed provided the diffusivity tensor is correctly represented. If sand and clay have an interface, it is possible for the clay to contain more water and still draw water from the sand. Thus, the moisture gradient does not tell the entire story of moisture transfer in soil. In this case, the derivative of the product of diffusivity and moisture gradient must provide the information on whether the moisture increases or decreases at a point. Because this equation must normally be solved on a computer, the necessity of keeping track of the different types of soil and the different values of diffusivity from point to point must be considered.

Advantages of Each Scheme. The two equations are derived using essentially the same assumptions. The advantage of the moisture gradient approach is its use of a diffusivity which lumps together permeability, water density, and slope of the suction-moisture curve. A single laboratory test on a sample with a moisture gradient can be used to assess the diffusivity term.

In the suction-gradient approach, permeability and slope of the suction-moisture curve are separated and average mass density of water is assumed to remain constant with time and distance. Numerical computations are done using suction values which vary widely from place to place rather than the corresponding small moisture differences, and this is more conducive to accuracy in the calculations. Two moisture-dependent tests are required to obtain data for this approach:

(1) a series of permeability tests, and

(2) a series of suction-moisture tests.

Each of these must be made at different moisture content levels to obtain the overall shape of the curve.

Equations of Flow in Cylindrical Coordinates

The net inflow to the cylindrical element shown in Fig 13(b) is equal to the storage in a particular time increment. Stated in differential form, the continuity condition is given as

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho u r) + \frac{1}{r}\frac{\partial}{\partial \beta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial(\rho\theta)}{\partial t} \qquad (5.22)$$

This equation is to be combined with the equations for velocity in cylindrical coordinates as given in Eq 5.18.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = - \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial r} \\ \frac{\partial H}{r\partial \beta} \\ \frac{\partial H}{\partial z} \end{bmatrix} \qquad (5.23)$$

and the three assumptions used to derive Eq 5.16 to get Eq 5.19 which expresses the change of suction with time in cylindrical coordinates.

$$\frac{\partial\tau}{\partial t} = \frac{\partial\tau}{\partial\theta}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left(k_{11}\frac{\partial\tau}{\partial r}+\frac{k_{12}}{r}\frac{\partial\tau}{\partial\beta}+k_{13}\frac{\partial\tau}{\partial z}+k_{13}\right)$$

$$+\frac{\partial\tau}{\partial\theta}\frac{1}{r}\frac{\partial}{\partial\beta}\left(k_{21}\frac{\partial\tau}{\partial r}+\frac{k_{22}}{r}\frac{\partial\tau}{\partial\beta}+k_{23}\frac{\partial\tau}{\partial z}+k_{23}\right)$$

$$+\frac{\partial\tau}{\partial\theta}\frac{\partial}{\partial z}\left(k_{31}\frac{\partial\tau}{\partial r}+\frac{k_{32}}{r}\frac{\partial\tau}{\partial\beta}+k_{33}\frac{\partial\tau}{\partial z}+k_{33}\right) \qquad (5.24)$$

The axially symmetric condition occurs when all derivatives with respect to β are equal to zero. The equation for this case is as follows:

$$\frac{\partial\tau}{\partial t} = \frac{\partial\tau}{\partial\theta}\left(\frac{1}{r}+\frac{\partial}{\partial r}\right)\left[k_{11}\frac{\partial\tau}{\partial r}+k_{13}\left(\frac{\partial\tau}{\partial z}+1\right)\right]$$

$$+\frac{\partial\tau}{\partial\theta}\frac{\partial}{\partial z}\left[k_{31}\frac{\partial\tau}{\partial r}+k_{33}\left(\frac{\partial\tau}{\partial z}+1\right)\right] \qquad (5.25)$$

The same remarks regarding the diffusivity approach apply to Eq 5.25 as to Eq 5.16. This differential equation has been included in the computer programs reported in Research Report Nos. 118-3 and 118-4.

Unsaturated Flow Equations

The flow of water in its three phases through unsaturated and expansible soil can be viewed in as complicated and microscopically detailed a manner as one wishes. The listing of complicating factors can tend to discourage effort in the analysis of unsaturated flow. By considering only the average measurable properties of laboratory-size pieces of the soil and by proposing simple but adequately comprehensive differential equations for the flow process, a good measure of satisfactory prediction can be achieved.

CHAPTER 6. CONCLUSIONS

In this report, a comprehensive theory of moisture movement in expansible porous media is presented. The phenomenon of moisture movement in clay is best envisioned by an analogy with current flow in electricity. The report is broken into three parts to discuss the soil analogs of electrical conductance, voltage, and something like inductance.

Chapter 1 considers soil water conductance or permeability and Chapter 3 discusses the voltage analog, soil suction. In Chapter 4, stress transfer coefficients α and χ are discussed. Because the condition of stress in water induces a condition of stress in the soil solids, the rather loose analogy with electrical induction has been drawn.

Permeability

Permeability cannot be separated from suction in actuality. It is a function of geometry, density, suction level, and absolute temperature. The character of permeability becomes altered as soil becomes less saturated. As soil dries, permeability tends away from a dependence on water pressure toward a dependence on a suction-induced rate process. Consideration is not given in this report to the effect of an activation entropy on permeability, but it certainly should be considered in those cases where entropy changes at constant temperature. Examples of this are common enough: moisture transfer near an ice lens and moisture transfer at a wetting surface in very dry soil in which hydration energy is liberated. As long as the process is isothermal, the energy jumps that take place at interfaces such as described above can be described as an equivalent suction jump.

The dependence of permeability upon the water driving forces indicates that moisture movement in soils is a kind of self-diffusion process. Clay soils provide their own "voltage," so to speak, and provide their own "resistance" to flow of water. The movement of moisture in clay is a self-regulating process.

Soil Suction

Suction is a collection of several distinguishable tendencies of soil to take on water. Among these tendencies, or driving forces, the most prevalent force potentials are

(1) water pressure gradients,
(2) osmotic attraction,
(3) capillary attraction,
(4) temperature gradients,
(5) electrical voltage, and
(6) vapor pressure gradients.

Suction is the effective sum of these potentials weighted by the ability of each to move water. The size of suction depends, just as does permeability, on the state of the soil. Suction is normally higher if soil is dry, freshly remolded or compacted, or in its virginal drying state. Suction is lower when soil is wet, aged by many wetting and drying cycles, undisturbed in the natural condition, or undergoing a wetting process.

Hysteresis is involved in the suction-moisture relation as is internal bonding of soil particles; and, in general, the current value of suction of soil is a function of that soils' complete history.

The α and χ Coefficients

When soil is made wetter, suction is decreased and the soil volume normally increases. The effect is as if some internal tension or negative pore pressure is relieved and the soil springs open, becomes softer, and is easier for water to flow through. The amount of suction release that is equivalent to a mechanical internal tension release which is wholly effective in changing total soil volume is treated by the factors α and χ . Much discussion has been devoted to these factors in the literature. Some confusion has resulted in proposing different symbols for α and χ . Perhaps the greatest confusion comes from how they are defined and used. In this report four common definitions of α and χ are investigated and some important distinctions are made between the factors:

(1) α_{FS} - a volume ratio from a free swell test,
(2) α_P - a pressure ratio from a constant water content test,

(3) χ_E - an equilibrium value of the factor showing how much water stress is effective in causing some mechanical effect, and

(4) χ_T - a time rate of change of stress ratio.

At one time or another two or more of these factors have been equated, but it is demonstrated in this report that this is not necessarily valid.

Derivations are presented using the χ_E , α_{FS} , and α_P factors to assess the change of suction with water content due to the restraint of the soil and confining pressure. The expressions are different in effectively saturated and effectively unsaturated soils. In the former, water is assumed incompressible; and virtually all that is needed is a change of suction, a degree of saturation, and a "compressibility" coefficient for soil. In effectively unsaturated soil, relative vapor pressure, saturation, compressibility of soil, and air are among the requirements for computing the effect of soil compressibility on suction change.

Speculation

Two efforts at theoretical speculation are included in this report. Neither are proven theoretical notions, but both are built on experimental indications. One of these uses the notion of "probability of aggregation" and an approach similar to those contained in statistical mechanics to explain the shape of the suction-moisture curve.

The second speculative effort is presented in Chapter 4 regarding the shape of the swelling pressure-total volume-water volume surface. The shape of the surface for compacted soil is drawn from swelling data on Taylor clay tested at The University of Texas. It is somewhat different from the shape of the surface for an illitic clay reported by Olson (Ref 38). Progress in predicting swelling clay behavior will come when this surface can be fairly well defined. The definition of this surface requires some carefully planned tests including measurement of swelling pressures in the unsaturated range.

This report provides a theoretical basis for further studies of swelling clays and the processes of moisture movement in them. Equations are proposed and questions are raised, the answers to which will lead to a more comprehensive and accurate knowledge of the structural behavior of expansive clay.

Application of Results to Engineering Practice

This report has been developed solely to provide background for subsequent developments within this project. It will provide background information for subsequent reports and computer programs for analyzing swelling clay problems. As such it is an important building block in the solution of the swelling clay problem. The results, however, are not intended for immediate direct use within the Highway Department or the Bureau of Public Roads except by other researchers and soils engineers interested in studying the swelling clay problem.

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