AN ANALYTICAL SOLUTION OF THE IMPACT BEHAVIOR OF LUMINAIRE SUPPORT ASSEMBLIES

by

J. E. Martinez
Engineering Research Associate

Research Report Number 75-9
Supplementary Studies in Highway Illumination
Research Project Number 2-8-64-75

Sponsored by

THE TEXAS HIGHWAY DEPARTMENT
In Cooperation with the
U. S. DEPARTMENT OF TRANSPORTATION
FEDERAL HIGHWAY ADMINISTRATION
BUREAU OF PUBLIC ROADS

August 1967

TEXAS TRANSPORTATION INSTITUTE
TEXAS A&M UNIVERSITY
COLLEGE STATION, TEXAS

FOREWORD

The information contained herein was developed on Research Project 2-8-64-75 entitled "Supplementary Studies in Highway Illumination," which is a cooperative research project sponsored jointly by the Texas Highway Department and the U. S. Department of Transportation, Federal Highway Administration, Bureau of Public Roads. The broad objective of this project is to (a) study methods to evaluate and compare continuous highway illumination systems, (b) study the visibility characteristics for high level lighting and driver requirements for rural interchange lighting, (c) evaluate contemporary luminaire supports for safety and develop break-away bases to enhance roadside safety. This report covers the specific objective of developing a mathematical model of a luminaire support assembly that is impacted by a vehicle.

ACKNOWLEDGEMENTS

The author gratefully acknowledges the assistance of Barry E. Morgan, senior student in Aerospace Engineering, Texas A&M University, who worked closely with the author in compiling the parameter study.

The opinions, findings, and conclusions expressed in this paper are those of the author and not necessarily those of the Bureau of Public Roads.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF FIGURES (Continued)

LIST OF TABLES

NOTATION

A, B, C	=	Principal mass moments of inertia at the mass center.
D1X, D1Y, D1Z	=	Direction cosines between the 1 axis and the X, Y, Z axes respectively.
D2X, D2Y, D2Z	=	Direction cosines between the 2 axis and the X, Y, Z axes respectively.
D3X, D3Y, D3Z	=	Direction cosines between the 3 axis and the X, Y, Z axes respectively.
FFXX, FFYY	=	Frictional forces in the XX and YY directions respectively.
FN	=	The normal force.
FS	=	The spring force.
FSXX, FSYY	=	The components of the spring force in the XX and YY directions respectively.
FX, FY, FZ	=	Resultant forces in the X, Y, Z directions respectively.
F1, F2, F3	=	Resultant forces in the 1, 2 and 3 directions respectively.
g	=	Acceleration due to gravity.
h	=	Time increment.
HLEN	=	Length of the hood of the vehicle.
HHV, HTRV, HTV	=	Coordinates of the hood, trunk and top of the vehicle respectively.
K	=	Spring constant of the vehicle.
$\bar{L}$	=	Angular momentum vector.
$\dot{\bar{L}}$	=	Time rate of change in the angular momentum vector.
M	=	Mass of the post.

M_v	=	Mass of the vehicle.
P	=	Typical point on the post.
S1, S2, S3	=	Translations of the post center of mass in the 1, 2 and 3 directions respectively.
SV	=	Displacement of the vehicle.
$\bar{T}$	=	The torque vector.
TX, TY, TZ	=	The torques about the X, Y and Z axes respectively.
T1, T2, T3	=	The torques about the 1, 2 and 3 axes respectively.
VV	=	Velocity of the vehicle.
$\dot{V}V$	=	Acceleration of the vehicle.
V1, V2, V3	=	Velocities of the post center of mass in the 1, 2 and 3 directions respectively.
$\dot{V}1, \dot{V}2, \dot{V}3$	=	Component accelerations of the post center of mass in the 1, 2 and 3 directions respectively.
$\omega 1, \omega 2, \omega 3$	=	Angular velocities of the post about the 1, 2 and 3 axes respectively.
$\dot{\omega}1, \dot{\omega}2, \dot{\omega}3$	=	Component angular accelerations of the post about the 1, 2 and 3 axes respectively.
XX, YY, ZZ	=	A fixed right-handed coordinate system having its XX-YY plane where vehicle motion takes place and its XX axis in the direction of the highway.
X, Y, Z	=	A fixed right-handed coordinate system having its axes coinciding with the initial position of the principal 1, 2, 3 axes and obtained by rotating the XX, YY, ZZ system on angle δ about the XX axis.
XV, YV, ZV	=	A fixed right-handed coordinate system obtained by rotating the XX, YY, ZZ system on angle α about the-ZZ axis.
XCM, YCM, ZCM	=	The translations of the post center of mass in the X, Y and Z directions respectively.

XP, YP, ZP	=	The translations of the point "P" as measured in the X, Y, Z coordinate system.
XPO, YPO, ZPO	=	The initial coordinates of the point P as measured in the X, Y, Z coordinate system.
XXP, YYP, ZZP	=	The translations of the point "P" as measured in the XX, YY, ZZ coordinate system.
XVP, YVP, ZVP	=	The translations of the point "P" as measured in the XV, YV, ZV coordinate system.
YLFEN	=	The YY or YV coordinate of the left fender of the vehicle.
YRFEN	=	The YY or YV coordinate of the right fender of vehicle
1, 2, 3	=	A moving right-handed coordinate system having its 1, 2 and 3 axes along principal directions of the post.
α	=	Angle the XX, YY, ZZ coordinate system is rotated about -ZZ axis to obtain XV, YV, ZV coordinate system.
δ	=	Angle of XX, YY, ZZ coordinate system is rotated about XX axis to obtain X, Y, Z coordinate system.
θ, ϕ, ψ	=	The Eulerian angles.
$\dot{\theta}, \dot{\phi}, \dot{\psi}$	=	The time-rate of change of the Eulerian angles.
ξ, η, ζ, χ	=	The rotation parameters.

CHAPTER I

INTRODUCTION

1.1 General Background

Modern freeways require adequate lighting facilities and in order to meet lighting requirements, light posts sometimes have to be located near the edge of the traffic lane. Single post sign standards and stop light posts are also necessary. Each of these post installations are often located so as to constitute a safety hazard, and collision with these posts can cause fatalities.

An obvious solution to the problem is the relocation of the post. This is not always feasible so the engineer has to resort to other means to eliminate this safety hazard. The method of developing supports that will limit impact forces to tolerable limits has been suggested as another solution to the problem. A design showing considerable merit is the "break-away" luminaire support post that disengages the post from its foundation upon impact.

Since posts are usually quite massive, they could, after impact, be knocked into the path of the vehicle, or onto the highway.

The work presented in this research is directed toward developing a post model that upon impact will be knocked out of the vehicle path and also not land on the highway causing an unsafe condition for other motorists. This can be accomplished by a suitable location of the mass center of the post assembly which means the mass will have to be distributed in a certain fashion.

In order to develop a concept into a design that can be utilized under field conditions, it is necessary to investigate the behavior under various conditions. For the problem in question, this entails investigation under different conditions of vehicle impact with various vehicle sizes and velocities. It is also necessary to study the behavior of the various "break-away" features of the posts. Current techniques involve a full-scale crash test for each sign and vehicle parameter.

Samson, Rowan, Olson and Tidwell[1] draw the following conclusion in the summary statement of their report:

"The thorough observation of the high speed film has clearly indicated the phenomenological behavior of the several structural supports tested. These observations have also created an insight into the formulation of a mathematical model for expressing the behavior quantitatively."

Edwards[2] has investigated the solution to the case where a post is simulated using a discrete mass system. This method is based on a distributed mass system consisting of a discrete member of concentrated masses connected by assumed massless elastic link elements. The dynamical equations written for this model express the relations between mass point displacements and accelerations in terms of post parameters and external actions. These equations are solved using a numerical integration technique. The solution assumes motion to take place in a plane but it is possible to extend it to three-dimentional motion in order that it may handle the more general situation which is usually the case.

The work presented in this research is part of a larger project on sign and light post behavior.

1.2 Objectives

The objectives of this research are:

(1) To establish an analytical model that will describe the motion of a rigid body under the influence of gravity and time dependent forces.

(2) To apply numerical integration techniques to obtain a solution to the equations of motion.

(3) To investigate the stability of the numerical solution.

(4) To attempt to correlate theoretical results with experimental data obtained from the impact of a vehicle on sign and light post systems.

1.3 Literature Review

The direct solution of the equations of motion for a rigid body subjected to time dependent forces presents a formidable task. The equations of motion for a body rotating about a fixed point are presented by several authors.[3,4,5,6,7,8,9] The special case of the motion of a rigid body with a fixed point under no forces is presented in works on analytical dynamics.[3,4,5] The problem is treated by two methods -- the descriptive and the analytic. The descriptive method, or method of Poinsot, gives a good qualitative idea of the motion. In the case where the body has an axis of dynamical symmetry, the description is particularly simple.

The analytical method, like that of Poinsot, makes use of the fact that for the special case considered the kinetic energy and angular momentum are constants.

The angular velocities and displacements of the body are obtained using elliptic functions. From the periodic property of the elliptic functions, it is seen that the motion as a whole is not periodic.

The general motion of a rigid body consists of motion of the mass center plus motion relative to the mass center and the equations for this motion are given by Synge.[3] It is illogical to suppose that the determination of the general motion always divides into two parts -- a problem in particle dynamics and a problem in the dynamics of a body with a fixed point. Constraints make the two problems interlock, and complications arise. A general plan cannot be given for the solution of all such problems and the method of solution depends upon the particular problem under consideration.

Even if a rigorous solution to a problem does exist, its use to obtain numerical results may often be tedious and time-consuming. This condition has led in recent years to the rapid development of numerical methods of analysis such as those discussed by Karman,[10] Salvadori[11] and Johnson,[12] and machine methods of computation.

For the dynamics problem, a numerical solution consists in obtaining numerical values of the displacement and velocity at discrete times. These displacement and velocity values are obtained by a step-by-step integration procedure of the equations of motion of the system, starting with the necessary initial conditions and evaluating the conditions at the end of a discrete time interval. These values are then the basis for calculation of the velocity and displacement at successive discrete times.

Historically, the development of numerical-integration methods has resulted from the efforts of individuals searching for the solution to specific problems in science or engineering. These researchers often devised methods of solution based on the physical behavior of the system in question, but with little regard for mathematical rigor. As the need for numerical methods of analysis has increased, mathematicians have become interested in the problem and provided a mathematical classification of the available procedures placing emphasis on the subject of errors, convergence, and stability of the various numerical-integration methods.

CHAPTER II

THE GENERAL MATHEMATICAL MODEL

2.1 Development of the equations of motion

Choose $\bar{i}$, $\bar{j}$, $\bar{k}$ to be a triad of unit orthogonal vectors in a moving frame of reference S', which rotates with angular velocity $\bar{\omega}$ relative to a Newtonian frame S.

Any vector $\bar{V}$ may be expressed in the form

$$\bar{V} = V1\ \bar{i} + V2\ \bar{j} + V3\ \bar{k} \tag{2.1}$$

Now it is desired to determine the rate of change of $\bar{V}$ as estimated by an observer in the Newtonian frame S. It must be emphasized that not only do V1, V2 and V3 vary, but also the vectors $\bar{i}$, $\bar{j}$ and $\bar{k}$.

Differentiation with respect to time t, of equation (2.1) gives

$$\frac{d\bar{V}}{dt} = \frac{dV1}{dt}\bar{i} + \frac{dV2}{dt}\bar{j} + \frac{dV3}{dt}\bar{k} + \frac{V1d\bar{i}}{dt} + \frac{V2d\bar{j}}{dt} + \frac{V3d\bar{k}}{dt} \tag{2.2}$$

Let $\bar{i}$, $\bar{j}$ and $\bar{k}$ be unit vectors fixed in a rigid body S' which rotates with angular velocity $\bar{\omega}$. One may think of $\bar{i}$, $\bar{j}$ and $\bar{k}$ as the position vectors of a particle "B" of this body relative to a base point "A", the origin of $\bar{i}$, $\bar{j}$ and $\bar{k}$. The derivative $\frac{d\bar{i}}{dt}$ is now the velocity of "B" relative to "A", with the same reasoning applying to $\bar{j}$ and $\bar{k}$.

Therefore

$$\frac{d\bar{i}}{dt} = \bar{\omega} \times \bar{i}$$

$$\frac{d\bar{j}}{dt} = \bar{\omega} \times \bar{j} \tag{2.3}$$

$$\frac{d\bar{k}}{dt} = \bar{\omega} \times \bar{k}$$

Now define

$$\frac{\Delta\bar{V}}{\Delta t} = \frac{dV1}{dt}\,\bar{i} + \frac{dV2}{dt}\,\bar{j} + \frac{dV3}{dt}\,\bar{k} \tag{2.4}$$

Substitution of equations (2.3) and (2.4) into (2.2) yields

$$\frac{d\bar{V}}{dt} = \frac{\Delta\bar{V}}{\Delta t} + V1\ (\bar{\omega} \times \bar{i}) + V2\ (\bar{\omega} \times \bar{j}) + V3\ (\bar{\omega} \times \bar{k})$$

or

$$\frac{d\bar{V}}{dt} = \frac{\Delta\bar{V}}{\Delta t} + \bar{\omega} \times (V1\bar{i} + V2\bar{j} + V3\bar{k}) \tag{2.5}$$

Substitution of equation (2.1) into (2.5) yields

$$\frac{d\bar{V}}{dt} = \frac{\Delta\bar{V}}{\Delta t} + \bar{\omega} \times \bar{V}$$

Thus the rate of change of a vector as estimated by an observer in the Newtonian frames "S" is

$$\frac{d\bar{V}}{dt} = \frac{\Delta\bar{V}}{\Delta t} + \bar{\omega} \times \bar{V} \tag{2.6}$$

$\frac{d\bar{V}}{dt}$ consists of two parts.

The first part, $\frac{\Delta\bar{V}}{\Delta t}$, is the rate of change of $\bar{V}$ as measured by an observer moving with S' and is commonly referred to as the "rate of growth," since, in calculating it, one thinks of the vector as changing or growing, whereas $\bar{i}$, $\bar{j}$ and $\bar{k}$ remain constant. The second term, $\bar{\omega} \times \bar{V}$, is due to the rotation of the triad $\bar{i}$, $\bar{j}$, $\bar{k}$ and may be called the "rate of transport." Thus, for a rotating frame, the rate of change of a vector equals the rate of growth plus the rate of transport.

Consider now obtaining the equations of motion of a rigid body. Let $\bar{F}$ denote the total external force and $\bar{T}$ the total moment of the external forces about the mass center. The acceleration $\bar{A}$ of the mass center relative to a Newtonian frame is given by the elementary equation

$$\bar{F} = \frac{d}{dt} (M\bar{V})$$

or if the mass, M, is independent of time

$$\bar{F} = M\bar{A} \tag{2.7}$$

where M is the mass of this body. For motion relative to the mass center

$$\frac{d\bar{L}}{dt} = \bar{T} \tag{2.8}$$

where $\bar{L}$ is the angular momentum about the mass center.

Consider resolving $\bar{A}$, $\bar{F}$, $\frac{d\bar{L}}{dt}$, and $\bar{T}$ along a principal triad $\bar{i}$, $\bar{j}$, $\bar{k}$ at the mass center. The triad is permanently a principal triad fixed in the body and having an angular velocity $\bar{\omega}$.

From equation (2.6)

$$\bar{A} = \frac{\Delta \bar{V}}{\Delta t} + \bar{\omega} \times \bar{V}$$

where

$$\bar{V} = V1\bar{i} + V2\bar{j} + V3\bar{k}$$

and $\bar{V}$ is now the velocity of the mass center.

Also,

$$\bar{\omega} = \omega 1\bar{i} + \omega 2\bar{j} + \omega 3\bar{k}$$

and

$$\frac{\Delta \bar{V}}{\Delta t} = \dot{V}1\bar{i} + \dot{V}2\bar{j} + \dot{V}3\bar{k}$$

where a dot denotes a rate of change with respect to time.

$\bar{\omega} \times \bar{V}$ may be written as

$$\bar{\omega} \times \bar{V} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ \omega 1 & \omega 2 & \omega 3 \\ V1 & V2 & V3 \end{vmatrix}$$

Thus, expanding the determinant,

$$\bar{\omega} \times \bar{V} = (\omega 2\ V3 - \omega 3\ V2)\ \bar{i} + (\omega 3\ V1 - \omega 1\ V3)\ \bar{j}$$

$$+ (\omega 1\ V2 - \omega 2\ V1)\ \bar{k} \qquad (2.9)$$

Using equations (2.7) and (2.9), the equations for the acceleration $\bar{A}$ of the mass center take the form:

$$M\ (\dot{V1} - V2\ \omega3 + V3\ \omega2) = F1$$
$$M\ (\dot{V2} - V3\ \omega1 + V1\ \omega3) = F2 \qquad (2.10)$$
$$M\ (\dot{V3} - V1\ \omega2 + V2\ \omega1) = F3$$

Consider now the equations of motion relative to the mass center.

Let A, B and C be the principal mass moments of inertia at the mass center so that

$$\bar{L} = A\ \omega1\ \bar{i} + B\ \omega2\ \bar{j} + C\ \omega3\ \bar{k} \qquad (2.11)$$

As before,

$$\dot{\bar{L}} = \frac{\Delta\bar{L}}{\Delta t} + \bar{\omega} \times \bar{L}$$

$$\dot{\bar{L}} = A\ \dot{\omega}1\ \bar{i} + B\ \dot{\omega}2\ \bar{j} + C\ \dot{\omega}3\ \bar{k} + (\omega1\ \bar{i} + \omega2\ \bar{j} + \omega3\ k) \times (A\ \omega1\ \bar{i} + B\ \omega2\ \bar{j} + C\ \omega3\ \bar{k})$$

or

$$\dot{\bar{L}} = (A\ \dot{\omega}1 - B\ \omega2\ \omega3 + C\ \omega3\ \omega2)\ \bar{i} + (B\ \dot{\omega}2 - C\ \omega3\ \omega1 + A\ \omega1\ \omega3)\ \bar{j} + (C\ \dot{\omega}3 - A\ \omega1\ \omega2 + B\ \omega2\ \omega1)\ \bar{k}$$

Thus the equations of motion relative to the mass center become

$$A\ \dot{\omega}1 - (B-C)\ \omega2\ \omega3 = T1$$
$$B\ \dot{\omega}2 - (C-A)\ \omega3\ \omega1 = T2$$
$$C\ \dot{\omega}3 - (A-B)\ \omega1\ \omega2 = T3$$

These equations are also known as Euler's equations of motion for a rigid body with a fixed point. The fixed point here being the mass center of the body. The motion of the body relative to the mass center is exactly the same as if the mass center were fixed and the same forces were acting. Thus, the six equations for the components of velocity of the mass center and the components of angular velocity of the body are:

$$M\,(\dot{V}1 - V2\,\omega 3 + V3\,\omega 2) = F1$$
$$M\,(\dot{V}2 - V3\,\omega 1 + V1\,\omega 3) = F2$$
$$M\,(\dot{V}3 - V1\,\omega 2 + V2\,\omega 1) = F3$$

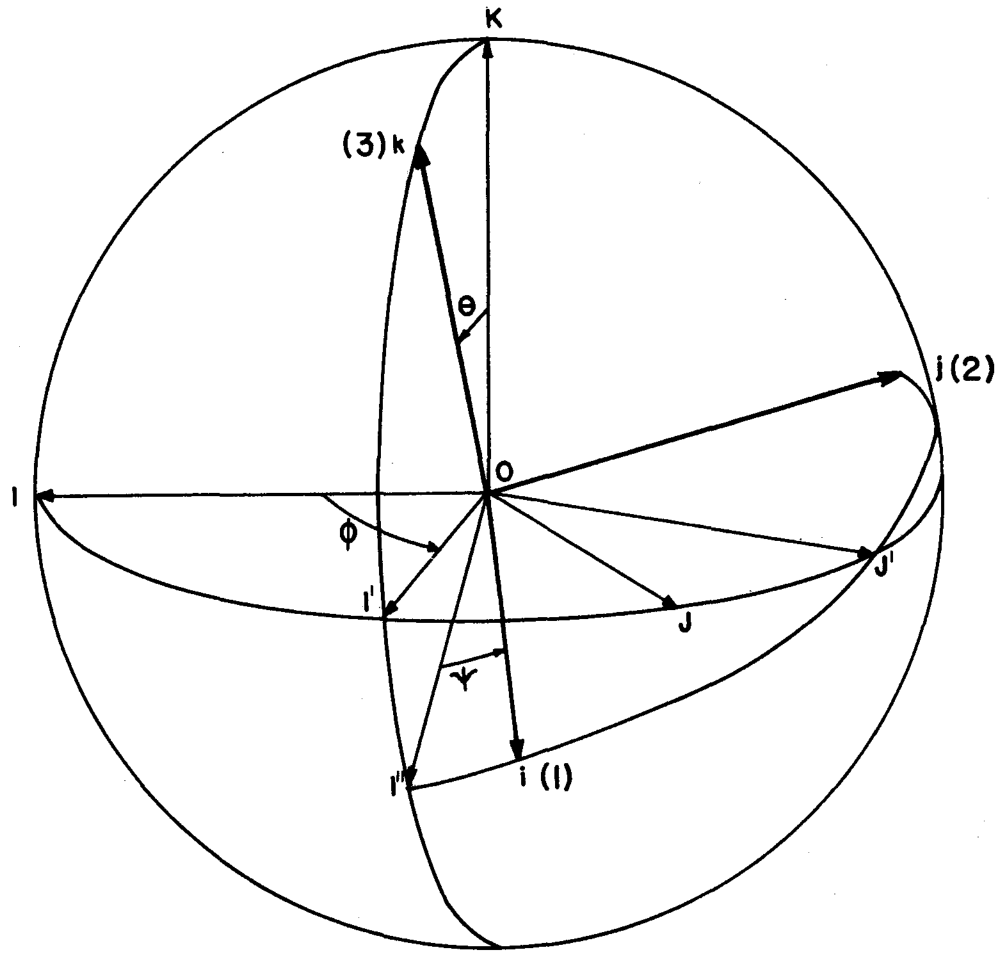
$$\text{(2.12)}$$

$$A\,\dot{\omega}1 - (B-C)\,\omega 2\,\omega 3 = T1$$
$$B\,\dot{\omega}2 - (C-A)\,\omega 3\,\omega 1 = T2$$
$$C\,\dot{\omega}3 - (A-B)\,\omega 1\,\omega 2 = T3$$

2.2 The Eulerian Angles

Consider the problem of describing the position of a rigid body which is free to turn about a point 0. (Figure 1.) A line "L" can be fixed in the body passing through 0, then the body can merely turn about "L". One could assign the angle through which the body has turned about "L" from some initial position, and a final position is completely determined.

To describe the direction of "L" and the angle of rotation, one needs to specify three parameters; the most convenient parameters are the Eulerian angles which will now be discussed.



THE EULERIAN ANGLES

FIGURE I

Figure 1 shows two unit orthogonal right-handed triads (i, j, k) and (I, J, K) at the point O. The triad (i, j, k) is fixed in a rigid body which can turn about "O", and the triad (I, J, K) is a fixed frame of reference. Let the direction of "K" be that of the line "L" mentioned above.

The first Eulerian angle θ is the angle between k and K. The second angle, ϕ is the angle between the plane (k, K) and the plane (K, I). The third angle, ψ is the angle between the plane (k, i) and the plane (K, k). The angles θ and ϕ, being the usual polar angles, fix (k); ψ is the angle of rotation about k. It is evident that θ, ϕ, and ψ determine the position of (i, j, k) and hence the position of the entire body.

To determine when the angles are to be counted as positive or negative, one takes an initial position in which (i, j, k) coincide with (I, J, K) then bring the body to the general position shown in Figure 1 by applying the following rotations in order:

(1) A rotation $\phi\ \bar{K}$; this brings the movable triad (i, j, k) into coincidence with (I', J', K).

(2) A rotation $\theta\ \bar{J}'$; this brings (i, j, k) into coincidence with (I", J', k).

(3) A rotation $\psi\ \bar{k}$; this brings (i, j, k) into the required final position.

Thus, all possible orientations of the body can be obtained by assigning values to θ, ϕ, and ψ in the ranges

$$0 \leq \theta \leq \pi; \quad 0 \leq \phi < 2\pi; \quad 0 \leq \psi < 2\pi$$

The Eulerian angles θ, ϕ, and ψ form a set of generalized coordinates for a rigid body with a fixed point. They can also be used as part of a set of generalized coordinates for a rigid body free to move in space.

Referring again to Figure 1, one may regard "0" as a base point in the body and (I, J, K) as a triad of unit vectors carried by "0" and remaining parallel to axes fixed in a frame of reference. The Cartesian coordinates x, y, z of "0" together with the Eulerian angles θ, ϕ, and ψ, describe the configuration of the body completely. Since the numbers x, y, z, θ, ϕ, and ψ can be varied independently, without violating the rigidity of the body, it is clear that a rigid body, free to move in space, has six degrees of freedom.

A table of scalar products representing the direction cosines of the vectors ($\bar{i}$, $\bar{j}$, $\bar{k}$) relative to ($\bar{I}$, $\bar{J}$, $\bar{K}$) or vice versa, according to the way the table is read, will now be developed.

Consider the rotation $\phi\bar{K}$ about the K axis. This gives Table 1.

Now, consider the rotation $\theta\bar{J}'$ about the J' axis and obtain Table 2.

Finally, consider the rotation $\psi\bar{k}$ about the k axis and obtain Table 3.

Let the first table represent the rotation matrix D, the second E, and the third F.

Thus, matrix equations may be written as

$$\begin{aligned} \{X'\} &= [D]\,\{X\} \\ \{X''\} &= [E]\,\{X'\} \\ \{X\} &= [F]\,\{X''\} \end{aligned} \tag{2.13}$$

TABLE 1. REPRESENTATION OF THE ROTATION $\phi\bar{K}$

	I	J	K
I'	COS ϕ	SIN ϕ	0
J'	−SIN ϕ	COS ϕ	0
K'	0	0	1

TABLE 2. REPRESENTATION OF THE ROTATION $\theta\bar{J}'$

	I'	J'	K'
I"	COS θ	0	−SIN θ
J"	0	1	0
K	SIN θ	0	COS θ

TABLE 3. REPRESENTATION OF THE ROTATION $\psi\bar{K}$

	I"	J"	K
X	COS ψ	SIN ψ	0
Y	−SIN ψ	COS ψ	0
Z	0	0	1

Where

$$\{X''\} = [E]\ [D]\ \{X\}$$

and

$$[X] = [F]\ [E]\ [D]\ \{X\} \qquad (2.14)$$

Now consider premultiplying matrix E by matrix F such that

$$\begin{bmatrix} \text{Cos } \psi & \text{Sin } \psi & 0 \\ -\text{Sin } \psi & \text{Cos } \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \text{Cos } \theta & 0 & -\text{Sin } \theta \\ 0 & 1 & 0 \\ \text{Sin } \theta & 0 & \text{Cos } \theta \end{bmatrix} = [F]\ [E]$$

The product matrix is given by

$$\begin{bmatrix} \text{Cos } \psi \ \ \text{Cos } \theta & \text{Sin } \psi & -\text{Sin } \theta \ \ \text{Cos } \psi \\ -\text{Sin } \psi \ \ \text{Cos } \theta & \text{Cos } \psi & \text{Sin } \theta \ \ \text{Sin } \psi \\ \text{Sin } \theta & 0 & \text{Cos } \theta \end{bmatrix} = [FE]$$

Now consider post multiplying the product matrix [FE] by [D] such that

$$\begin{bmatrix} \text{Cos } \psi \ \ \text{Cos } \theta & \text{Sin } \psi & -\text{Sin } \theta \ \ \text{Cos } \psi \\ -\text{Sin } \psi \ \ \text{Cos } \theta & \text{Cos } \psi & \text{Sin } \theta \ \ \text{Sin } \psi \\ \text{Sin } \theta & 0 & \text{Cos } \theta \end{bmatrix} \begin{bmatrix} \text{Cos } \phi & \text{Sin } \phi & 0 \\ -\text{Sin } \phi & \text{Cos } \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = [FE][D]$$

The product matrix is given by

$$\begin{bmatrix} \text{Cos}\,\theta\ \text{Cos}\,\phi\ \text{Cos}\,\psi & \text{Cos}\,\psi\ \text{Cos}\,\theta\ \text{Sin}\,\phi & \\ -\text{Sin}\,\phi\ \text{Sin}\,\psi & +\text{Cos}\,\phi\ \text{Sin}\,\psi & -\text{Sin}\,\theta\ \text{Cos}\,\psi \\ -\text{Sin}\,\psi\ \text{Cos}\,\theta\ \text{Cos}\,\phi & -\text{Sin}\,\psi\ \text{Sin}\,\phi\ \text{Cos}\,\theta & \\ -\text{Sin}\,\phi\ \text{Cos}\,\psi & +\text{Cos}\,\phi\ \text{Cos}\,\psi & \text{Sin}\,\theta\ \text{Sin}\,\psi \\ \text{Sin}\,\theta\ \text{Cos}\,\phi & \text{Sin}\,\theta\ \text{Sin}\,\phi & \text{Cos}\,\theta \end{bmatrix} = [FED]$$

The matrix [FED] is now used to obtain Table 4 which represents a table of scalar products representing the direction cosines of the vectors ($\bar{i}$, $\bar{j}$, $\bar{k}$) relative to ($\bar{I}$, $\bar{J}$, $\bar{K}$) or vice versa according to the way the table is read.

The ($\bar{i}$, $\bar{j}$, $\bar{k}$) vectors correspond to the (1, 2, 3) directions as shown in Figure 1. It will now be useful to develop a relationship for the angular velocities about these axes in terms of the angular velocities $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$ and the Eulerian angles θ, ϕ and ψ.

The rotations by which the axes 0 (I, J, K) were moved to their final position were through ϕ about $O\bar{K}$, through θ about $O\bar{J}'$ and through ψ about $O\bar{k}$. One can imagine these three rotations to be carried out simultaneously, the angular velocities of the body being $\dot{\phi}$ about $O\bar{K}$, $\dot{\theta}$ about $O\bar{J}'$ and $\dot{\psi}$ about $O\bar{k}$. The components about $O\bar{i}$, $O\bar{j}$, and $O\bar{k}$ of the angular velocity $\dot{\phi}$ are readily obtained from the [FED] matrix as

$$-\text{Sin}\,\theta\ \text{Cos}\,\psi\ \dot{\phi},\quad \text{Sin}\,\theta\ \text{Sin}\,\psi\ \dot{\phi},\ \text{and}\ \text{Cos}\,\theta\ \dot{\phi}.$$

TABLE 4. TABLE OF DIRECTION COSINES RELATING THE ROTATING AXES TO THE FIXED AXES BY MEANS OF THE EULERIAN ANGLES

	I	J	K
i	$-\text{SIN}\,\phi\,\text{SIN}\,\psi$ $+\text{COS}\,\theta\,\text{COS}\,\phi\,\text{COS}\,\psi$	$\text{COS}\,\phi\,\text{SIN}\,\psi$ $+\text{COS}\,\theta\,\text{SIN}\,\phi\,\text{COS}\,\psi$	$-\text{SIN}\,\theta\,\text{COS}\,\psi$
j	$-\text{SIN}\,\phi\,\text{COS}\,\psi$ $-\text{COS}\,\theta\,\text{COS}\,\phi\,\text{SIN}\,\psi$	$\text{COS}\,\phi\,\text{COS}\,\psi$ $-\text{COS}\,\theta\,\text{SIN}\,\phi\,\text{SIN}\,\psi$	$\text{SIN}\,\theta\,\text{SIN}\,\psi$
k	$\text{SIN}\,\theta\,\text{COS}\,\phi$	$\text{SIN}\,\theta\,\text{SIN}\,\phi$	$\text{COS}\,\theta$

The angular velocity components of $\dot{\theta}$ are obtained from the [F] matrix and are

$$\text{Sin } \psi \ \dot{\theta}, \quad \text{Cos } \psi \ \dot{\theta}, \text{ and } 0.$$

The angular velocity ψ lies along the (3) axis and needs no transformation.

Combining the three results, the scalar equations relating the angular velocities $\omega 1$, $\omega 2$ and $\omega 3$ about the 1, 2 and 3 axes and the angular velocities $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$ and the Euler angles θ, ϕ and ψ, are obtained and given by equations (2.16).

$$\begin{aligned}
\omega 1 &= \text{Sin } \psi\dot{\theta} - \text{Sin } \theta \text{ Cos } \psi\dot{\phi} \\
\omega 2 &= \text{Cos } \psi\dot{\theta} + \text{Sin } \theta \text{ Sin } \psi\dot{\phi} \qquad (2.15) \\
\omega 3 &= \text{Cos } \theta\dot{\phi} + \dot{\psi}
\end{aligned}$$

2.3 Derivation of the Rotation Formula

The theorem formulated by Euler in 1776 asserts, of a body with one point 0 fixed, that any displacement is a rotation. In other words, any change of orientation of the body can be achieved by a rotation about some axis through 0. Euler's theorem is equivalent to saying that in any two orientations of the body, there is one line OL fixed in the body whose direction and sense remains invariant.

Consider expressing the shift in position from R to S, of a particle fixed in the body; the coordinates of R relative to a fixed set of axes OX, Y, Z are (Y1, Y2, Y3) and the coordinates of S are

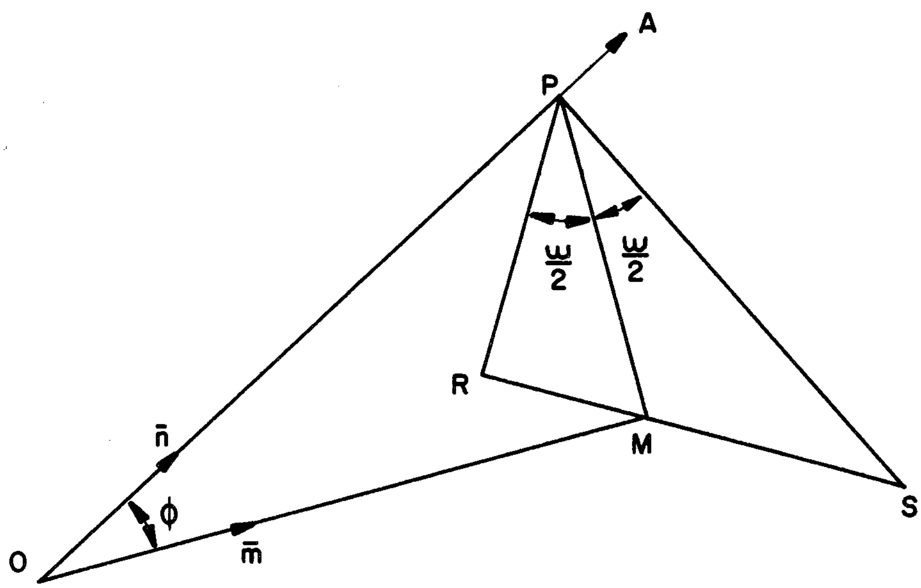


FIGURE 2 DISPLACEMENT OF A POINT

(X1, X2, X3).

Let $\bar{T}$ be the rotation vector in Figure 2. The line OA is in the direction of the vector $\bar{T}$, M is the mid-point of RS, RS is perpendicular to the plane of $\bar{T}$ and $\bar{m}$ since $(1/2)\overline{RS} = \bar{T} \times \bar{m}$ and P is the point where the plane through R perpendicular to OA meets OA. Now,

$\bar{m} = (1/2)(\bar{X} + \bar{Y})$ where $\bar{X}$ and $\bar{Y}$ refer to S and R respectively.

Also, $\overline{MS} = (1/2)(\bar{X} - \bar{Y}) = \bar{T} \times \bar{m}$ (2.16)

or

$$|\overline{MS}| = |\bar{T}|\,|\bar{m}| \quad \text{Sin}\ \phi = |T|\ |PM|$$

Therefore,

$$|\bar{T}| = \left|\frac{MS}{PM}\right|$$

or

$$|\bar{T}| = \text{Tan}\ \frac{\omega}{2} \qquad (2.17)$$

Thus, the shift from R to S has been achieved by a rotation through an angle ω about OA.

The rotation vector can be expressed as

$$\bar{T} = (\text{Tan}\ \frac{\omega}{2})\ \bar{n} \qquad (2.18)$$

Where $\bar{n}$ is a unit vector along the axis of rotation and ω is the angle of rotation.

Equation (2.16) may be rewritten in a more useful form as

$$\bar{S} - \bar{r} = \bar{T} \times (\bar{S} + \bar{r}) \qquad (2.19)$$

where the position vector of a particle of the body relative to the fixed axes before the displacement is designated by $\bar{r}$ and the position vector of the same particle of the body after the displacement is represented by $\bar{S}$.

At this stage it will be useful to solve equation (2.19) for $\bar{S}$.

Multiply each side of equation (2.19) by $\bar{T}$ to form the vector product

$$\bar{T} \times (\bar{S} - \bar{r}) = \bar{T} \times \left\{ \bar{T} \times (\bar{S} \times \bar{r}) \right\} .$$

Since $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\, \bar{b} - (\bar{b} \cdot \bar{a})\, \bar{c}$

$$\bar{T} \times (\bar{S} - \bar{r}) = [\bar{T} \cdot (\bar{S} + \bar{r})]\, \bar{T} - (\bar{T} \cdot \bar{T})\, (\bar{S} + \bar{r})$$

$$\bar{T} \cdot \bar{T} = t^2$$

and $\bar{T} \cdot (\bar{S} + \bar{r}) = 2\, (\bar{T} \cdot \bar{r})$

Thus,

$$\bar{T} \times (\bar{S} - \bar{r}) = 2\, (\bar{T} \cdot \bar{r})\, \bar{T} - t^2\, (\bar{S} + \bar{r}) \qquad (2.20)$$

Adding equations (2.20) and (2.19) gives

$$\bar{T} \times \bar{S} - (\bar{T} \times \bar{r}) + \bar{S} - \bar{r} = 2\, (\bar{T} \cdot \bar{r})\, \bar{T} - t^2\, (\bar{S} + \bar{r})$$
$$+ (\bar{T} \times \bar{S}) + (\bar{T} \times \bar{r})$$

or

$$\bar{S}\ (1 + t^2) + \bar{r}\ (t^2 - 1) = 2\ (\bar{T} \times \bar{r}) + 2\ (\bar{T} \cdot \bar{r})\ \bar{T}$$

Adding 2 $\bar{r}$ to both sides yields

$$(\bar{S} + \bar{r})\ (1 + t^2) = 2 \left\{ \bar{r} + (\bar{T} \cdot \bar{r})\ \bar{T} + \bar{T} \times \bar{r} \right\}$$

or

$$\bar{S} + \bar{r} = \left(\frac{2}{1+t^2}\right) \left\{ \bar{r} + (\bar{T} \cdot \bar{r})\ \bar{T} + (\bar{T} \times \bar{r}) \right\} \quad (2.21)$$

Equation (2.21) is the rotation formula.

Consider now that the rotation vector $\bar{T}$ makes angles α, β, γ, with the OX, Y, Z axes respectively, and that the point R has initial coordinates Xo, Yo and Zo before the rotation ω.

With the above in mind, it can be said that

$$t = \operatorname{Tan} \frac{\omega}{2}$$

$$\bar{T} = (\operatorname{Tan} \frac{\omega}{2})\ (\operatorname{Cos} \alpha\ \bar{i}, + \operatorname{Cos} \beta\ \bar{j} + \operatorname{Cos} \gamma\ \bar{k}$$

and

$$\bar{r} = Xo\ \bar{i} + Yo\ \bar{j} + Zo\ \bar{k}$$

where $\bar{i}$, $\bar{j}$, $\bar{k}$ are unit vectors along the X, Y and Z axes, respectively.

$$\bar{T} \times \bar{r} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \operatorname{Tan} \frac{\omega}{2} \operatorname{Cos} \alpha & \operatorname{Tan} \frac{\omega}{2} \operatorname{Cos} \beta & \operatorname{Tan} \frac{\omega}{2} \operatorname{Cos} \gamma \\ Xo & Yo & Zo \end{vmatrix}$$

Therefore

$$\bar{T} \times \bar{r} = (Zo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \beta - Yo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \gamma)\ \bar{i}$$

$$+ (Xo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \gamma - Zo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha)\ \bar{j}$$

$$+ Yo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha - Xo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \beta\ )\ \bar{k} \tag{2.22}$$

$$(\bar{T} \cdot \bar{r}) = Xo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha + Yo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \beta + Zo\ \mathrm{Tan}\ \frac{\omega}{2}\ \mathrm{Cos}\ \gamma \tag{2.23}$$

$$(\bar{T} \cdot \bar{r})\ \bar{T} = (Xo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}^2\ \alpha + Yo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \beta$$

$$+ Zo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \gamma)\ \bar{i}$$

$$+ (Xo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \beta + Yo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}^2\ \beta$$

$$+ Zo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \beta\ \mathrm{Cos}\ \gamma)\ \bar{j}$$

$$+ (Xo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \gamma + Yo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \beta\ \mathrm{Cos}\ \gamma$$

$$+ Zo\ \mathrm{Tan}^2\ \frac{\omega}{2}\ \mathrm{Cos}^2\ \gamma)\ \bar{k} \tag{2.24}$$

$$\frac{2}{1 + t^2} = \frac{2}{1 + \mathrm{Tan}^2\ \frac{\omega}{2}} = \frac{2}{\mathrm{Sec}^2\ \frac{\omega}{2}} = 2\ \mathrm{Cos}^2\ \frac{\omega}{2} \tag{2.25}$$

Substituting the above expressions into equation (2.21), the expression for $\bar{S}$ becomes

$$\bar{S} = 2(Xo\ \mathrm{Sin}^2\ \frac{\omega}{2}\ \mathrm{Cos}^2\ \alpha + Yo\ \mathrm{Sin}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \beta + Zo\ \mathrm{Sin}^2\ \frac{\omega}{2}\ \mathrm{Cos}\ \alpha\ \mathrm{Cos}\ \gamma$$

$$+ Zo \sin\frac{\omega}{2} \cos\frac{\omega}{2} \cos\beta - Yo \sin\frac{\omega}{2} \cos\frac{\omega}{2} \cos\gamma + Xo \cos^2\frac{\omega}{2} - \frac{Xo}{2}) \bar{i}$$

$$+ 2(Xo \sin^2\frac{\omega}{2} \cos\alpha \cos\beta + Yo \sin^2\frac{\omega}{2} \cos^2\beta + Zo \sin^2\frac{\omega}{2} \cos\beta \cos\gamma$$

$$+ Xo \sin\frac{\omega}{2} \cos\frac{\omega}{2} \cos\gamma - Zo \sin\frac{\omega}{2} \cos\frac{\omega}{2} \cos\alpha + Yo \cos^2\frac{\omega}{2} - \frac{Yo}{2}) \bar{j}$$

$$+ 2(Xo \sin^2\frac{\omega}{2} \cos\alpha \cos\gamma + Yo \sin^2\frac{\omega}{2} \cos\beta \cos\gamma + Zo \sin^2\frac{\omega}{2} \cos^2\gamma$$

$$+ Yo \sin\frac{\omega}{2} \cos\frac{\omega}{2} \cos\alpha - Xo \sin\frac{\omega}{2} \cos\frac{\omega}{2} \cos\beta + Zo \cos^2\frac{\omega}{2} - \frac{Zo}{2}) \bar{k}$$

Let X, Y and Z be the components of $\bar{S}$ in the x, y and z directions respectively.

Then,

$$X = 2 \sin^2\frac{\omega}{2} (Xo \cos^2\alpha + Yo \cos\alpha \cos\beta + Zo \cos\alpha \cos\gamma + \frac{\cos^2\frac{\omega}{2}}{\sin^2\frac{\omega}{2}} Xo)$$

$$+ 2 \sin\frac{\omega}{2} \cos\frac{\omega}{2} (Zo \cos\beta - Yo \cos\gamma) - Xo$$

or adding and subtracting 2 Xo

$$X = Xo - 2 \sin^2\frac{\omega}{2} \left[Xo \sin^2\alpha - Yo \cos\alpha \cos\beta - Zo \cos\alpha \cos\gamma \right]$$
$$+ 2 \sin\frac{\omega}{2} \cos\frac{\omega}{2} \left[Zo \cos\beta - Yo \cos\gamma \right] \qquad (2.26)$$

Similarly,

$$Y = Yo - 2 \sin^2\frac{\omega}{2} \left[Yo \sin^2\beta - Zo \cos\beta \cos\gamma - Xo \cos\beta \cos\alpha \right]$$
$$+ 2 \sin\frac{\omega}{2} \cos\frac{\omega}{2} \left[Xo \cos\gamma - Zo \cos\alpha \right] \qquad (2.27)$$

$$Z = Zo - 2\,\text{Sin}^2\frac{\omega}{2}\left[Zo\,\text{Sin}^2\gamma - Xo\,\text{Cos}\,\gamma\,\text{Cos}\,\alpha - Yo\,\text{Cos}\,\gamma\,\text{Cos}\,\beta\right]$$

$$+ 2\,\text{Sin}\frac{\omega}{2}\,\text{Cos}\frac{\omega}{2}\left[Yo\,\text{Cos}\,\alpha - Xo\,\text{Cos}\,\beta\right] \qquad (2.28)$$

2.4 The Rotation Parameters

It was shown in section 2.3 that the coordinates (X, Y, Z) of the new position of a point whose original coordinates were (Xo, Yo, Zo) can be expressed by equations (2.29), (2.30) and (2.31), when the rigid body is rotated through an angle ω about a line through the origin, whose direction-angles are α, β, γ.

$$X = Xo - 2\,\text{Sin}^2\frac{\omega}{2}\,(Xo\,\text{Sin}^2\alpha - Yo\,\text{Cos}\,\gamma\,\text{Cos}\,\beta - Zo\,\text{Cos}\,\alpha\,\text{Cos}\,\gamma)$$

$$+ 2\,\text{Sin}\frac{\omega}{2}\,\text{Cos}\frac{\omega}{2}\,(Zo\,\text{Cos}\,\beta - Yo\,\text{Cos}\,\gamma) \qquad (2.29)$$

$$Y = Yo - 2\,\text{Sin}^2\frac{\omega}{2}\,(Yo\,\text{Sin}^2\beta - Zo\,\text{Cos}\,\beta\,\text{Cos}\,\gamma - Xo\,\text{Cos}\,\beta\,\text{Cos}\,\alpha)$$

$$+ 2\,\text{Sin}\frac{\omega}{2}\,\text{Cos}\frac{\omega}{2}\,(Xo\,\text{Cos}\,\gamma - Zo\,\text{Cos}\,\alpha) \qquad (2.30)$$

$$Z = Zo - 2\,\text{Sin}^2\frac{\omega}{2}\,(Zo\,\text{Sin}^2\gamma - Xo\,\text{Cos}\,\gamma\,\text{Cos}\,\alpha - Yo\,\text{Cos}\,\gamma\,\text{Cos}\,\beta)$$

$$+ 2\,\text{Sin}\frac{\omega}{2}\,\text{Cos}\frac{\omega}{2}\,(Yo\,\text{Cos}\,\alpha - Xo\,\text{Cos}\,\beta) \qquad (2.31)$$

Now introduce parameters ξ, η, ζ, χ, defined by the equations

$$\xi = \text{Cos}\,\alpha\,\text{Sin}\frac{\omega}{2}$$

$$\eta = \text{Cos}\,\beta\,\text{Sin}\frac{\omega}{2}$$

$$\zeta = \text{Cos}\,\gamma\,\text{Sin}\frac{\omega}{2}$$

$$\chi = \text{Cos}\ \frac{\omega}{2} \tag{2.32}$$

These parameters satisfy the relation $\xi^2 + \eta^2 + \zeta^2 + \chi^2 = 1$.

Substituting equation (2.19) into equations (2.16), (2.17) and (2.18), the equations for X, Y and Z become

$$X = (\xi^2 - \eta^2 + \zeta^2 + \chi^2)\ Xo + 2(\xi\eta - \zeta\chi)\ Yo + 2(\xi\zeta + \eta\chi) Zo$$

$$Y = 2(\xi\eta + \zeta\chi)\ Xo + (-\xi^2 + \eta^2 - \zeta^2 + \chi^2)\ Yo + 2\ (\eta\zeta - \xi\chi)\ Zo$$

$$Z = 2\ (\xi\zeta - \eta\chi)\ Xo + 2\ (\eta\zeta + \xi\chi)\ Yo + (-\xi^2 - \eta^2 + \zeta^2 + \chi^2)\ Zo \tag{2.33}$$

These equations may also be written in matrix form as

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} (\xi^2 - \eta^2 - \zeta^2 + \chi^2) & 2(\xi\eta - \zeta\chi) & 2(\xi\zeta + \eta\chi) \\ 2(\xi\eta + \zeta\chi) & (-\xi^2 + \eta^2 - \zeta^2 + \chi^2) & 2(\eta\zeta - \xi\chi) \\ 2(\xi\zeta - \eta\chi) & 2(\eta\zeta + \xi\chi) & (-\xi^2 - \eta^2 + \zeta^2 + \chi^2) \end{bmatrix} \begin{bmatrix} Xo \\ Yo \\ Zo \end{bmatrix}$$

If the coordinate axes are denoted by OX, Y, Z and if movable axes which originally coincide with these are brought into the position Ox, y, z by the given rotation, the direction-cosines of the two sets of axes with respect to each other are given by Table 5.

2.5 Connection of the Eulerian Angles with the Rotation Parameters

The relations between the Eulerian angles θ, ϕ, ψ and the parameters ξ, η, ζ, χ may be obtained by comparing the tables of direction-cosines given by Table 4 and Table 5.

TABLE 5. RELATION BETWEEN THE ROTATING AXES AND THE FIXED AXES IN TERMS OF THE ROTATION PARAMETERS

	X	Y	Z
x	$\xi^2-\eta^2-\zeta^2+\chi^2$	$2(\xi\eta+\zeta\chi)$	$2(\xi\zeta-\eta\chi)$
y	$2(\xi\eta-\zeta\chi)$	$-\xi^2+\eta^2-\zeta^2+\chi^2$	$2(\eta\zeta+\xi\chi)$
z	$2(\xi\zeta+\eta\chi)$	$2(\eta\zeta-\xi\chi)$	$-\xi^2-\eta^2+\zeta^2+\chi^2$

By comparison

$$2(\xi\zeta + \eta\chi) = \text{Cos } \phi \text{ Sin } \theta \tag{2.34}$$

$$2(\eta\zeta - \xi\chi) = \text{Sin } \phi \text{ Sin } \theta \tag{2.35}$$

$$2(\xi\zeta - \eta\chi) = [- \text{Sin } \theta \text{ Cos } \psi] \tag{2.36}$$

$$2(\eta\zeta + \xi\chi) = \text{Sin } \theta \text{ Sin } \psi \tag{2.37}$$

$$2(\xi\eta - \zeta\chi) = -[\text{Cos } \phi \text{ Cos } \theta \text{ Sin } \psi + \text{Sin } \phi \text{ Cos } \psi] \tag{2.38}$$

$$2(\xi\eta + \zeta\chi) = [\text{Sin } \phi \text{ Cos } \theta \text{ Cos } \psi + \text{Cos } \phi \text{ Sin } \psi] \tag{2.39}$$

From (2.34) and (2.36)

$$\xi\zeta = \frac{\text{Sin } \phi}{4} [\text{Cos } \phi - \text{Cos } \psi] \tag{2.40}$$

From (2.35) and (2.37)

$$\eta\zeta = \frac{\text{Sin } \theta}{4} [\text{Sin } \phi + \text{Sin } \psi] \tag{2.41}$$

From (2.40) and (2.41)

$$\xi = \eta \left(\frac{\text{Cos } \phi - \text{Cos } \psi}{\text{Sin } \phi + \text{Sin } \psi}\right) \tag{2.42}$$

From (2.38) and (2.39) after some trigonometric substitutions

$$\xi\eta = 1/4 [\text{Cos } \theta \text{ Sin } (\phi - \psi) + \text{Sin } (\psi - \phi)] \tag{2.43}$$

Substituting (2.43) into (2.42) and making some trigonometric substitutions

$$\eta^2 = \frac{1 - \text{Cos } \theta}{2} \left[\frac{1 + \text{Cos } (\theta - \psi)}{2} \right] \qquad (2.44)$$

By making use of a trigonometric identity, equation (2.39) may be written as

$$\eta = \text{Sin } \frac{\theta}{2} \text{ Cos } \left(\frac{\psi}{2} - \frac{\phi}{2}\right) \qquad (2.45)$$

Substituting (2.45) into (2.42) and again making use of trigonometric identities

$$\xi = \text{Sin } \frac{\theta}{2} \text{ Sin } \left(\frac{\psi - \phi}{2}\right) \qquad (2.46)$$

Equation (2.40) may be rewritten as

$$\zeta = -\left[\frac{\text{Sin } \theta}{2} \text{ Sin } \left(\frac{\phi + \psi}{2}\right) \text{ Sin } \left(\frac{\phi - \psi}{2}\right)\right] \frac{1}{\xi} \qquad (2.47)$$

Substituting (2.46) into (2.47), the expression for ζ becomes

$$\zeta = \text{Cos } \frac{\theta}{2} \text{ Sin } \frac{\phi + \psi}{2} \qquad (2.48)$$

Performing some trigonometric substitutions and combining equations (2.35) and (2.37), the expression for $\xi\chi$ becomes

$$\xi\chi = \frac{\text{Sin } \theta}{2} \text{ Cos } \left(\frac{\psi + \phi}{2}\right) \text{ Sin } \left(\frac{\psi - \phi}{2}\right) \qquad (2.49)$$

Substituting equation (2.46) into equation (2.49), the expression for χ becomes

$$\chi = \text{Cos}\,\frac{\theta}{2}\ \ \text{Cos}\,\left(\frac{\psi + \phi}{2}\right) \tag{2.50}$$

The equations for the rotation parameters in terms of the Eulerian angles are now given by

$$\begin{aligned} \xi &= \text{Sin}\,\frac{\theta}{2}\ \ \text{Sin}\,\left(\frac{\psi - \phi}{2}\right) \\ \eta &= \text{Sin}\,\frac{\theta}{2}\ \ \text{Cos}\,\left(\frac{\psi - \phi}{2}\right) \\ \zeta &= \text{Cos}\,\frac{\theta}{2}\ \ \text{Sin}\,\left(\frac{\psi + \phi}{2}\right) \\ \chi &= \text{Cos}\,\frac{\theta}{2}\ \ \text{Cos}\,\left(\frac{\psi + \phi}{2}\right) \end{aligned} \tag{2.51}$$

Thus, using equations (2.33) and (2.51), one may obtain the displacements due to a rotation about the center of mass, of a point in a rigid body having initial coordinates, Xo, Yo, Zo.

2.6 The Displacement of any Point of a Rigid Body

The displacement of any point of a rigid body is equal to the displacement of the center of mass plus the motion relative to the center or mass.

The equations for motion relative to the center of mass are given by equations (2.33) and (2.51).

Define XCM, YCM and ZCM to be the translations of the mass center in the x, y, and z directions, respectively. The expressions for the displacements of any point of the rigid body are now given by

$$XP = XCM + (\xi^2 - \eta^2 - \zeta^2 + \chi^2)xpo + 2(\xi\eta - \zeta\chi)ypo$$
$$+ 2(\xi\zeta + \eta\chi)zpo$$
$$YP = YCM + 2(\xi\eta + \zeta\chi)xpo + (-\xi^2 + \eta^2 - \zeta^2 + \chi^2)ypo \qquad (2.52)$$
$$+ 2(\eta\zeta - \xi\chi)zpo$$
$$ZP = ZCM + 2(\xi\zeta - \eta\chi)xpo + 2(\eta\zeta + \xi\chi)ypo + (-\xi^2 - \eta^2 + \zeta^2 + \chi^2)zpo$$

Where zpo, ypo and zpo are the initial coordinates of any point P and ξ, η, ζ, χ are defined by equations (2.51).

CHAPTER III

FORMULATION OF THE POST PROBLEM

3.1 Definition of Axes

The origin of the coordinate systems will be established at the mass center of the post assembly. Let XX, YY, ZZ be a fixed right-handed coordinate system having its XX-YY plane parallel to the plane of vehicular motion and its XX axis in the direction of the highway. Let X, Y, Z be a fixed right-handed coordinate system having its axes coinciding with the initial positions of the principal 1, 2, 3 axes and obtained by rotating the XX, YY, ZZ system an angle δ about the XX axis. The fixed directions I, J, K on Figure 1 correspond to X, Y and Z, respectively, and the moving axes (1, 2, 3) are the principal axes of the body. Let XV, YV, ZV be a fixed right-handed coordinate system obtained by rotating the XX, YY, ZZ system an angle α about the negative ZZ axis. The angle α is the angle that the path of travel of the vehicle makes with the XX axis.

3.2 Representation of the Idealized Vehicle

The model vehicle is assumed to be a single degree-of-freedom system consisting of a rigid mass and a massless spring as shown in Figure 3. The spring is assumed to be incapable of restitution and the rigid mass and its velocity simulate the momentum of the vehicle. The energy absorbed by the vehicle is obtained from the spring force-deformation relation.

The automobile is a highly redundant multidegree-of-freedom system composed of various types of structural elements. All these elements have certain energy absorbing characteristics and under impact forces, are capable of absorbing various amounts of energy. The total energy absorbed by the vehicle is the sum of the incremental energies absorbed by each of its components. It is hoped that satisfactory results can be obtained for the simple system as long as the spring is capable of absorbing an amount of energy equivalent to that of an actual vehicle.

Vehicle simulation is a vital part of the overall problem, but present research is concerned mainly with the development of a model to simulate the response of the "break-away" post and not to simulate vehicle response.

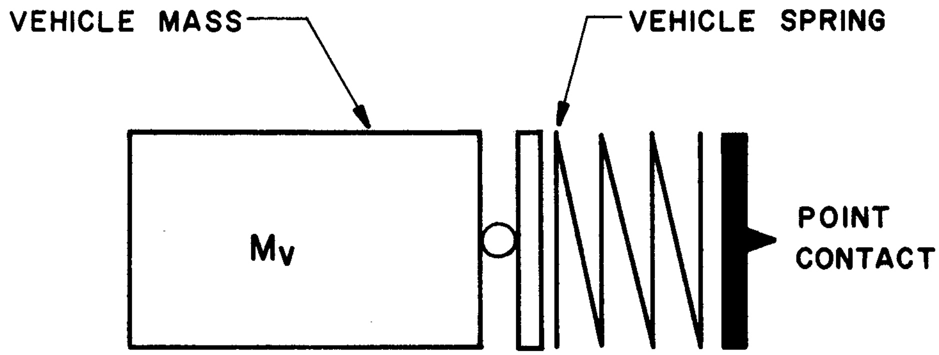
The mass of the vehicle is M_V and K represents the spring constant. The spring force is FS; thus the equation of motion for the vehicle becomes

$$M_V \dot{VV} + FS = 0$$

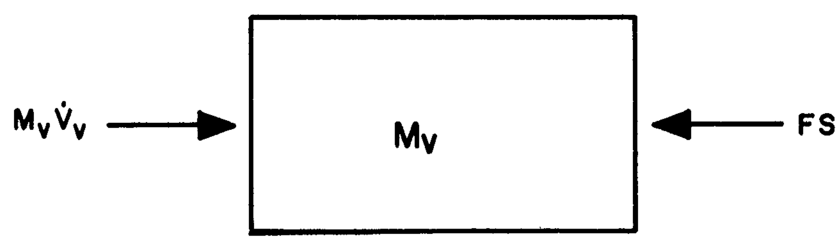
where VV is the vehicle velocity.

3.3 Definition of Forces

The forces that will be assumed to be acting on the post will be the frictional forces, the normal force, the gravity force and the spring force. The frictional forces will be such as to oppose motion and will be taken in the positive XX and negative YY directions. The normal force will be in the positive ZZ direction and



(a) SINGLE DEGREE-OF-FREEDOM VEHICLE MODEL



(b) VEHICLE FREE BODY

FIGURE 3 THE IDEALIZED VEHICLE

the gravity force in the negative ZZ direction. The spring force, which is due to vehicle impact, when considering the XX, YY plane may have components in the negative XX and positive YY directions or merely in the negative XX direction, depending on the angle of vehicle impact.

The frictional forces will be assumed to be acting at a point Q at the base of the post and will be designated by FFXX and FFYY, referring to forces in the XX and YY directions respectively. The normal force will also be acting at the point Q and will be designated by FN. The gravity force will, of course, be acting at the mass center and will be represented by mg. The spring force will be assumed to be acting at a point S on the post and its components will be represented by FSXX and FSYY.

With the above in mind, the equations for the summation of forces in the XX, YY and ZZ directions respectively, are given by

$$\begin{aligned} \Sigma\ FXX &= -\ FSXX + FFXX \\ \Sigma\ FYY &= FSYY - FFYY \\ \Sigma\ FZZ &= FN - mg \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} FSXX &= FS\ \text{Cos}\ \alpha \\ FSYY &= FS\ \text{Sin}\ \alpha \end{aligned} \tag{3.2}$$

Equations (3.1) hold for the particular way the problem is being formulated, as long as the XX, YY and ZZ axes originally coincide with the principal axes of the post. This is not generally the case, so it will be convenient to modify equations (3.1). Figure 4 represents the general situation for the light post under consideration.

The original position of the principal axes X, Y and Z are related to the XX, YY and ZZ system by the table of direction cosines, given by Table 6.

It is convenient to resolve the forces acting in the XX, YY and ZZ directions to the X, Y and Z system.

Using Table 6, the equations for the forces in the X, Y and Z system may be written as

$$\Sigma \text{ FX} = - \text{ FSXX} + \text{FFXX}$$

$$\Sigma \text{ FY} = \text{FSYY Cos } \delta + \text{FN Sin } \delta$$

$$- \text{ mg Sin } \delta - \text{FFY Cos } \delta$$

$$\Sigma \text{ FZ} = \text{FFYY Sin } \delta + \text{FN Cos } \delta$$

$$- \text{ mg Cos } \delta - \text{FSYY Sin } \delta$$

It is now desired to obtain expressions for F1, F2 and F3 in order that they may be used in equations (2.12).

To obtain these expressions, it is necessary to employ Table 4 which relates the 1, 2, 3 or principal directions of the body to the X, Y, Z axes.

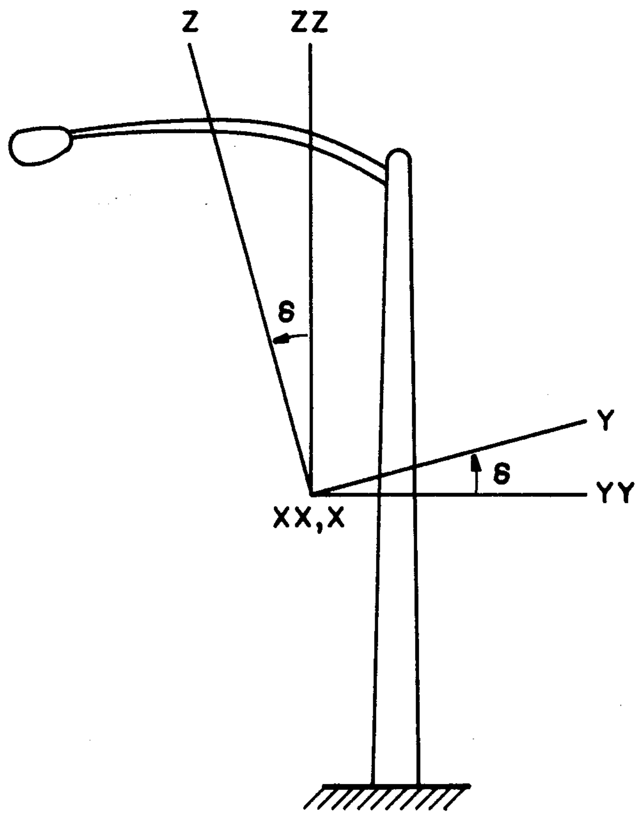


FIGURE 4 REPRESENTATION OF THE MASS CENTER OF THE POST

TABLE 6. RELATION BETWEEN ORIGINAL POSITION OF THE PRINCIPAL AXES AND THE BASE COORDINATE SYSTEM

	X	Y	Z
XX	1	0	0
YY	0	COS δ	$-$SIN δ
ZZ	0	SIN δ	COS δ

DEFINE:

$$D1X = -\sin\phi \sin\psi + \cos\theta \cos\phi \cos\psi$$

$$D1Y = \cos\phi \cos\psi + \cos\theta \sin\phi \cos\psi$$

$$D1Z = -\sin\theta \cos\psi$$

$$D2X = -\sin\phi \cos\psi - \cos\theta \cos\phi \sin\psi$$

$$D2Y = \cos\phi \cos\psi - \cos\theta \sin\phi \sin\psi \qquad (3.4)$$

$$D2Z = \sin\theta \sin\psi$$

$$D3X = \sin\theta \cos\phi$$

$$D3Y = \sin\theta \sin\phi$$

$$D3Z = \cos\theta$$

Using equations (3.4) and (3.3), the expressions for F1, F2 and F3 may be written as

$$F1 = (\Sigma\ FX)\ D1X + (\Sigma\ FY)\ D1Y + (\Sigma\ FZ)\ D1Z$$

$$F2 = (\Sigma\ FX)\ D2X + (\Sigma\ FY)\ D2Y + (\Sigma\ FZ)\ D2Z \qquad (3.5)$$

$$F3 = (\Sigma\ FX)\ D3X + (\Sigma\ FY)\ D3Y + (\Sigma\ FZ)\ D3Z$$

Equations (3.5) now constitute the right-hand side of the first set of equations (2.12).

3.4 Definition of Torques About the Mass Center

It is now desired to obtain the right-hand side of the second set of equations (2.12). The same approach that was followed for the forces will be employed here. Torques about the X, Y, and Z axes will be obtained first, and they will then be resolved to the 1, 2 and 3 directions.

Consider taking moments about the center of mass of the body. Let $\bar{r}$ be a position vector drawn from the mass center to a point on the body where a force $\bar{F}$ is acting.

$$\bar{r} = X\bar{i} + Y\bar{j} + Z\bar{k}$$

$$\bar{F} = FX\bar{i} + FY\bar{j} + FZ\bar{k} \tag{3.6}$$

The torque equation is defined by

$$\bar{T} = \bar{r} \times \bar{F}$$

or

$$\bar{T} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x & y & z \\ FX & FY & FZ \end{vmatrix} \tag{3.7}$$

Expanding the determinant, the torque equation becomes

$$\bar{T} = [(FZ)\ Y - (FY)\ Z]\ \bar{i} + [(FX)\ Z - (FZ)\ X]\ \bar{j} + [(FY)\ X - (FX)\ Y]\ \bar{k} \tag{3.8}$$

The vector equation (3.8) may be broken up into the three scalar equations given by equations (3.9).

$$TX = (FZ)\ Y - (FY)\ Z$$

$$TY = (FX)\ Z - (FZ)\ X \qquad (3.9)$$

$$TZ = (FY)\ X - (FX)\ Y$$

Here TX, TY, TZ represent torques about the x, y and z axes respectively.

Also, X, Y and Z in the right-hand side of equations (3.9) represent the moment arms and FX, FY and FZ represent the forces.

Let XS and YS and FS represent the moment arms to the spring forces and let XQ and YQ and ZQ represent the moment arms to the frictional forces and the normal force.

Substituting equations (3.3) into (3.9) and leaving out the mg term, the expressions for the torques are given by

$$TX = -\ (FSYY\ \mathrm{Sin}\ \delta)\ (YS) + (FFYY\ \mathrm{Sin}\ \delta + FN\ \mathrm{Cos}\ \delta\ )\ (YQ)$$

$$-\ (FSYY\ \mathrm{Cos}\ \delta)\ (ZS) + (FFYY\ \mathrm{Cos}\ \delta - FN\ \mathrm{Sin}\ \delta\ )\ (ZQ)$$

$$TY = -\ (FSXX)\ (ZS) + (FFXX)\ (ZQ) + (FSYY\ \ \mathrm{Sin}\ \delta\ )\ (XS)$$

$$-\ (FFYY\ \mathrm{Sin}\ \delta + FN\ \mathrm{Cos}\ \delta\ )\ XQ$$

$$TZ = -\ (FSYY\ \mathrm{Cos}\ \delta)\ (XS) + (FN\ \mathrm{Sin}\ \delta - FFYY\ \mathrm{Cos}\ \delta\ )\ (XQ)$$

$$+\ (FSXX)\ (YS) - (FFXX)\ (YQ) \qquad (3.10)$$

Using equations (3.4) the torques about the 1, 2 and 3 axes become

$$T1 = (TX)\ D1X + (TY)\ D1Y + (TZ)\ D1Z$$

$$T2 = (TX)\ D2X + (TY)\ D2Y + (TZ)\ D2Z \qquad (3.11)$$

$$T3 = (TX)\ D3X + (TY)\ D3Y + (TZ)\ D3Z$$

There now remains the problem of obtaining expressions for the moment arms that appear in the right-hand side of equations (3.10).

The displacement of any point in the rigid body is equal to the displacement of the center of mass plus the motion relative to the center of mass. Equations (2.52) give the displacements of any point of the rigid body.

Consider the Z-X plane and a point Q of the post P-Q as shown in Figure 5.

Now, consider the Z-Y plane and the same point Q of the post P-Q as shown in Figure 6.

Again, consider the Z-Y plane and the point P of the post P-Q having moved to position P' as shown in Figure 7.

From Figures 5, 6 and 7, it is clear that the moment arms from the center of mass for any time t, are equal to

$$(XQ - XCM),\ (YQ - YCM),\ \text{and}\ (ZQ - ZCM)$$

where Q denotes any point where a force may be acting.

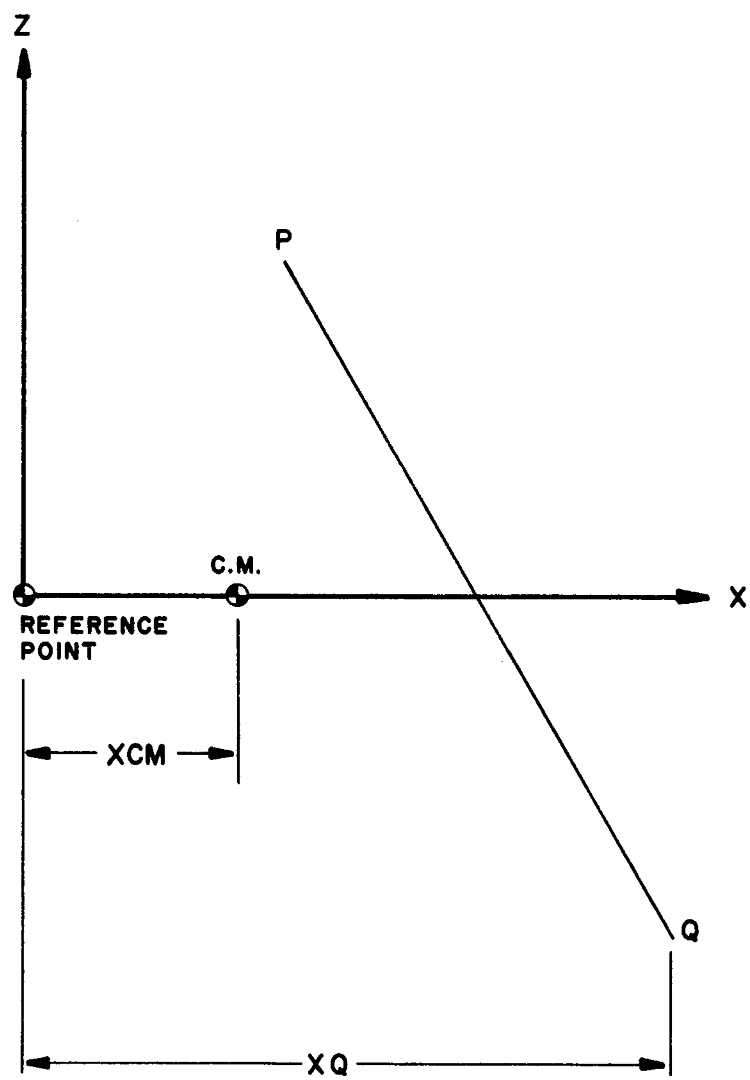


FIGURE 5 AN X-TRANSLATION IN THE X-Z PLANE

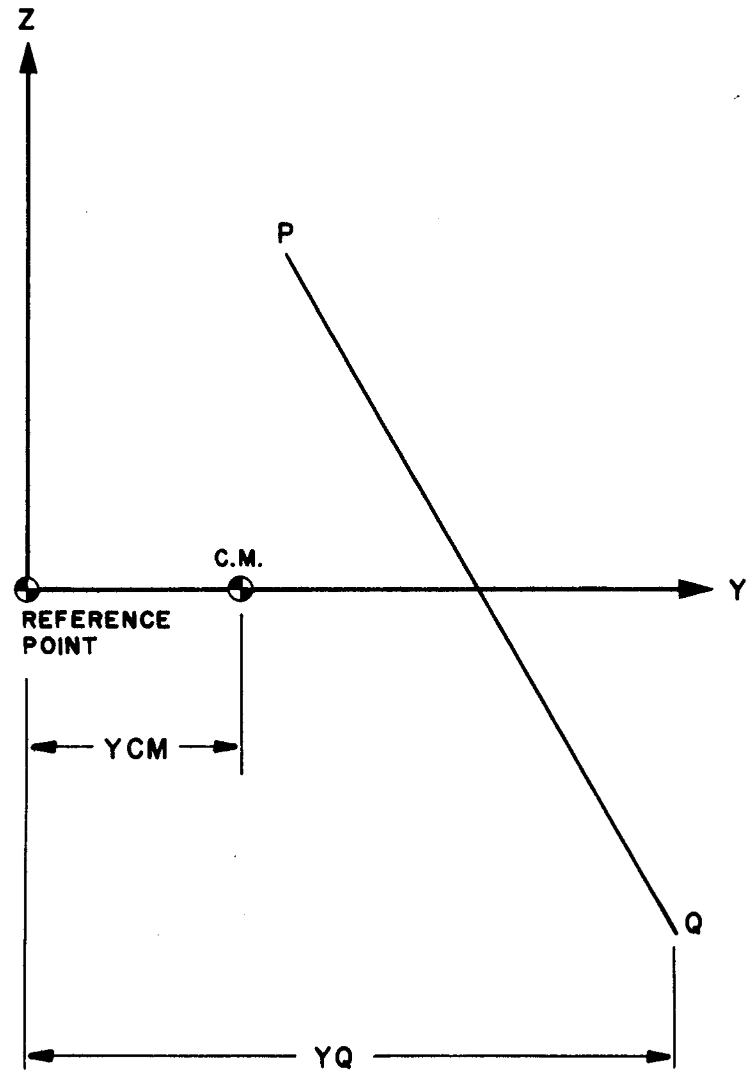


FIGURE 6 A Y–TRANSLATION IN THE Y–Z PLANE

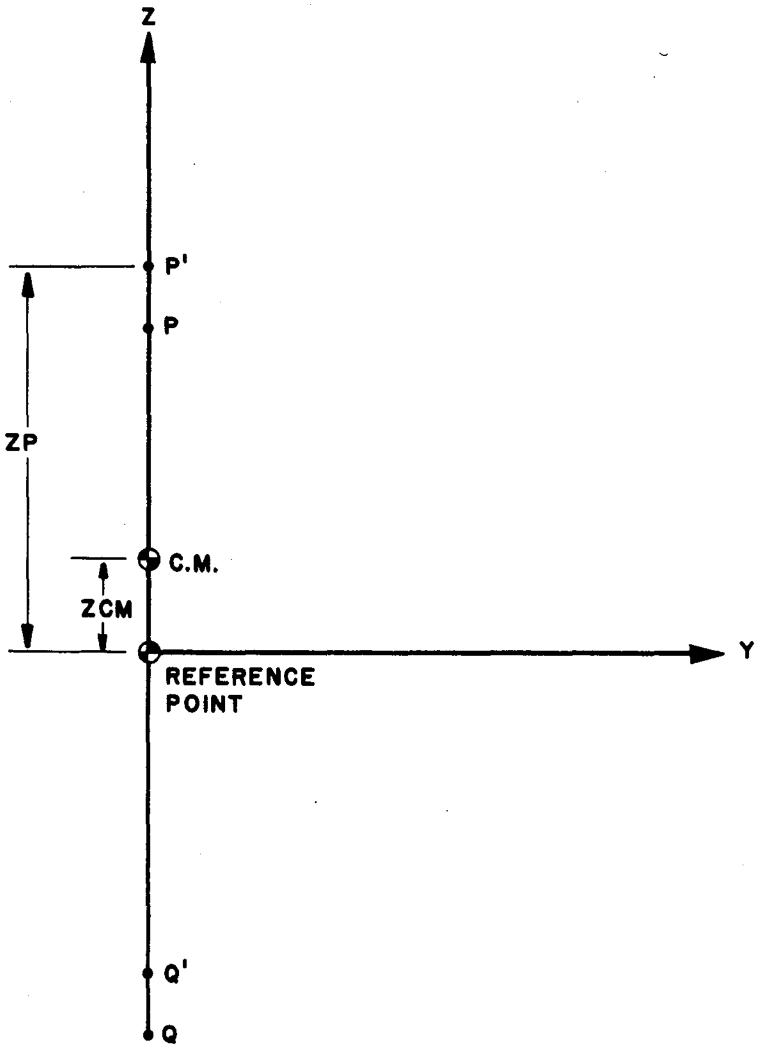


FIGURE 7 A Z TRANSLATION IN THE Y-Z PLANE

The equations for the torques about the X, Y and Z axes now become

$$TX = -(FSYY \sin\delta)\ (YS - YCM) + (FFYY \sin\delta + FN \cos\delta)\ (YQ - YCM)$$

$$-(FSYY \cos\delta)\ (ZS - ZCM) + (FFYY \cos\delta - FN \sin\delta)\ (ZQ-ZCM)$$

$$- TXO$$

$$TY = -(FSXX)\ (ZS - ZCM) + (FFXX)\ (ZQ - ZCM) + (FSYY \sin\delta)\ (XS - XCM)$$

$$-(FFYY \sin\delta + FN \cos\delta)\ (XQ - XCM)$$

$$TZ = (FSYY \cos\delta)\ (XS - XCM) + (FN \sin\delta - FFY \cos\delta)\ (XQ - XCM)$$

$$+ (FSXX)\ (YS - YCM) - (FFXX)\ (YQ - YCM) \qquad (3.12)$$

where TXO is the torque that is required to put the post initially in equilibrium (see Figure 3).

The values of T1, T2 and T3 are obtained by substituting equations (3.12) into equations (3.11).

3.5 Definition of the Spring Force

Consider Figure 8 and the XX-YY plane. The vehicle is approaching the post along the line V which makes an angle α with the XX axis. The point "S" on the post, where it will be assumed that the vehicle strikes the post, will have X, Y and Z displacements as given by equations (2.52). At this stage, it is convenient to obtain expressions for the translations of the point "S" in the plane of

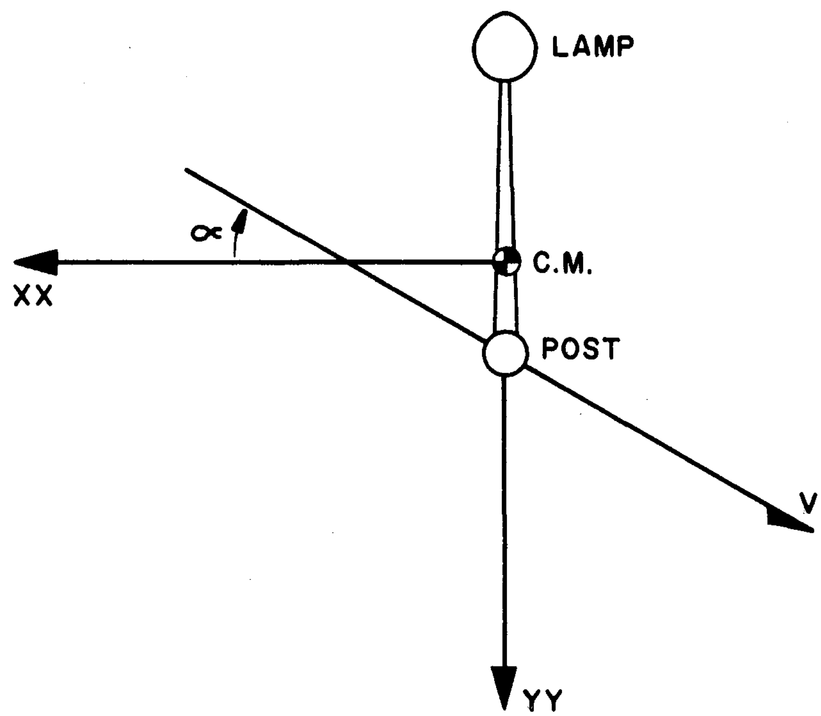


FIGURE 8 DEFINITION OF VEHICLE MOTION

travel of the vehicle. From Figure 4, it is clear that the Y and Z displacements obtained from equations (2.39) are off by the rotation δ to the displacements that occur in the YY and ZZ directions.

As before, defining by XX, YY and ZZ, the coordinate system that has its XX-YY plane in the same plane as vehicle motion is taking place and employing Table 6, it is possible to transform the displacements obtained from equations (2.52) to the XX, YY and ZZ directions.

The equations for the displacements are given by

$$\begin{aligned} XXS &= XS \\ YYS &= YS \cos \delta - ZS \sin \delta \qquad (3.13) \\ ZZS &= YS \sin \delta + ZS \cos \delta \end{aligned}$$

Now consider a coordinate axes transformation as shown in Figure 9.

XV is along the line of vehicle motion and YV is normal to this direction. Table 7 now relates the XX, YY and ZZ axes to the XV, YV and ZV system.

The equations for displacements in the XV, YV and ZV system are given by

$$\begin{aligned} XV &= XX \cos \alpha - YY \sin \alpha \\ YV &= XX \sin \alpha + YY \cos \alpha \qquad (3.14) \\ ZV &= ZZ \end{aligned}$$

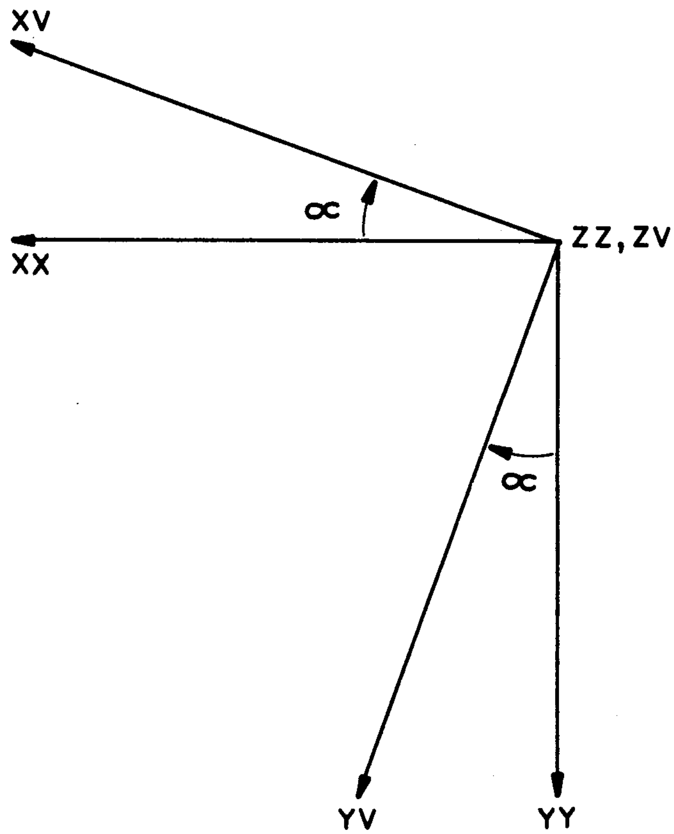


FIGURE 9 AXES DEFINITION FOR VEHICLE MOTION

TABLE 7. RELATION BETWEEN VEHICULAR AND BASE COORDINATE SYSTEMS

	XX	YY	ZZ
XV	COS α	$-$SIN α	0
YV	SIN α	COS α	0
ZV	0	0	1

Let XVS be the displacement of the point S of the post in the XV direction and let SV be the displacement of the center of mass of the vehicle in the XV direction.

Consider Figure 10 and the post PQ having moved to position P'Q'. It is clear that the spring force FS is given by

$$FS = K\ [(SVO - SV) - (XVO - XVS)] \qquad (3.15)$$

where K is the spring constant of the vehicle.

3.6 Summary of Equations

<u>Equations of Motion</u>

$$M(\dot{V}1 - V2\ \omega 3 + V3\ \omega 2) = F1$$

$$M(\dot{V}2 - V3\ \omega 1 + V1\ \omega 3) = F2 \qquad (3.16)$$

$$M(\dot{V}3 - V1\ \omega 2 + V2\ \omega 1) = F3$$

$$A\ \dot{\omega}1 - (B - C)\ \omega 2\ \omega 3 = T1$$

$$B\ \dot{\omega}2 - (C - A)\ \omega 3\ \omega 1 = T2 \qquad (3.17)$$

$$C\ \dot{\omega}3 - (A - B)\ \omega 1\ \omega 2 = T3$$

$$Mv\ \dot{V}V + FS = 0 \qquad (3.18)$$

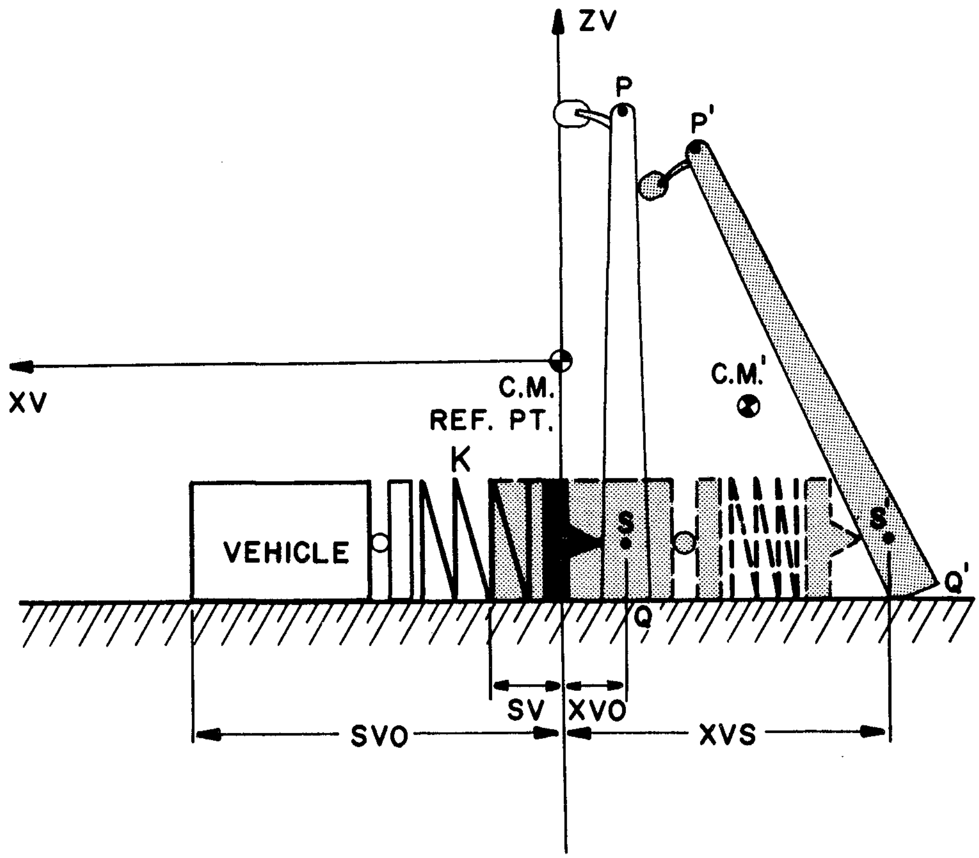


FIGURE 10 REPRESENTATION OF THE SPRING FORCE

Equations for Angular Velocities

$$\omega 1 = \text{Sin}\ \psi\ \dot{\theta} - \text{Sin}\ \theta\ \text{Cos}\ \psi\ \dot{\phi}$$

$$\omega 2 = \text{Cos}\ \psi\ \dot{\theta} + \text{Sin}\ \theta\ \text{Sin}\ \psi\ \dot{\phi} \qquad (3.19)$$

$$\omega 3 = \text{Cos}\ \theta\ \dot{\phi} + \dot{\psi}$$

Rotation Parameters

$$\xi = \text{Sin}\ \frac{\theta}{2}\ \text{Sin}\ \left(\frac{\psi - \phi}{2}\right)$$

$$\eta = \text{Sin}\ \frac{\theta}{2}\ \text{Cos}\ \left(\frac{\psi - \phi}{2}\right)$$

$$\zeta = \text{Cos}\ \frac{\theta}{2}\ \text{Sin}\ \left(\frac{\psi + \phi}{2}\right) \qquad (3.20)$$

$$\chi = \text{Cos}\ \frac{\theta}{2}\ \text{Cos}\ \left(\frac{\psi + \phi}{2}\right)$$

Translations of a Point P in X, Y and Z Directions

$$XP = XCM + (\xi^2 - \eta^2 - \zeta^2 + \chi^2)\ XPO + 2(\xi\eta - \zeta\chi)\ YPO + 2(\xi\zeta + \eta\chi)\ ZPO$$

$$YP = YCM + 2(\xi\eta + \zeta\chi)\ XPO + (-\xi^2 + \eta^2 - \zeta^2 + \chi^2)\ YPO + 2(\eta\zeta - \xi\chi)\ ZPO \qquad (3.21)$$

$$ZP = ZCM + 2(\xi\zeta - \eta\chi)\ XPO + 2(\eta\zeta + \xi\chi)\ YPO + (-\xi^2 - \eta^2 + \zeta^2 + \chi^2)\ ZPO$$

Direction Cosines

$$D1X = -\operatorname{Sin}\phi \operatorname{Sin}\psi + \operatorname{Cos}\theta \operatorname{Cos}\phi \operatorname{Cos}\psi$$

$$D1Y = \operatorname{Cos}\phi \operatorname{Sin}\psi + \operatorname{Cos}\theta \operatorname{Sin}\phi \operatorname{Cos}\psi \qquad (3.22)$$

$$D1Z = -\operatorname{Sin}\theta \operatorname{Cos}\psi$$

$$D2X = -\operatorname{Sin}\phi \operatorname{Cos}\psi - \operatorname{Cos}\theta \operatorname{Cos}\phi \operatorname{Sin}\psi$$

$$D2Y = \operatorname{Cos}\phi \operatorname{Cos}\psi - \operatorname{Cos}\theta \operatorname{Sin}\phi \operatorname{Sin}\psi \qquad (3.23)$$

$$D2Z = \operatorname{Sin}\theta \operatorname{Sin}\psi$$

$$D3X = \operatorname{Sin}\theta \operatorname{Cos}\phi$$

$$D3Y = \operatorname{Sin}\theta \operatorname{Sin}\phi \qquad (3.24)$$

$$D3Z = \operatorname{Cos}\theta$$

Translations of a Point P in XX, YY and ZZ Directions

$$XXP = XP$$

$$YYP = (YP) \operatorname{Cos}\delta - (ZP) \operatorname{Sin}\delta \qquad (3.25)$$

$$ZZP = (YP) \operatorname{Sin}\delta + (ZP) \operatorname{Cos}\delta$$

Translations of a Point P in XV, YV and ZV Directions

$$XVP = (XXP)\ \text{Cos}\ \alpha - (YYP)\ \text{Sin}\ \alpha$$

$$XVP = (XXP)\ \text{Sin}\ \alpha + (YYP)\ \text{Cos}\ \alpha \qquad (3.26)$$

$$ZVP = ZZP$$

Spring Force

$$FS = K\ [(SVO - SV) - (XVO - XVS)]$$

$$FSXX = FS\ \text{Cos}\ \alpha \qquad (3.27)$$

$$FSYY = FS\ \text{Sin}\ \alpha$$

Summation of Forces in X, Y and Z Directions

$$FX = - FSXX + FFXX$$

$$FY = FSYY\ \text{Cos}\ \delta + FN\ \text{Sin}\ \delta$$

$$- mg\ \text{Sin}\ \delta - FFYY\ \text{Cos}\ \delta \qquad (3.28)$$

$$FZ = FFYY\ \text{Sin}\ \delta + FN\ \text{Cos}\ \delta$$

$$- mg\ \text{Cos}\ \delta - FSYY\ \text{Sin}\ \delta$$

Forces in 1, 2 and 3 Directions

$$F1 = (FX)\ D1X + (FY)\ D1Y + (FZ)\ D1Z$$

$$F2 = (FX)\ D2X + (FY)\ D2Y + (FZ)\ D2Z \qquad (3.29)$$

$$F3 = (FX)\ D3X + (FY)\ D3Y + (FZ)\ D3Z$$

Torques About the X, Y and Z Axes

$$TX = -(FSYY \sin\delta)(YS - YCM) + (FFYY \sin\delta + FN \cos\delta)(YQ - YCM)$$
$$- (FSYY \cos\delta)(ZS - ZCM) + (FFYY \cos\delta - FN \sin\delta)(ZQ - ZCM)$$
$$- TXO$$

$$TY = -(FSXX)(ZS - ZCM) + (FFXX)(ZQ - ZCM) + (FSYY \sin\delta)(XS - XCM)$$
$$- (FFYY \sin\delta + FN \cos\delta)(XQ - XCM) \qquad (3.30)$$

$$TZ = (FSYY \cos\delta)(XS - XCM) + (FN \sin\delta - FFYY \cos\delta)(XQ - XCM)$$
$$+ (FSXX)(YS - YCM) - (FFXX)(YQ - YCM)$$

Torques About the 1, 2 and 3 Axes

$$T1 = (TX)\,D1X + (TY)\,D1Y + (TZ)\,D1Z$$

$$T2 = (TX)\,D2X + (TY)\,D2Y + (TZ)\,D2Z \qquad (3.31)$$

$$T3 = (TX)\,D3X + (TY)\,D3Y + (TZ)\,D3Z$$

Equations (3.16) through (3.31) are used to describe the motion of the system while the post and the vehicle are in contact.

3.7 Post Loses Contact with the Vehicle

It is assumed that after the displacement of the point "S" on the post becomes greater than the displacement of the vehicle, the post and the vehicle are no longer in contact. After contact is lost, the post is essentially a rigid body moving in space under the influence of gravity.

Since the only force present is gravity, equations (3.28) now become

$$FX = 0$$

$$FY = -\ mg\ \mathrm{Sin}\ \delta \tag{3.32}$$

$$FZ = -\ mg\ \mathrm{Cos}\ \delta$$

and the equations for F1, F2 and F3 are given by

$$F1 = (FY)\ D1Y + (FZ)\ D1Z$$

$$F2 = (FY)\ D2Y + (FZ)\ D2Z \tag{3.33}$$

$$F3 = (FY)\ D3Y + (FZ)\ D3Z$$

The torque about the center of mass is equal to zero, so the equations of motion become

$$M(\dot{V}1 - V2\ \omega 3 + V3\ \omega 2) = F1$$

$$M(\dot{V}2 - V3\ \omega 1 + V1\ \omega 3) = F2 \tag{3.34}$$

$$M(\dot{V}3 - V1\ \omega 2 + V2\ \omega 1) = F3$$

$$A\ \dot{\omega}1 - (B - C)\ \omega 2\ \omega 3 = 0$$

$$B\ \dot{\omega}2 - (C - A)\ \omega 3\ \omega 1 = 0 \tag{3.35}$$

$$C\ \dot{\omega}3 - (A - B)\ \omega 1\ \omega 2 = 0$$

Equations (3.19) through (3.26) remain unchanged.

3.8 Trajectory of the Post

In section (3.7) the equations that determine the motion of the post after it loses contact with the vehicle are given. It is now desired to know where the post will first hit on its return path to the ground. The possibilities are that (1) it hits the vehicle, (2) is knocked out of the vehicle path, or (3) is knocked high enough into space that the vehicle passes under the post before the post strikes the ground.

It will be assumed that the vehicle travels at constant velocity after it loses contact with the post and travels in the same direction as when it first contacted the post. To determine if the post strikes the vehicle, account must be kept of the displacements of various points on the post and on the vehicle.

The displacements of the points on the post will first be resolved to the XV, YV and ZV directions (see Figure 9) and compared to the displacements of the hood, top, and trunk of the vehicle to determine where the post has hit the vehicle.

Figures 11 and 12 show the vehicle and the various names used to describe the position of the vehicle. To facilitate checking the vehicle displacements against the post displacements, equations (3.36), (3.37) and (3.38) are used.

$$XBUMP = SV - HLEN \tag{3.36}$$

$$XENTOP = SV + TLEN \tag{3.37}$$

$$XTAIL = ZENTOP + TRLEN \tag{3.38}$$

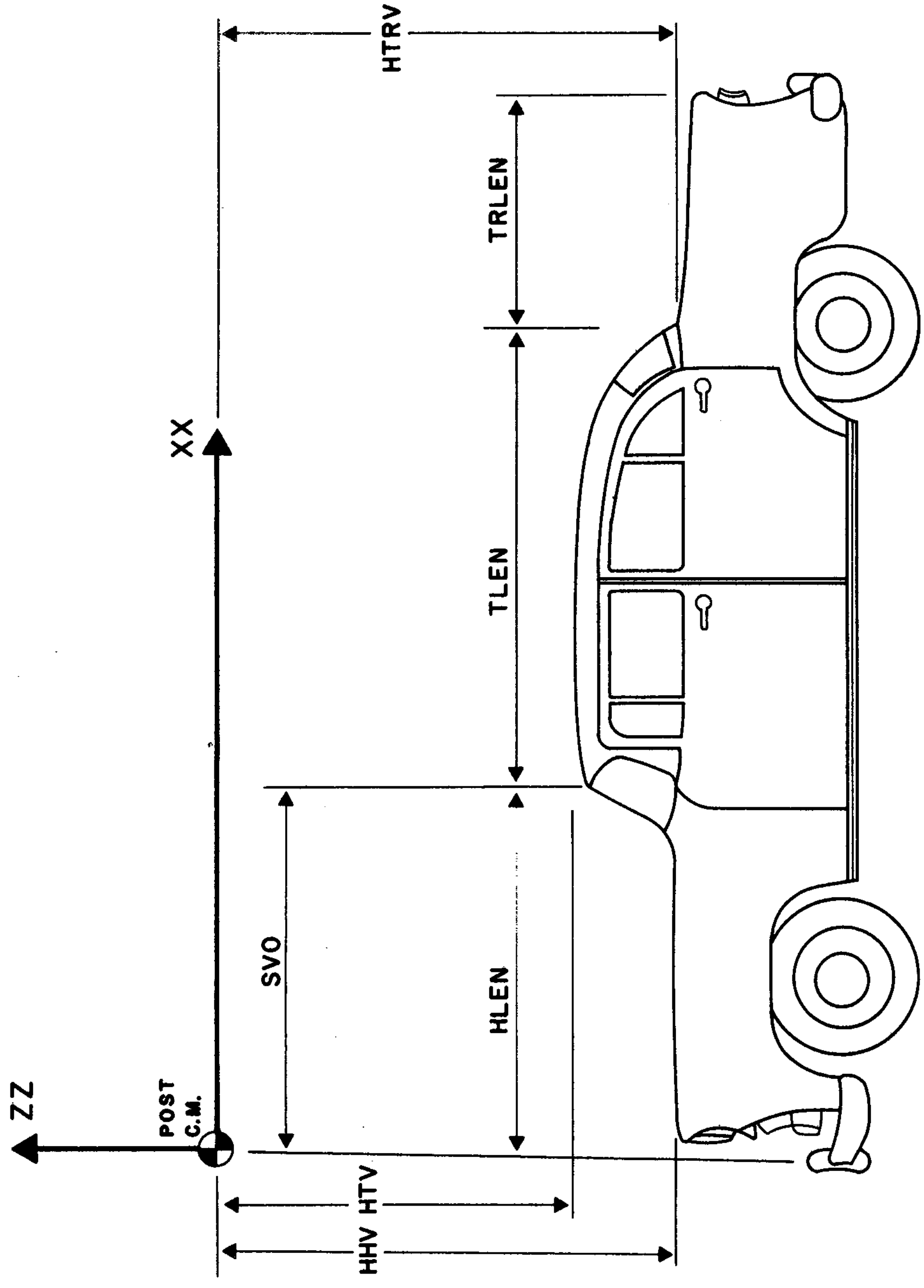


FIGURE 11 THE MODEL VEHICLE

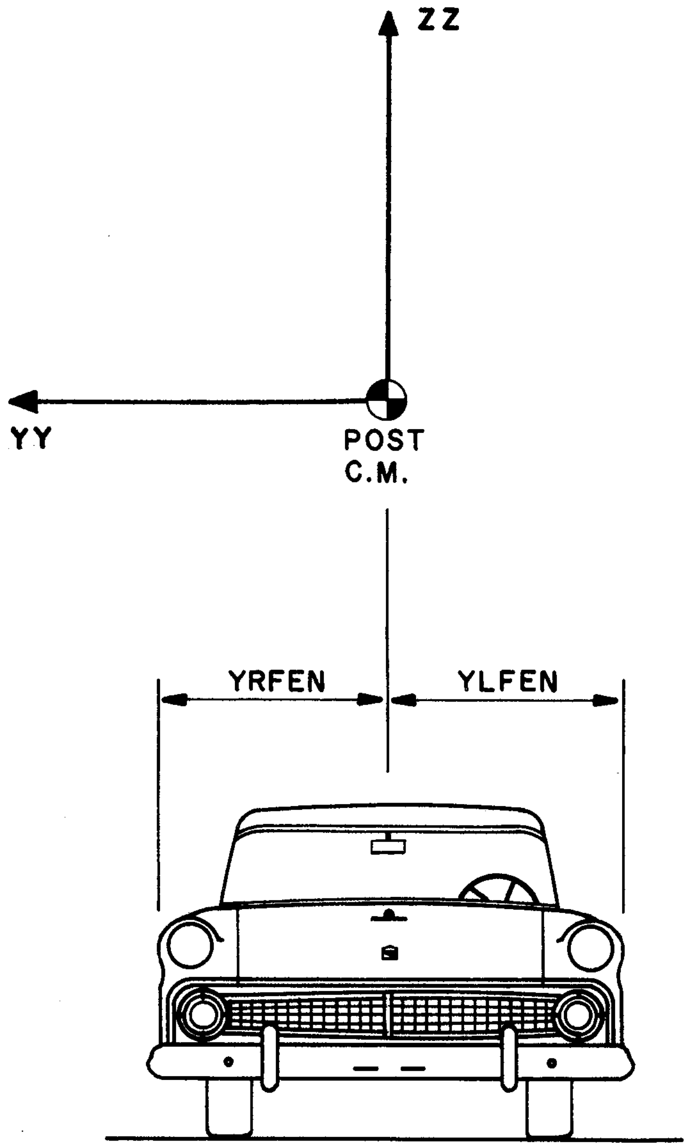


FIGURE 12 THE MODEL VEHICLE

The quantities XBUMP, XENTOP and XTAIL are the displacements of the front bumper, the end of the top, and the rear bumper of the vehicle, respectively, and are in the XV direction. The term SV is the displacement of the assumed center of mass of the vehicle and this point is assumed to remain directly below the end of the hood of the vehicle. The lengths of the hood, top and trunk of the vehicle are represented by HLEN, TLEN and TRLEN, respectively, and for this phase of the problem, may be assumed to remain constant. The quantities HHV, HTV and HTRV are the ZZ or ZV coordinates measured from the initial position of the mass center of the post to the hood, top and trunk of the vehicle, respectively, and they too will remain constant during the motion.

The terms YLFEN and YRFEN (see Figure 12) represent the coordinates, measured from the initial position of the post center of mass, of the left and right fenders of the vehicle, respectively. They are measured in the YV direction and remain constant.

CHAPTER IV

SOLUTION OF THE EQUATIONS OF MOTION

4.1 Discussion of Numerical Techniques Employed

Whenever a technical problem, as the one under consideration, leads to differential equations which cannot be integrated in closed form, approximate methods of solution must be employed. Initial value and boundary value problems involving either partial differential equations or ordinary differential equations, as is the case here, may be solved by such methods.

In recent years, numerical methods for the solution of differential equations have become extremely popular because modern technical problems lead to complicated equations seldom solvable in closed form and because electronic computers have become widely available.

The numerical solution of differential equations consists essentially in obtaining the numerical values of the unknown integral at some pivotal points, spaced along the time axis, for example, for the set of ordinary differential equations being considered. To obtain the pivotal values of the integral f, the derivatives of f appearing in the differential equation are approximated either by the derivatives of n th degree parabolas passing through a certain number of pivotal points, or by Taylor expansions of the unknown function f.

Consider Figure 13 and the given values Vo, V_1, V_2 -- V_L, V_i, V_R -- V_{n-2}, V_{n-1}, V_n of a function V(t) at the pivotal points of its interval of definition, evenly spaced by h. One calls the first

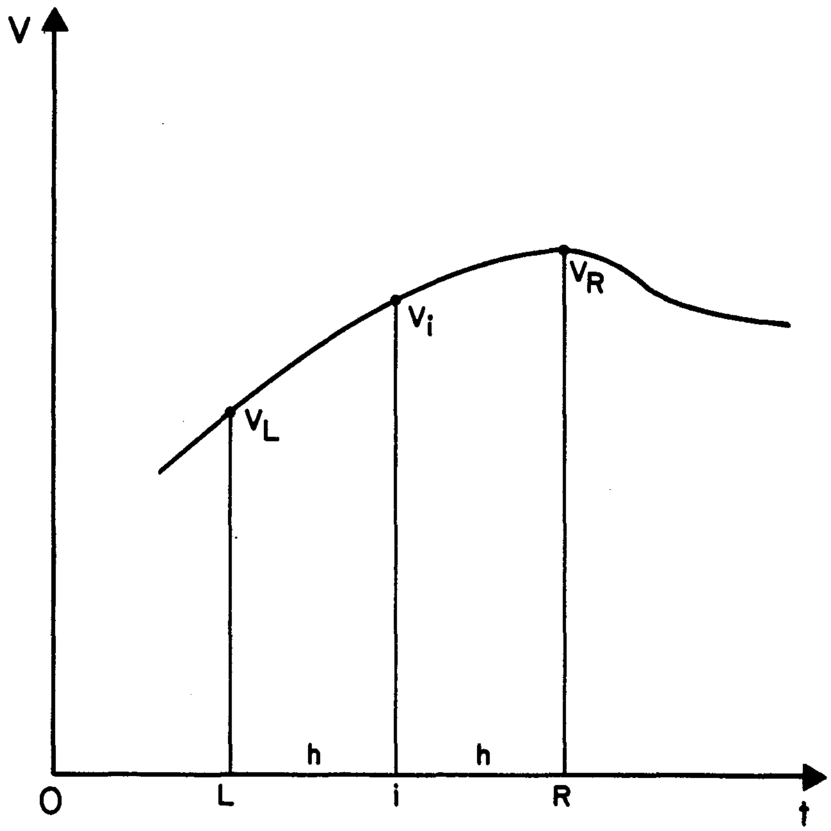


FIGURE 13 REPRESENTATION OF EQUALLY SPACED POINTS USED IN FINITE DIFFERENCE DERIVATION

forward difference of V at i the difference

$$\Delta V_i = V_R - V_i$$

Consider now the Taylor expansion of V(t + h) about t:

$$V(t + h) = V(t) + \frac{h}{1!}\dot{V}(t) + \frac{h^2}{2!}\ddot{V}(t) + \frac{h^3}{3!}\dddot{V}(t) + \text{-----} \tag{4.1}$$

Using the symbol D to indicate derivatives of V, (4.1) becomes

$$V(t + h) = V(t) + \frac{h}{1!} DV(t) + \frac{h^2}{2!} D^2 V(t) + \frac{h^3}{3!} D^3 V(t) + \text{-----} \tag{4.2}$$

or

$$V(t + h) = (1 + \frac{h}{1!} D + \frac{h^2}{2!} D^2 + \frac{h^3}{3!} D^3 + \text{-----})\, V(t) \tag{4.3}$$

By means of the series expansion of e^x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \text{-----}$$

The differential operator on the right-hand side of equation (4.3) may be written symbolically as

$$1 + \frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \text{-----} = e^{hD} \tag{4.4}$$

and hence V(t + h) may also be written symbolically as

$$V(t + h) = e^{hD}\, V(t) \tag{4.5}$$

Setting $t = t_i$ and indicating $V(t_i + h)$ by V_R and $V(t_i)$ by V_i equation (4.5) becomes

$$V_R = e^{hD} V_i \tag{4.6}$$

The first forward difference ΔV_i may now be written by means of equation (4.5) as

$$\Delta V_i = V_R - V_i = (e^{hD} - 1) V_i \tag{4.7}$$

or by means of equation (4.4)

$$\Delta V_i = \left[\frac{hD}{1!} + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \frac{h^4D^4}{4!} + \text{-----} \right] V_i \tag{4.8}$$

If h is very small, as will be the case for the problem under consideration, only the first term in equation (4.8) need be retained. Thus,

$$\Delta V_i = V_R - V_i = hD\, V_i$$

or

$$\dot{V}_i = \frac{V_R - V_i}{h} \tag{4.9}$$

There are many formulas for numerical integration, but for most engineering applications, the trapezoidal rule is quite adequate. To recall its derivation, let the required integral be

$$\int_a^b f(t)\, dt$$

and let the range or integration be divided into n equal parts; the ordinates at the points of subdivision being

$$f_o = f(a),\ f_1,\ f_2 \ldots .,\ f_{n-1},\ f_n = f(b)$$

as shown in Figure 14.

A definite integral can always be interpreted as an area, and thus any method of approximating an area is essentially a method of approximating a definite integral. If the arc of V = f(t) is replaced over each subinterval, $t_{i+1} - t_i$ by its chord and the sum of the areas of the resulting trapezoids is taken as an approximation to the true area under V = f(t), the trapezoidal rule results.

Making use of the fact that the area of a trapezoid is equal to the average of the parallel sides times the perpendicular distance between them.

$$A_1 = \left(\frac{f_o + f_1}{2}\right) h$$

$$A_2 = \left(\frac{f_1 + f_2}{2}\right) h$$

- -

$$A_{n-1} = \left(\frac{f_{n-2} + f_{n-1}}{2}\right) h$$

$$A_n = \left(\frac{f_{n-1} + f_n}{2}\right) h$$

or adding

$$A = h\ (1/2\ f_o + f_1 + f_2 + \text{-----} + f_{n-1} + 1/2\ f_n) \tag{4.10}$$

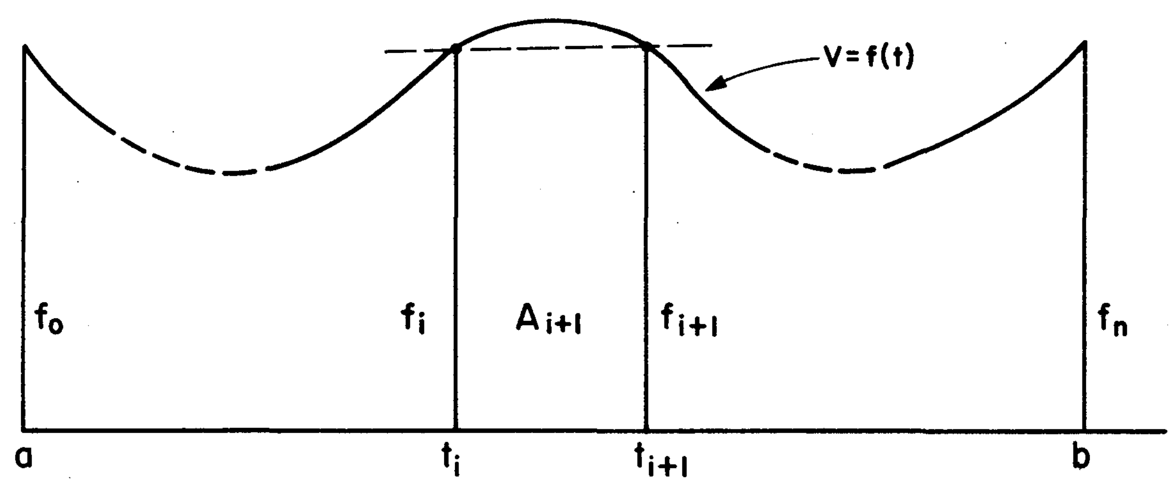


FIGURE 14 REPRESENTATION OF CURVE USED IN TRAPEZOIDAL RULE DERIVATION

which is the trapezoidal rule.

Equations (4.9) and (4.10) will be employed to solve the equations of motion obtained in Chapter III.

4.2 Application of the Numerical Techniques to the Equations of Motion

Equations (3.18) and (3.19) involve first derivatives of the velocities so equation (4.9) can be used along with (3.18) and (3.19) to solve for the velocities. After the velocities are obtained, the trapezoidal rule and equation (4.10) are employed to solve for the displacements.

Letting R be replaced by i + 1 in equation (4.9) and substituting into equations (3.17), (3.18) and (3.19) the expressions for the velocities are obtained as

$$\frac{V1_{i+1} - V1_i}{h} = \frac{F1_i}{M} + V2_i\ \omega3_i - V3_i\ \omega2_i$$

$$\frac{V2_{i+1} - V2_i}{h} = \frac{F2_i}{M} + V3_i\ \omega1_i - V1_i\ \omega3_i$$

$$\frac{V3_{i+1} - V3_i}{h} = \frac{F3_i}{M} + V1_i\ \omega2_i - V2_i\ \omega1_i$$

$$\frac{\omega1_{i+1} - \omega1_i}{h} = \frac{T1_i}{A} + \frac{B-C}{A}\ (\omega2_i\ \omega3_i)$$

$$\frac{\omega2_{i+1} - \omega2_i}{h} = \frac{T2_i}{B} + \frac{C-A}{B}\ (\omega3_i\ \omega1_i)$$

$$\frac{\omega3_{i+1} - \omega3_i}{h} = \frac{T3_i}{C} + \frac{A-B}{C}\ (\omega1_i\ \omega2_i)$$

$$\frac{VV_{i+1} - VV_i}{h} = -\frac{FS_i}{Mv} \tag{4.11}$$

or

$$V1_{i+1} = h\left[\frac{F1_i}{M} + V2_i\ \omega3_i - V3_i\ \omega2_i\right] + V1_i$$

$$V2_{i+1} = h\left[\frac{F2_i}{M} + V3_i\ \omega1_i - V1_i\ \omega3_i\right] + V2_i$$

$$V3_{i+1} = h\left[\frac{F3_i}{M} + V1_i\ \omega2_i - V2_i\ \omega1_i\right] + V3_i$$

$$\omega1_{i+1} = \frac{h}{A}\left[T1_i + (B-C)\ \omega2_i\ \omega3_i\right] + \omega1_i$$

$$\omega2_{i+1} = \frac{h}{B}\left[T2_i + (C-A)\ \omega3_i\ \omega1_i\right] + \omega2_i$$

$$\omega3_{i+1} = \frac{h}{C}\left[T3_i + (A-B)\ \omega1_i\ \omega2_i\right] + \omega3_i$$

$$VV_{i+1} = -\frac{h}{Mv}\ (FS_i) + VV_i$$

Assuming that all velocities for a time increment ahead can be obtained from the values of a time behind from equations (4.11), there now remains the problem of obtaining the Eulerian angles θ, ϕ and ψ from equations (3.19).

From equations (3.19)

$$\omega1_{i+1} = (\text{Sin}\ \psi_{i+1})\ \dot{\theta}_{i+1} - (\text{Sin}\ \theta_{i+1}\ \text{Cos}\ \psi_{i+1})\ \dot{\phi}_{i+1}$$

$$\omega2_{i+1} = (\text{Cos}\ \psi_{i+1})\ \dot{\theta}_{i+1} + (\text{Sin}\ \theta_{i+1}\ \text{Sin}\ \psi_{i+1})\ \dot{\phi}_{i+1}$$

$$\omega3_{i+1} = (\text{Cos}\ \theta_{i+1})\ \dot{\phi}_{i+1} + \dot{\psi}_{i+1} \tag{4.12}$$

Also, from equations (4.10) and the trapezoidal rule

$$
\begin{aligned}
(\dot{\theta}_{i+1} + \dot{\theta}_i)\frac{h}{2} + \theta_i &= \theta_{i+1} \\
(\dot{\phi}_{i+1} + \dot{\phi}_i)\frac{h}{2} + \phi_i &= \phi_{i+1} \qquad (4.13) \\
(\dot{\psi}_{i+1} + \dot{\psi}_i)\frac{h}{2} + \psi_i &= \psi_{i+1}
\end{aligned}
$$

or

$$
\begin{aligned}
\dot{\theta}_{i+1} &= (\theta_{i+1} - \theta_i)\frac{2}{h} - \dot{\theta}_i \\
\dot{\phi}_{i+1} &= (\phi_{i+1} - \phi_i)\frac{2}{h} - \dot{\phi}_i \qquad (4.14) \\
\dot{\psi}_{i+1} &= (\psi_{i+1} - \psi_i)\frac{2}{h} - \dot{\psi}_i
\end{aligned}
$$

At this stage it will be useful to solve equations (4.12) for $\dot{\theta}_{i+1}$, $\dot{\phi}_{i+1}$ and $\dot{\psi}_{i+1}$.

Let

$$
\begin{aligned}
A_{11} &= \text{Sin } \psi_{i+1} \\
A_{12} &= -\text{Sin } \theta_{i+1} \text{ Cos } \psi_{i+1} \\
A_{21} &= \text{Cos } \psi_{i+1} \\
A_{22} &= \text{Sin } \theta_{i+1} \text{ Sin } \psi_{i+1} \qquad (4.15) \\
A_{32} &= \text{Cos } \theta_{i+1} \\
A_{33} &= 1
\end{aligned}
$$

Thus,

$$\omega 1_{i+1} = (A_{11})\ \dot{\theta}_{i+1} + (A_{12})\ \dot{\phi}_{i+1} \qquad (4.16)$$

$$\omega 2_{i+1} = (A_{21})\ \dot{\theta}_{i+1} + (A_{22})\ \dot{\phi}_{i+1} \qquad (4.17)$$

$$\omega 3_{i+1} = (A_{32})\ \dot{\phi}_{i+1} + \dot{\psi}_{i+1} \qquad (4.18)$$

Now multiplying (4.16) by A_{21} and (4.17) by A_{11}, and subtracting, the expression for $\dot{\phi}$ is given by

$$\dot{\phi}_{i+1} = \frac{(A_{21})\ \omega 1_{i+1} - (A_{11})\ \omega 2_{i+1}}{(A_{21})\ (A_{12}) - (A_{11})\ (A_{22})} \qquad (4.19)$$

From equation (4.17)

$$\dot{\theta}_{i+1} = \frac{\omega 2_{i+1}}{A_{21}} - \left(\frac{A_{22}}{A_{21}}\right) \dot{\phi}_{i+1} \qquad (4.20)$$

or

$$\dot{\theta}_{i+1} = \frac{\omega 2_{i+1}}{A_{21}} - \frac{A_{22}}{A_{21}} \left[\frac{(A_{21})\ \omega 1_{i+1} - (A_{11})\ \omega 2_{i+1}}{(A_{21})\ (A_{12}) - (A_{11})\ (A_{22})} \right] \qquad (4.21)$$

From equation (4.18)

$$\dot{\psi}_{i+1} = \omega 3_{i+1} - (A_{32})\ \dot{\phi}_{i+1}$$

or

$$\dot{\psi}_{i+1} = \omega 3_{i+1} - A_{32} \left[\frac{(A_{21})\ \omega 1_{i+1} - (A_{11})\ \omega 2_{i+1}}{(A_{21})\ (A_{12}) - (A_{11})\ (A_{22})} \right] \qquad (4.22)$$

Substituting (4.15) into equations (4.19), (4.21) and (4.22)

$$\dot{\phi}_{i+1} = \frac{(\text{Cos } \psi_{i+1})\ \omega 1_{i+1} - (\text{Sin } \psi_{i+1})\ \omega 2_{i+1}}{-\text{Cos}^2\ \psi_{i+1}\ \text{Sin } \theta_{i+1} - \text{Sin}^2\ \psi_{i+1}\ \text{Sin } \theta_{i+1}}$$

or

$$\dot{\phi}_{i+1} = -\left[\frac{(\text{Cos } \psi_{i+1})\ \omega 1_{i+1} - (\text{Sin } \psi_{i+1})\ \omega 2_{i+1}}{\text{Sin } \theta_{i+1}}\right] \tag{4.23}$$

$$\dot{\theta}_{i+1} = \frac{\omega 2_{i+1}}{\text{Cos } \psi_{i+1}} + \frac{\text{Sin } \psi_{i+1}}{\text{Cos } \psi_{i+1}}\left[(\text{Cos } \psi_{i+1})\omega 1_{i+1} - (\text{Sin } \psi_{i+1})\ \omega 2_{i+1}\right] \tag{4.24}$$

$$\dot{\psi}_{i+1} = \omega 3_{i+1} + \frac{\text{Cos } \theta_{i+1}}{\text{Sin } \theta_{i+1}}\left[(\text{Cos } \psi_{i+1})\ (\omega 1_{i+1}) - (\text{Sin } \psi_{i+1}) \times (\omega 2_{i+1})\right] \tag{4.25}$$

Equations (4.23), (4.24) and (4.25) have a common factor in which the only Eulerian angle present is ψ.

Define

$$\Psi_{i+1} = (\text{Cos } \psi_{i+1})\ \omega 1_{i+1} - (\text{Sin } \psi_{i+1})\ \omega 2_{i+1} \tag{4.26}$$

so that

$$\dot{\phi}_{t+1} = -\frac{1}{\text{Sin } \theta_{i+1}}\ (\Psi_{i+1}) \tag{4.27}$$

$$\dot{\theta}_{i+1} = \frac{\omega 2_{i+1}}{\text{Cos } \psi_{i+1}} + \frac{\text{Sin } \psi_{i+1}}{\text{Cos } \psi_{i+1}}\ (\Psi_{i+1}) \tag{4.28}$$

$$\dot{\psi}_{i+1} = \omega 3_{i+1} + \frac{\cos\theta_{i+1}}{\sin\theta_{i+1}} (\Psi_{i+1}) \qquad (4.29)$$

Substituting (4.14) into (4.27), (4.28) and (4.29)

$$(\phi_{i+1} - \phi_i)\frac{2}{h} - \dot{\phi}_i = -\frac{1}{\sin\theta_{i+1}} (\Psi_{i+1})$$

$$(\theta_{i+1} - \theta_i)\frac{2}{h} - \dot{\theta}_i = \frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + \frac{\sin\psi_{i+1}}{\cos\psi_{i+1}} (\Psi_{i+1})$$

$$(\psi_{i+1} - \psi_i)\frac{2}{h} - \dot{\psi}_i = \omega 3_{i+1} + \frac{\cos\theta_{i+1}}{\sin\theta_{i+1}} (\Psi_{i+1})$$

or

$$\theta_{i+1} = \frac{h}{2}\left[\frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + \frac{\sin\psi_{i+1}}{\cos\psi_{i+1}} (\Psi_{i+1}) + \dot{\theta}_i\right] + \theta_i \qquad (4.30)$$

$$\psi_{i+1} = \frac{h}{2}\left[\omega 3_{i+1} + \frac{\cos\theta_{i+1}}{\sin\theta_{i+1}} (\Psi_{i+1}) + \dot{\psi}_i\right] + \psi_i \qquad (4.31)$$

$$\phi_{i+1} = \frac{h}{2}\left[-\frac{1}{\sin\theta_{i+1}} (\Psi_{i+1}) + \dot{\phi}_i\right] + \phi_i \qquad (4.32)$$

Substituting equation (4.30) into equation (4.31), an equation where ψ_{i+1} is the only unknown, is obtained as

$$\psi_{i+1} = \frac{h}{2}\left\{\omega 3_{i+1} + \left[\frac{\cos\left[\frac{h}{2}\left(\frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + \frac{\sin\psi_{i+1}}{\cos\psi_{i+1}} (\Psi_{i+1}) + \dot{\theta}_i\right) + \theta_i\right]}{\sin\left[\frac{h}{2}\left(\frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + \frac{\sin\psi_{i+1}}{\cos\psi_{i+1}} (\Psi_{i+1}) + \dot{\theta}_i\right) + \theta_i\right]}\right] \Psi_{i+1} + \dot{\psi}_i\right\} + \psi_i \qquad (4.33)$$

Then,

$$\theta_{i+1} = \frac{h}{2}\left[\frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + \frac{\sin\psi_{i+1}}{\cos\psi_{i+1}}(\Psi_{i+1}) + \dot{\theta}_i\right] + \theta_i \qquad (4.34)$$

$$\phi_{i+1} = \frac{h}{2}\left[-\frac{1}{\sin\theta_{i+1}}(\Psi_{i+1}) + \dot{\phi}_i\right] + \phi_i \qquad (4.35)$$

Consider simplifying the term

$$\frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + \frac{\sin\psi_{i+1}}{\cos\psi_{i+1}}\left[(\cos\psi_{i+1})\,\omega 1_{i+1} - (\sin\psi_{i+1})\,\omega 2_{i+1}\right] + \dot{\theta}_i$$

Expanding, one obtains

$$\frac{\omega 2_{i+1}}{\cos\psi_{i+1}} + (\sin\psi_{i+1})\,\omega 1_{i+1} - \frac{\sin^2\psi_{i+1}}{\cos\psi_{i+1}}(\omega 2_{i+1}) + \dot{\theta}_i$$

or

$$\frac{\omega 2_{i+1}}{\cos\psi_{i+1}}\left[1-\sin^2\psi_{i+1}\right] + \sin\psi_{i+1}\,(\omega 1_{i+1}) + \dot{\theta}_i$$

Further simplifying one has

$$(\omega 2_{i+1})\ (\cos\psi_{i+1}) + (\omega 1_{i+1})\ (\sin\psi_{i+1}) + \dot{\theta}_i$$

The expressions for the Eulerian angles now become

$$\psi_{i+1} = \frac{h}{2}\left\{\omega 3_{i+1} + \left[\cot\left[\frac{h}{2}(\omega 2_{i+1}\cos\psi_{i+1} + \omega 1_{i+1}\sin\psi_{i+1} + \dot{\theta}_i) + \theta_i\right]\right]\Psi_{i+1} + \dot{\psi}_i\right\} + \psi_i \qquad (4.36)$$

$$\theta_{i+1} = \frac{h}{2}\left[\omega 2_{i+1} \operatorname{Cos} \psi_{i+1} + \omega 1_{i+1} \operatorname{Sin} \psi_{i+1} + \dot{\theta}_i\right] + \theta_i \qquad (4.37)$$

$$\phi_{i+1} = \frac{h}{2}\left[- \frac{1}{\operatorname{Sin} \theta_{i+1}} (\Psi_{i+1}) + \dot{\phi}_i\right] + \phi_i \qquad (4.38)$$

where

$$\Psi = \operatorname{Cos} \psi_{i+1} (\omega 1_{i+1}) - \operatorname{Sin} \psi_{i+1} (\omega 2_{i+1}) \qquad (4.39)$$

After the Eulerian angles are obtained from equations (4.36) through (4.38), the angular velocities $\dot{\theta}, \dot{\phi}$ and $\dot{\psi}$ are computed from equations (4.27), (4.28) and (4.29). Having obtained values of velocities and angular displacements, it is now desired to obtain values of the translations of the center of mass in the 1, 2 and 3 directions.

Assuming $V1_{i+1}$, $V2_{i+1}$, and $V3_{i+1}$ have been obtained from equations (4.11), denoting by S1, S2, and S3, the translations in the 1, 2 and 3 directions respectively and again employing the trapezoidal rule, the expressions for the translations of the mass center become

$$S1_{i+1} = \frac{h}{2} (V1_{i+1} + V1_i)$$

$$S2_{i+1} = \frac{h}{2} (V2_{i+1} + V2_i) \qquad (4.40)$$

$$S3_{i+1} = \frac{h}{2} (V3_{i+1} + V3_i)$$

The values obtained from equations (4.38) may be resolved to the X, Y and Z directions by means of Table 4 and equations (3.23), (3.24) and (3.25), so that the expressions for the translations of the center of mass become

$$XCM_{i+1} = (S1_{i+1})\ (D1X_{i+1}) + (S2X_{i+1})\ (D2X_{i+1}) + (S3_{i+1})\ (D3X_{i+1}) + XCM_i$$

$$YCM_{i+1} = (S1_{i+1})\ (D1Y_{i+1}) + (S2Y_{i+1})\ (D2Y_{i+1}) + (S3_{i+1})(D3Y_{i+1}) + YCM_i$$

$$ZCM_{i+1} = (S1_{i+1})\ (D1Z_{i+1}) + (S2_{i+1})\ (D2Z_{i+1}) + (S3_{i+1})\ (D3Z_{i+1}) + ZCM_i$$

(4.41)

Equations (4.41) resolved to the XX, YY and ZZ directions are given by

$$XXCM_{i+1} = XCM_{i+1} \qquad (4.42)$$

$$YYCM_{i+1} = (YCM_{i+1})\ \text{Cos}\ \delta - (ZCM_{i+1})\ \text{Sin}\ \delta$$

$$ZZCM_{i+1} = (YCM_{i+1})\ \text{Sin}\ \delta + (ZCM_{i+1})\ \text{Cos}\ \delta$$

The rotation parameters are calculated from equations (3.21) and become

$$\xi_{i+1} = \text{Sin}\ \frac{\theta_{i+1}}{2}\ \ \text{Sin}\ \left(\frac{\psi_{i+1} - \phi_{i+1}}{2}\right)$$

$$\eta_{i+1} = \text{Sin}\ \frac{\theta_{i+1}}{2}\ \ \text{Cos}\ \left(\frac{\psi_{i+1} - \phi_{i+1}}{2}\right)$$

$$\zeta_{i+1} = \text{Cos}\ \frac{\theta_{i+1}}{2}\ \ \text{Sin}\ \left(\frac{\psi_{i+1} + \phi_{i+1}}{2}\right)$$

$$\chi_{i+1} = \text{Cos}\left(\frac{\theta_{i+1}}{2}\right)\text{Cos}\left(\frac{\psi_{i+1} + \phi_{i+1}}{2}\right)$$

(4.43)

The translations of any point P of the post are computed next by employing equations (3.22), (4.41) and (4.43). These translations are given by

$$XP_{i+1} = XCM_{i+1} + (\xi^2_{i+1} - \eta^2_{i+1} - \zeta^2_{i+1} + \chi^2_{i+1})XPO + 2(\xi_{i+1}\,\eta_{i+1} - \zeta_{i+1}\,\chi_{i+1})YPO + 2(\xi_{i+1}\,\zeta_{i+1} + \xi_{i+1}\,\chi_{i+1})ZPO$$

$$YP_{i+1} = YCM_{i+1} + 2(\xi_{i+1}\,\eta_{i+1} + \zeta_{i+1}\,\chi_{i+1})XPO + (-\xi^2_{i+1} + \eta^2_{i+1} - \zeta^2_{i+1} + \chi^2_{i+1})YPO + 2(\eta_{i+1}\,\zeta_{i+1} - \xi_{i+1}\,\chi_{i+1})ZPO$$

$$ZP_{i+1} = ZCM_{i+1} + 2(\xi_{i+1}\,\zeta_{i+1} - \eta_{i+1}\,\chi_{i+1})XPO + 2(\eta_{i+1}\,\zeta_{i+1} + \xi_{i+1}\,\chi_{i+1})YPO + (-\xi^2_{i+1} - \eta^2_{i+1} + \zeta^2_{i+1} + \chi^2_{i+1})ZPO$$

(4.44)

Equations (4.44) may be resolved to the XX, YY and ZZ directions by using the same transformation that was used to obtain equations (4.42) Thus

$$XXP_{i+1} = XP_{i+1}$$

$$YYP_{i+1} = (YP_{i+1})\,\text{Cos}\,\delta - (ZP_{i+1})\,\text{Sin}\,\delta$$

$$ZZP_{i+1} = (YP_{i+1}) \text{ Cos } \delta + (ZP_{i+1}) \text{ Cos } \delta \tag{4.45}$$

Equations (4.45) may now be resolved to the XV, YV and ZV directions by employing Table 7.

These equations become

$$\begin{aligned} XVP_{i+1} &= (XXP_{i+1}) \text{ Cos } \alpha - (YYP_{i+1}) \text{ Sin } \alpha \\ YVP_{i+1} &= (XXP_{i+1}) \text{ Sin } \alpha + (YYP_{i+1}) \text{ Cos } \alpha \\ ZVP_{i+1} &= ZZP_{i+1} \end{aligned} \tag{4.46}$$

The values of the forces and the torques at time i+1 may be obtained by using equations (3.29), (3.30), (3.31) and (3.32).

All the quantities desired have now been calculated for the time i+1 by knowing the values at time i. The values at time i+2 are obtained from the values at time i+1 and so on. At the start of the numerical procedure, the values at time i would correspond to the initial values at time equal to zero and the quantities obtained for time equal to i+1 would correspond to the values at the end of the first time increment, h.

In this section, the above has been done for the equations that govern while the post and the vehicle are in contact. The same technique applied to the solution of the equations that describe the motion when the post and the vehicle are no longer in contact except that the equations given in section 3.7 are used and the initial conditions for this set of equations are obtained from the terminal values of the first set.

CHAPTER V

CORRELATION WITH TEST RESULTS

In the Spring of 1967, a testing program was initiated to obtain information that could be used in the development and verification of the mathematical model. The test that was used for the purpose of correlation involved a 6 in. standard weight pipe that was 9 ft. long and had a triangular base with "break-away" characteristics. Instrumentation for the test employed: (1) An accelerometer mounted on the crash vehicle, (2) High-speed motion picture cameras, (3) A tach-generator mounted on the vehicle and driven by the differential of the vehicle so that the vehicular velocity could be recorded, (4) A clock used to help determine the vehicular velocity, and (5) A tape switch, secured to the post, which gives an indication of impact by simultaneously giving a deflection on the recorder and flashing a bulb for the benefit of the cameras.

The data from the test were analyzed and used to demonstrate the feasibility of the mathematical model. Successful correlation of the model with test data made it reasonable to assume that use of the model is feasible.

5.1 Philosophy of the Correlation

Before the model can be used, it is necessary to have a knowledge of the various parameters that are used as input information to the computer program. Accurate information can be obtained for the material properties of the post, but limited information is available for predicting the frictional resisting forces at the base of the post and also the energy-absorption characteristics of the vehicle.

In order to make a better prediction of the spring constant of the front end of the test vehicle, a leaf spring was attached to the front of the vehicle as is shown in Figure 15. After being carefully greased, the leaf spring was tested statically at the Civil Engineering Testing Laboratory, and constants were obtained as shown in Figure 17.

In order to approach a condition of minimum frictional resistance, the bolts which fasten the post to the base were tightened only enough to hold the post erect, and the base was carefully greased. It was hoped that by having a spring with a known constant at the front end of the vehicle, and a condition approaching that of negligible resistance at the base of the post, the model would come closer to predicting the actual situation occurring in the crash test.

The previous discussion shows some of the difficulties that are encountered in attempting to simulate a particular test.

5.2 Model Parameters

Slip-base force. Limited information is available for the resistance offered by the slip base. For the purpose of correlation, it was assumed that the frictional resistance was negligible since the slip surfaces had been carefully greased and the bolts that hold the post to the base were tightened only enough to hold the post erect.

Vehicle spring constant. Two ranges were used for the vehicle spring constant as can be seen from Figure 17. The leaf spring employed, which is shown in Figure 16, had a dual set of leaves and the point

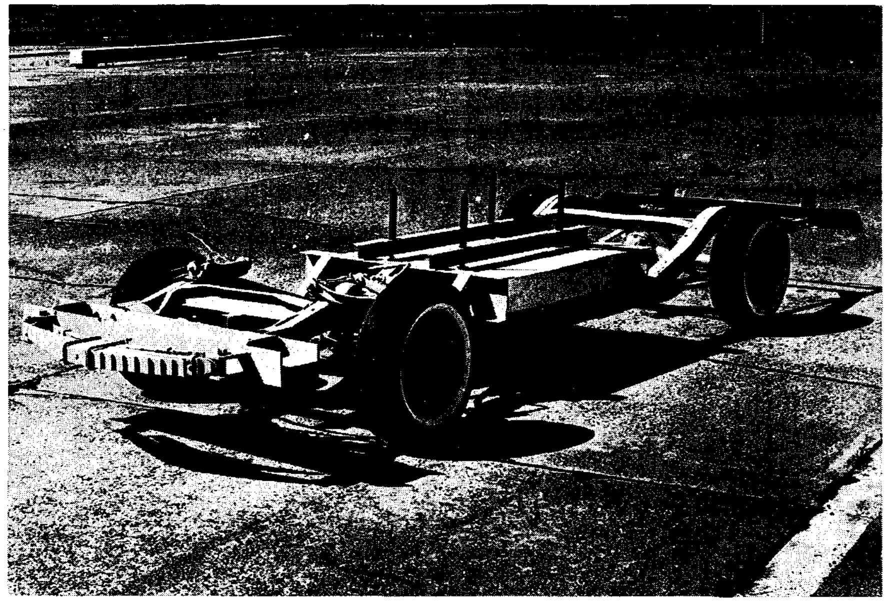


FIGURE 15 THE CRASH TEST VEHICLE

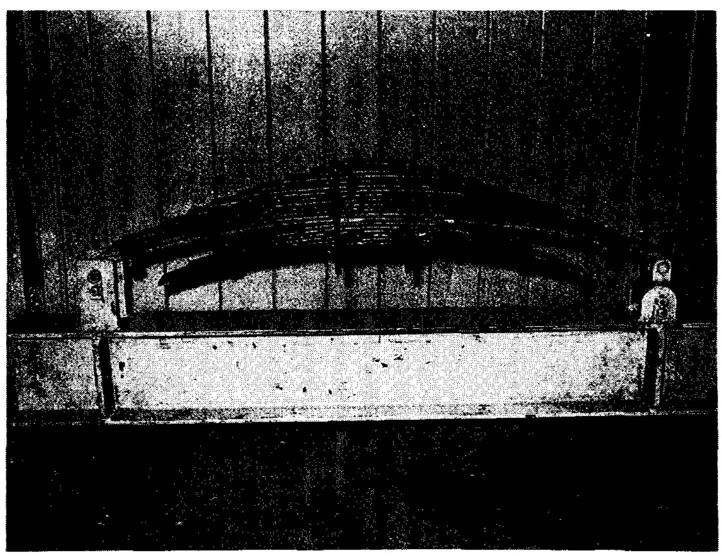


FIGURE 16 THE LEAF SPRING USED ON THE CRASH TEST VEHICLE

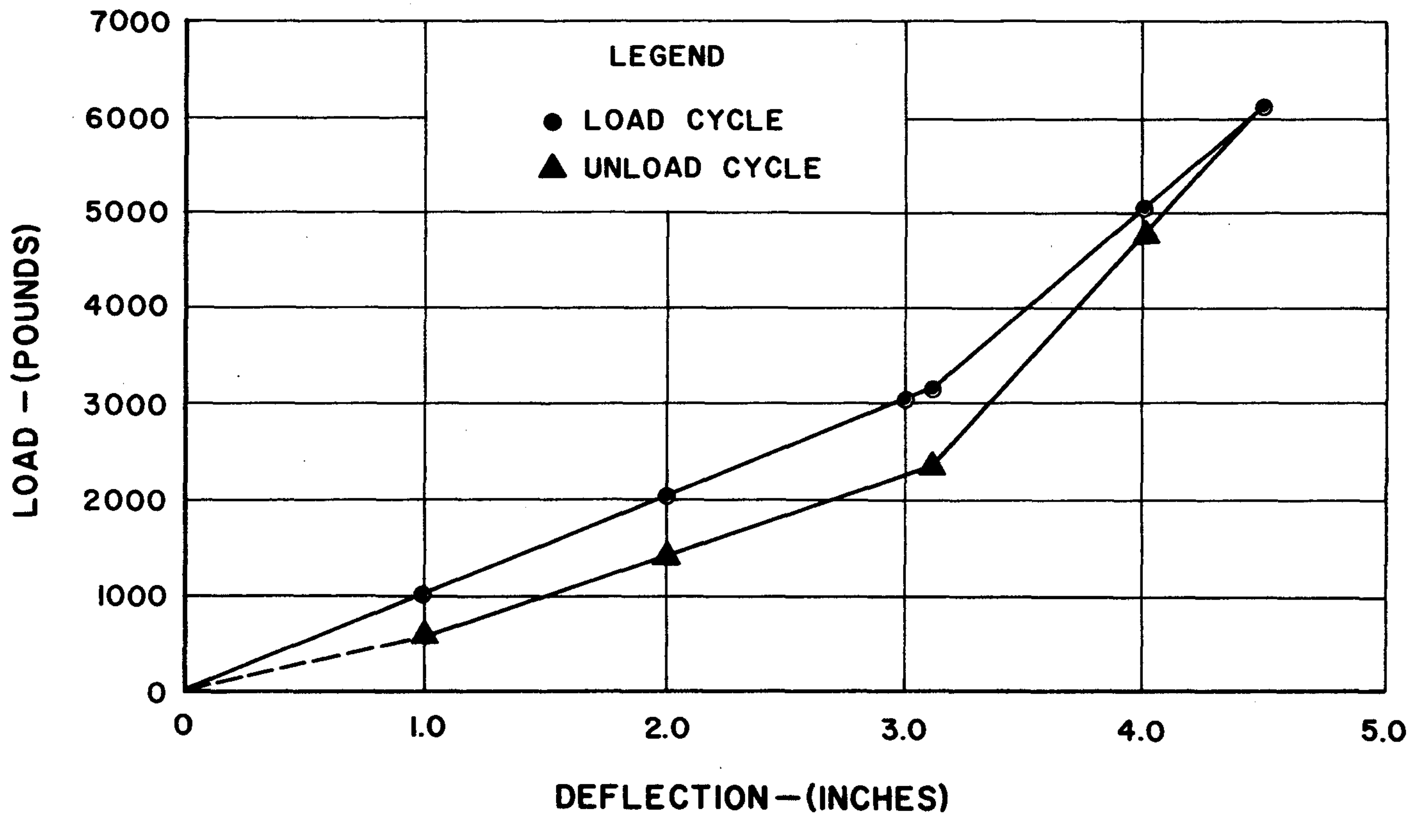


FIGURE 17 FORCE–DEFORMATION CURVE FOR LEAF SPRING USED ON CRASH TEST VEHICLE

of impact was assumed to be at the middle of the spring. This was the point for which the spring constants had been obtained in the static test.

The static test of the spring revealed that the spring had a certain spring constant for defections between 0 and 3.0 in. and a different constant for deflections between 3.0 in. and 4.5 in. This is due to the fact that in this second range the second set of leaves is engaged and the spring becomes stiffer. The spring also gave different force-deformation curves for the loading and unloading cycles. This can be explained by considering the manner in which the frictional forces between the leaves act for the two cycles. It should also be mentioned that the two sets of leaves were fastened together by means of U - clamps and the threaded end of the U - bolts came into contact with the I - beam to which the spring assembly was attached for a spring displacement of 4.5 in. This fact had to be accounted for in the mathematical model.

Vehicle weight. The vehicle used in the test is shown in Figure 15 and consisted of the frame of a 1955 Ford complete with front and rear axles and wheels. The vehicle had a concrete slab weighing approximately 1000 lbs. and the leaf spring attached to it as shown. The entire assembly was found to weigh approximately 2800 lbs.

Vehicle speed. The vehicle speed at impact was 40 ft./sec.

The post. The post used for the test was a 6 in., standard weight, ASA Schedule No. 40 pipe, 9 ft. long. The pipe had a triangular plate with 12 in. sides welded to one end and a circular plate with a 10.75

in. diameter welded to the other end. Both plates were 0.75 in. thick and had three bolt cutouts to allow the post to be bolted to a base.

The mass of the post and plates was found to be 215 $lbs._m$ and the mass moment of inertia at the mass center was calculated to be $62 \frac{lbs._f\text{-}sec.^2}{ft.}$ for an axis perpendicular to the longitudinal axis of the post. The location of the center of mass was calculated to be 4.63 ft. from the end with the triangular plate and along the longitudinal axis of the post.

5.3 The Correlation

The correlation was obtained by use of the high-speed films of the crash test and a Vanguard Motion Analyzer. The analyzer is used to take information such as displacements and events from the high-speed photographic record of a test.

Table 8 shows information obtained by means of the motion analyzer at two critical times. The two times are when the post and the vehicle lose contact and when the post strikes the vehicle on its return path to the ground.

XCM and ZCM refer to translations of the mass center of the post in the XX and ZZ directions, respectively. V1 and V3 refer to linear velocities of the post mass center along the principal directions, θ and $\dot{\theta}$ refer to the angular displacement and velocity, respectively, of the post mass center and SV is the distance the vehicle has travelled. HIT is the distance from the front end of the vehicle to the point on the vehicle where the post strikes after being impacted.

TABLE 8. CRASH TEST DATA AT TWO CRITICAL TIMES

	70 MILLISECONDS	270 MILLISECONDS
θ (DEG.)	25.0	122.0
SV (FT.)	3.2	12.60
XCM (FT.)	- 1.50	- 7.10
ZCM (FT.)	+ 0.01	- 0.19
V1 (FT./SEC.)	- 28.5	–
V3 (FT./SEC.)	- 12.5	–
$\dot{\theta}$ (RAD/SEC)	8.75	8.75

To assure that the set of equations describing the motion after the post and the vehicle lose contact were consistent with the actual situation, values obtained from the motion analyzer at the beginning of this stage of motion were used as input information to the computer program for the mathematical model. A good agreement was seen to exist and a comparison is made in Table 9 at the time the post strikes the vehicle.

The different values of the spring constants obtained from Figure 17 and shown below were used in the equation for the spring force in the model.

Loading cycle:

C = 12000 lbs./ft.

C = 26400 lbs./ft.

Unloading cycle:

C = 38400 lbs./ft.

C = 12000 lbs./ft.

A constant spring force was applied after the displacement of the spring exceeded 4.5 in. and the U-bolts came into contact with the I-beam to which the spring assembly was attached. This force was assumed to be equal to the maximum value developed in the second set of springs when being subjected to the condition imposed by the U-bolts. This force was kept present until displacements reached values such that the spring started to unload. A comparison between model and crash test results using this approach is shown in Table 10 and Figure 18.

TABLE 9. COMPARISON OF MODEL DATA WITH CRASH TEST DATA AT 270 MILLISECONDS

	MODEL	CRASH TEST
HIT (FT.)	7.55	7.83
θ (DEG.)	121.0	122.0
XCM (FT.)	7.61	7.10
ZCM (FT.)	0.27	0.19
SV (FT.)	11.5	11.04
TIME (MILLISEC.)	273.0	270.0

TABLE 10. COMPARISON OF MODEL DATA WITH CRASH TEST DATA AT 70 MILLISECONDS

	MODEL	CRASH TEST
TIME (MILLISEC.)	71.8	70.0
θ (DEG.)	26.0	25.0
$\dot{\theta}$ (RAD/SEC)	12.50	8.75
XCM (FT.)	- 1.20	- 1.50
ZCM (FT.)	- 0.04	+ 0.01
V1 (FT./SEC.)	-28.75	-28.50
V3 (FT./SEC.)	- 15.47	- 12.50
SV (FT.)	2.80	3.20

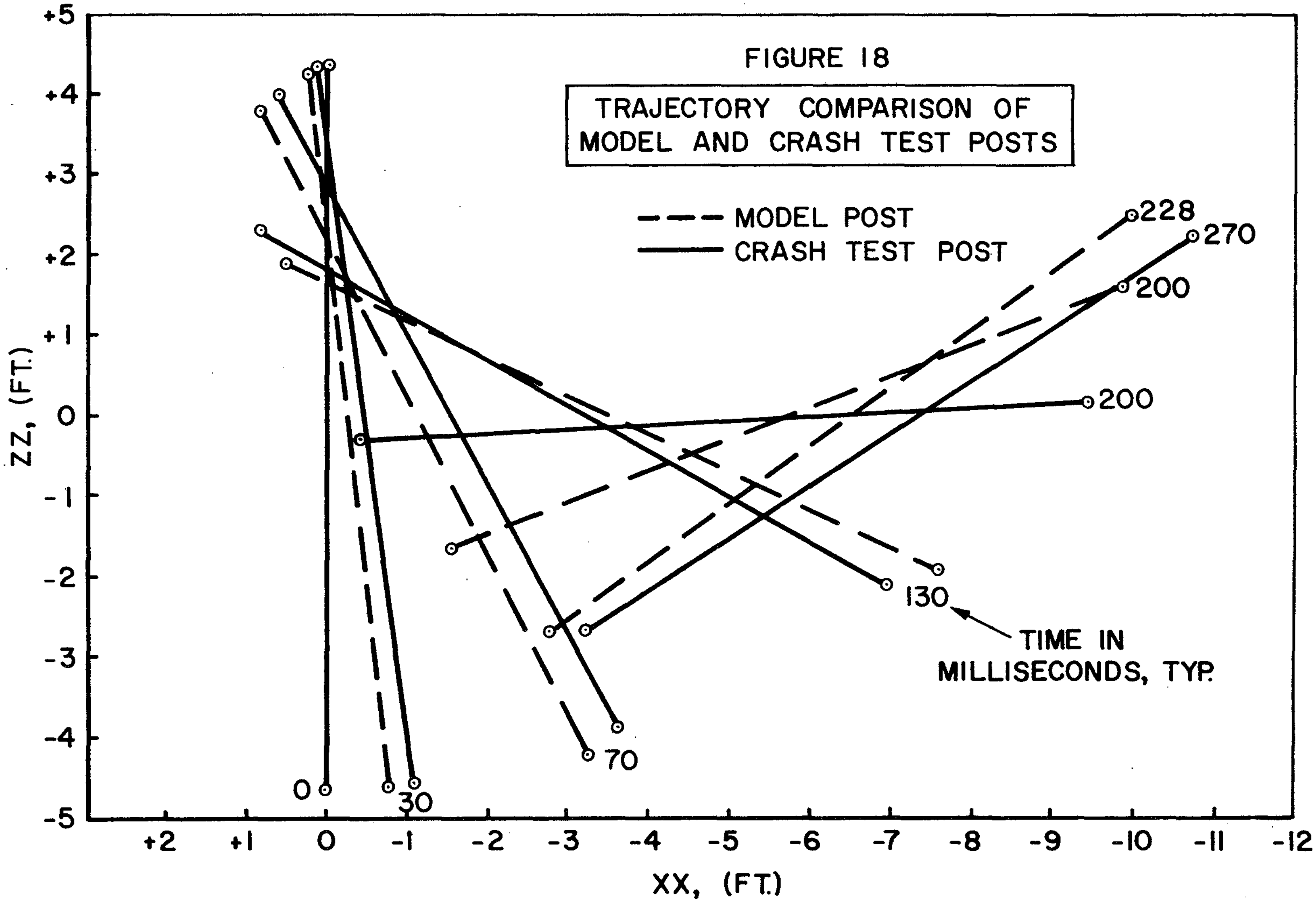


FIGURE 18

TRAJECTORY COMPARISON OF MODEL AND CRASH TEST POSTS

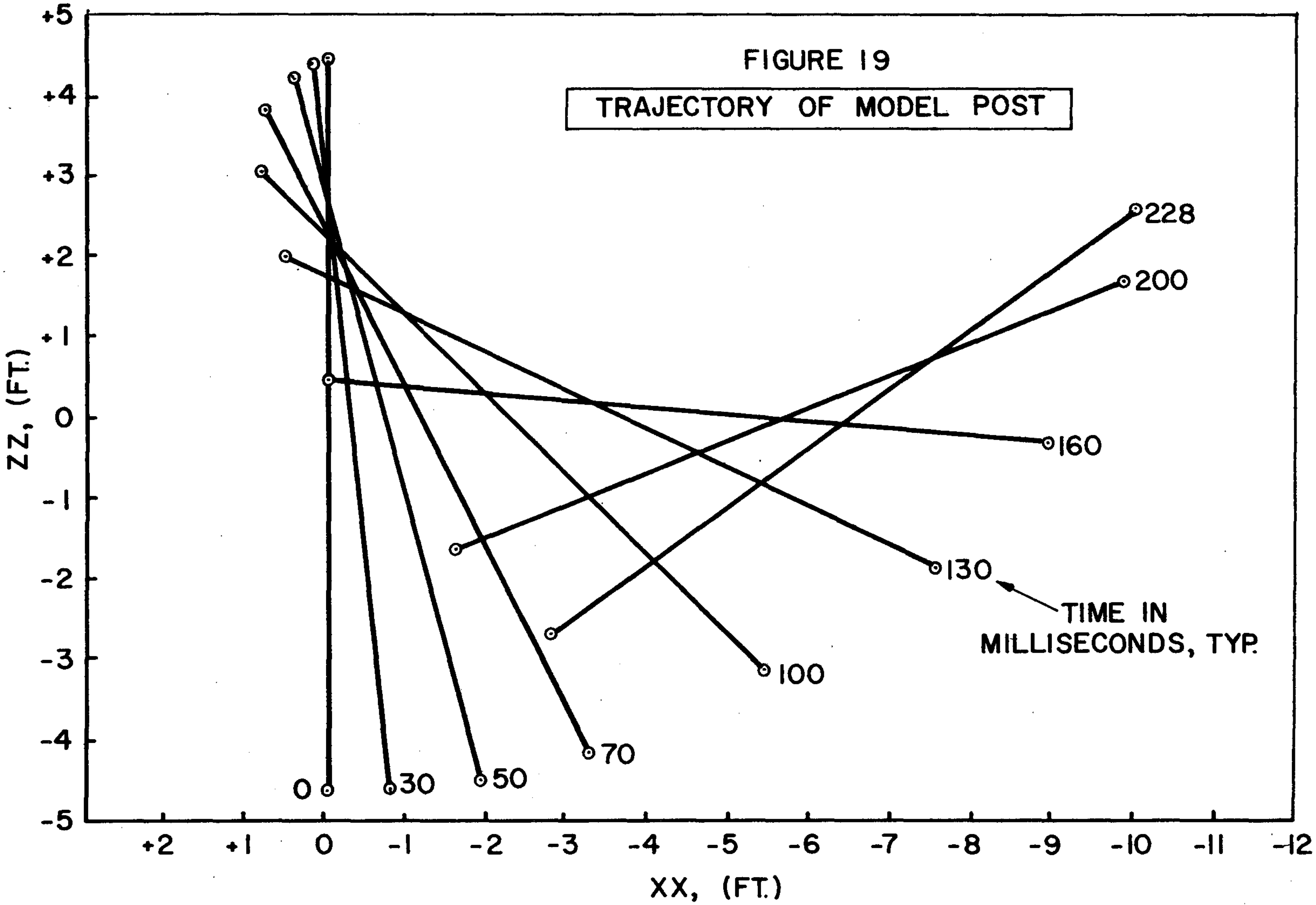


FIGURE 19

TRAJECTORY OF MODEL POST

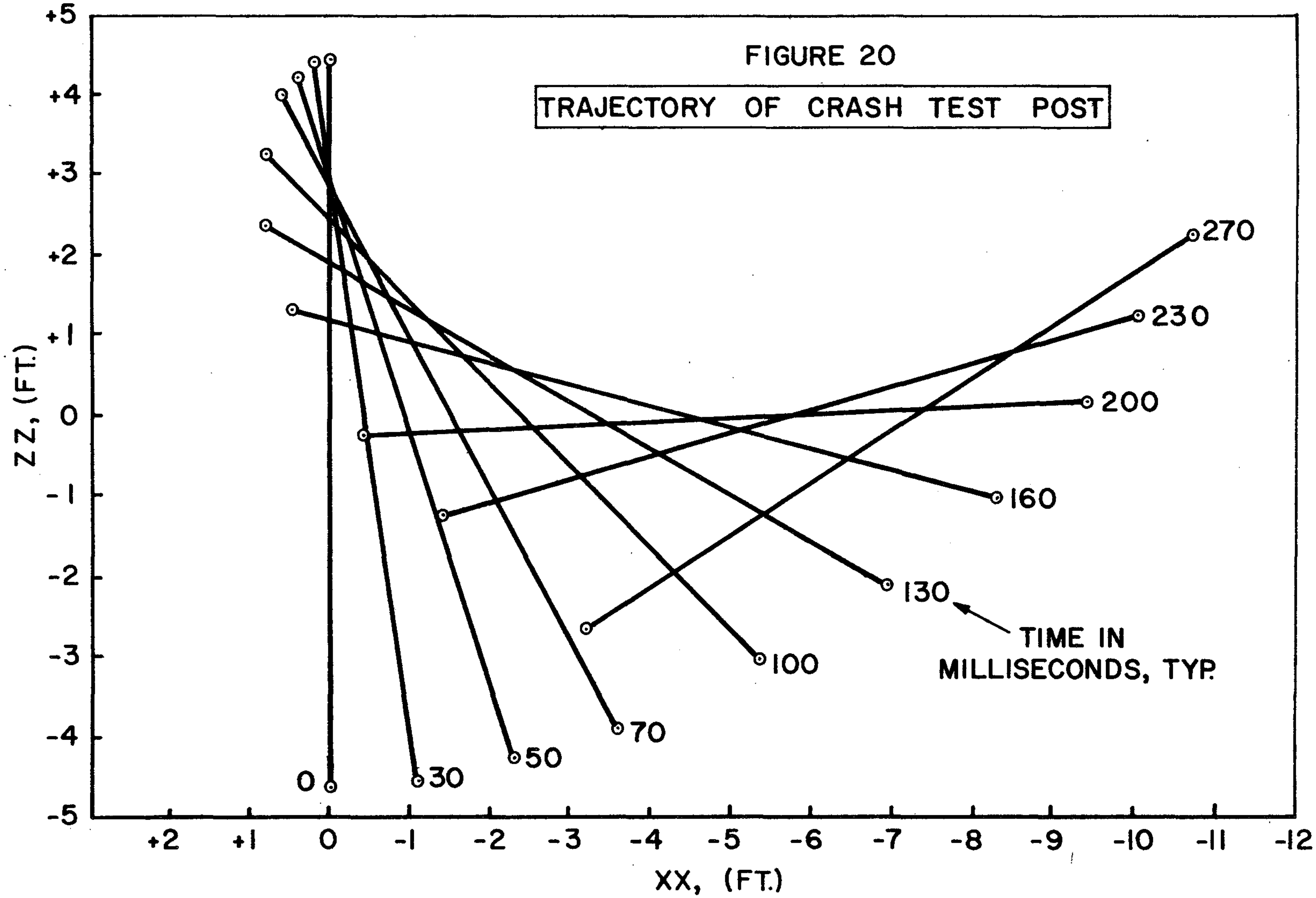


FIGURE 20

TRAJECTORY OF CRASH TEST POST

The biggest discrepancy in Table 10 between model and crash test results is seen for the value of the angular velocity. This discrepancy accounts for the further disagreement between the model and the real post behavior in the second stage of motion. Table 9 contrasts the behavior of the model and the real post when values obtained from the motion analyzer at the beginning of the second stage of motion are used as input information to the computer program for the mathematical model. A good correlation is seen to exist.

The purpose of this chapter is to present the behavior pattern that can be expected from the model under the assumptions made and to try and obtain a correlation. An attempt to find values of the unknown parameters that will force the model to fit the test data will not be shown.

5.4 Correlation for Non-Planer Motion

The non-planer motion case occurs when the behavior of the luminaire support assembly complete with luminaire and luminaire support arm and post is considered.

The model has verified the phenomenological behavior of this type of motion and like the crash test shows that:

(1) The luminaire support arm rotates clockwise when the post is struck by the vehicle in the manner described in Chapter III.

(2) For vehicular speeds of 30 miles/hr and 40 miles/hr, the vehicle is seen to pass under the post and not be struck by the post on its return path to the ground.

(3) The rotation of the support arm for the vehicular speeds

mentioned previously is large enough so that the luminaire comes to rest at the edge of the highway and therefore does not cause an unsafe condition for other motorists.

(4) Variations of up to 15 degrees in the angle Alpha do not significantly affect the trajectory of the luminaire-support assembly for the speeds considered.

5.5 Conclusions Based on Correlation

From the correlations in this chapter, it may be concluded that the mathematical model can reasonably simulate the behavior of a breakaway post that is assumed to behave like a rigid body. Even when no attempt is made to find values of the unknown parameters that would force the model to match the test data, a reasonable phenomenological simulation of the behavior of the post can be expected.

The assumptions made in the mathematical model for the stage of motion where the post and the vehicle are in contact require some modification. It can be seen in Figure 18 that the model post lags the crash test post in this initial phase of the motion and overtakes it in the final phase. This is due to the fact that the conditions imposed on the model bring about a longer application of the larger forces than is actually the case. The larger force produces a larger torque about the mass center of the post and this larger torque brings about a higher angular velocity.

A careful study of the high-speed photographic films of the crash test reveals that after the second set of leaf springs have reached their maximum deflected position, the post seems to ride the front end of the vehicle. This can best be observed to occur for about 20 milliseconds. An assumption of this nature is not made in the model.

The model assumes that after the post and the vehicle lose contact the post is essentially a rigid body moving in space under the influence of gravity and having a constant angular velocity. This assumption was verified by employing post displacements and velocities obtained from the motion analyzer. The displacements and velocities were used as input information to the computer program for the mathematical model. A good correlation was seen to exist.

C H A P T E R V I

PARAMETER STUDY

The study presented in this chapter was conducted to illustrate the value of the mathematical model and to investigate the crash-dynamic effects of some of the parameters of the luminaire support assembly. No effort was made to force the model to fit the limited test data that were available and the findings are based on the assumption that the luminaire support post assembly behaves as a rigid body. It was also assumed that the assembly had a base exhibiting break-away characteristics, and that a constant frictional resisting force of 900 lbs. was applied at the base. The force remained present during a base translation of one inch.

Three different luminaire support posts were employed in the study. They included a 9.5" x 4" x 36' - 8.5" steel post, an 8" x 6" x 30' - 0" aluminum post and an 8" x 4.1" x 27' x 9" steel support post with twin luminaires. The mass of the luminaire was taken to be 35 lbs_m and the luminaire support arm, in all cases, was taken to have a length of 10.5 ft. and a mass of 83.5 lbs_m. The different support assemblies are shown in Figure 21.

A vehicle having a mass of 3200 lb_m and the dimensions of a 1955 Ford Sedan was used for this study. The investigation was carried out for vehicular velocities of 20 miles and 40 miles per hour. The values of the vehicular approach angle α (see Figure 9) used in the investigation were 0°, 15°, and 30°.

6.1 General Discussion

In order to facilitate the interpretation of the results obtained from the mathematical model it was deemed necessary to define a new coordinate system. This coordinate system has its origin at the base of the support of the post and is determined by translating the XX, YY, ZZ coordinate system defined in Chapter III. The letters, P, Q, R, and U refer to different points on the luminaire support post assembly and are defined in Figure 21. The values XPL, YPL, ZPL, etc., presented in Tables 11 and 12 are coordinates of the points defined in Figure 21 and with respect to the XL, YL, ZL coordinate system. These coordinates, depending upon which occurrence takes place first, are taken at a time when the post, on its return path, has struck either the vehicle or the ground. Figures 22 through 29 show the posts at various positions for different values of the angle α and the vehicular velocity.

6.2 Effect of the Vehicular Velocity

The reader may observe from Table 11 that for a vehicular velocity of 20 miles per hour the support post, with one exception, strikes the vehicle before striking the ground. Table 12 shows that for a vehicular velocity of 40 miles per hour the support assembly clears the vehicle in all cases.

The overall behavior of the support post assembly for the two vehicular velocities considered is very similar. In the case of a vehicular velocity of 20 miles per hour the slower moving vehicle imparts less energy to the support assembly and causes it to encounter the vehicle before striking the ground.

6.3 Effect of the Vehicular Approach Angle α

The study showed that an increase in the vehicular approach angle α caused the support post assembly to have a smaller absolute terminal coordinate in the XL direction and a larger one in the YL direction. This fact gives the assembly a tendancy to fall more in the direction away from the highway as the angle α is increased. This conclusion is obtained by assuming that the post does not first encounter the vehicle and the terminal coordinates are taken as the coordinates describing the position of the assembly when it strikes the ground.

6.4 Observations

The study presented in this chapter, although not broad in scope, indicates the value of the model and also some of the impact characteristics of the luminaire support post assembly.

From the values presented for point R in Tables 11 and 12 and from Figure 24, it is clear that regardless of the vehicular speed, type of post used, or vehicular approach angle, the tendancy is for the single luminaire support arm to have a negative rotation (according to the right hand rule) about the ZL axis. This rotation causes the support arm to rotate in a direction away from the highway after the post is impacted and not strike the highway causing an unsafe condition for other motorists.

The case involving the support post with twin luminaires shows that here the tendancy is for the rotation to be positive about the ZL axis. This is shown in Figure 29. A positive rotation causes point U to displace towards the highway but for the cases investigated this effect always causes the striking point to increase in the direction away from the assumed edge of the pavement. Tables 11 and 12 reveal that for these cases, the point R is the first to contact the ground. As the vehicle approach angle is increased, the terminal position of the point R increases in the direction away from the highway. The same effect had been noticed in the case of the single luminaire support assembly.

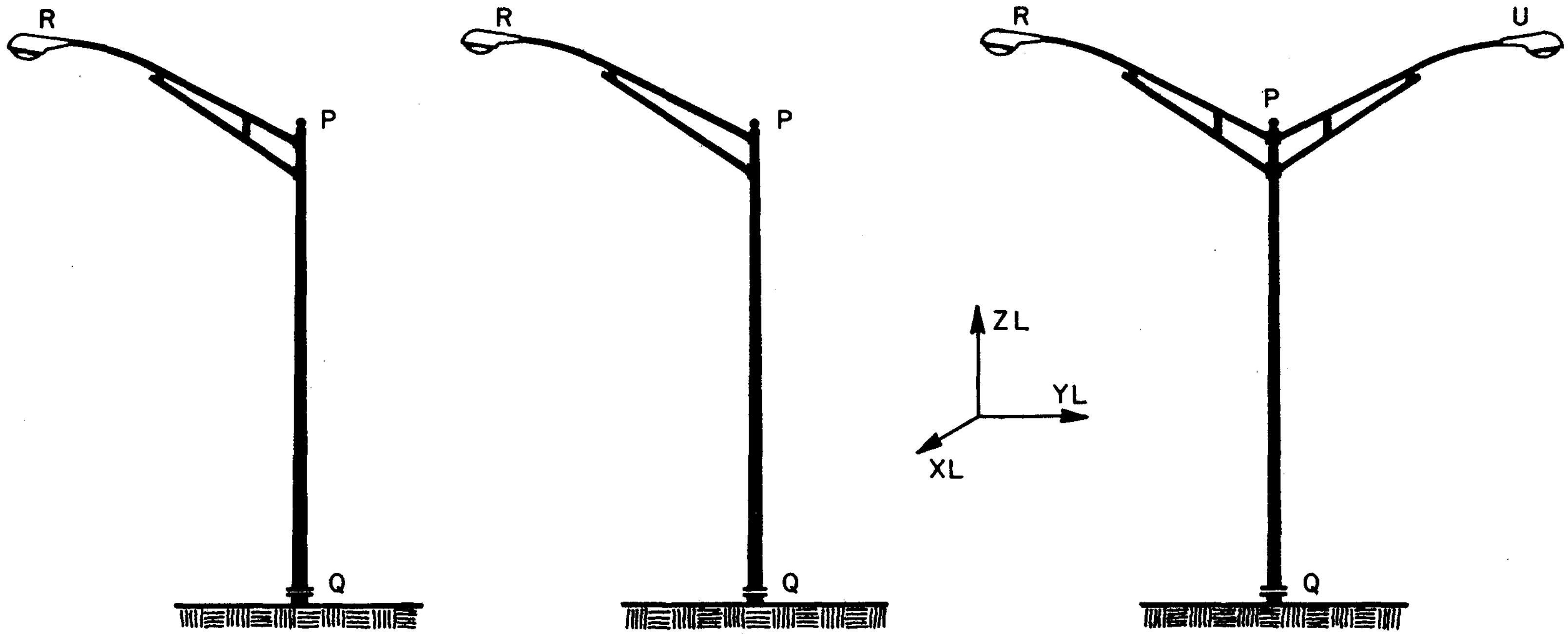
The values of the maximum vehicle deceleration presented in Tables 11 and 12 show that in all the cases considered the impact forces are kept within tolerable limits. This indicates the feasibility of the "break-away" luminaire support post design.

	ALUMINUM POST			STEEL POST			TWIN LUMINAIRE STEEL POST		
VEHICLE VELOCITY (MPH)	20			20			20		
WEIGHT OF LUMINAIRE AND SUPPORT ASSEMBLY (LBS)	316.63			485.07			502.45		
APPROACH ANGLE α (DEG)	0	15	30	0	15	30	0	15	30
DECREASE IN VEHICLE VELOCITY (MPH)	1.07	1.07	1.09	1.15	1.17	1.24	1.43	1.43	1.43
MAXIMUM VEHICLE DECELERATION (g's)	2.85	2.86	2.88	3.31	3.33	3.37	3.31	3.33	3.35
XPL (FT)	-0.48	-0.96	0.04	3.39	2.43	1.65	-4.33	-3.75	-2.74
YPL (FT)	-6.39	-4.35	-1.02	-6.96	-7.15	-6.34	0.0	1.68	2.67
ZPL (FT)	6.90	5.07	4.28	19.77	16.66	12.07	3.93	5.36	7.78
XRL (FT)	2.24	2.07	0.34	6.64	7.14	6.35	0.42	-0.76	-0.13
YRL (FT)	-1.51	-1.30	-6.59	0.30	-1.84	-4.53	-10.5	-8.12	-5.52
ZRL (FT)	16.14	15.40	13.53	27.62	25.30	22.05	3.76	0.08	0.0
XQL (FT)	-29.09	-27.87	-25.13	-26.51	-26.59	-25.29	-31.56	-30.43	-27.07
YQL (FT)	00.39	7.73	14.23	2.00	9.15	16.56	0.0	7.15	14.32
ZQL (FT)	3.76	5.04	6.32	0.46	1.19	2.23	4.91	4.58	3.98
XUL (FT)							0.42	2.56	3.15
YUL (FT)							10.5	9.57	6.79
ZUL (FT)							3.76	10.90	16.80
TIME TO HIT CAR (MILLISEC)	820.0	829.0	836.4	760.4	807.6	873.0	975.0	971.2	—
TIME TO HIT GROUND (MILLISEC)	—	—	—	—	—	—	—	—	992.0
COMMENTS	POST HITS TOP OF CAR	POST HITS TOP OF CAR	POST HITS TOP OF CAR	POST HITS FRONT OF CAR	POST HITS FRONT OF CAR	POST HITS FRONT OF CAR	POST HITS TOP OF CAR	POST HITS TOP OF CAR	POST HITS GROUND WITH POINT R

TABLE II A COMPARISON OF MODEL RESULTS FOR A VEHICULAR VELOCITY OF 20 MPH

	ALUMINUM POST			STEEL POST			TWIN LUMINAIRE STEEL POST		
VEHICLE VELOCITY (MPH)	40			40			40		
WEIGHT OF LUMINAIRE AND SUPPORT ASSEMBLY (LBS)	316.63			485.07			502.45		
APPROACH ANGLE α (DEG)	0	15	30	0	15	30	0	15	30
DECREASE IN VEHICLE VELOCITY (MPH)	1.90	1.91	1.94	2.05	2.05	2.06	2.56	2.58	2.62
MAXIMUM VEHICLE DECELERATION (g's)	5.52	5.54	5.59	6.58	6.61	6.70	6.60	6.62	6.71
XPL (FT)	-8.91	-11.28	-13.49	-4.31	-3.52	-5.13	-12.99	-8.10	-5.96
YPL (FT)	-6.76	-4.10	1.04	0.06	-0.97	-2.05	0.00	3.05	4.7
ZPL (FT)	5.81	3.37	0.00	8.82	10.48	8.88	3.49	8.72	10.81
XRL (FT)	-6.81	-5.62	-3.55	-0.30	-0.03	1.26	-9.68	-7.97	-5.50
YRL (FT)	2.15	4.49	5.17	-5.66	-3.14	0.56	-10.50	-4.58	0.52
ZRL (FT)	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.00	0.00
XQL (FT)	-35.40	-33.28	-28.77	-40.60	-38.43	-33.63	-32.01	-32.67	-30.16
YQL (FT)	-1.23	7.80	15.52	-1.45	9.74	20.09	0.00	7.02	15.12
ZQL (FT)	17.56	19.00	20.75	14.16	14.18	15.59	23.01	19.84	17.76
XUL (FT)							-9.68	0.33	2.01
YUL (FT)							10.50	9.30	5.25
ZUL (FT)							0.00	13.48	19.12
TIME TO HIT CAR (MILLISEC)									
TIME TO HIT GROUND (MILLISEC)	730.0	739.0	746.8	819.2	795.2	792.0	808.0	695.2	657.2
COMMENTS	POST HITS GROUND WITH POINT R	POST HITS GROUND WITH POINT R	POST HITS GROUND WITH POINT P	POST HITS GROUND WITH POINT R	POST HITS GROUND WITH POINT R	POST HITS GROUND WITH POINT R	POST HITS GROUND WITH POINTS R&U	POST HITS GROUND WITH POINT R	POST HITS GROUND WITH POINT R

TABLE 12 A COMPARISON OF MODEL RESULTS FOR A VEHICULAR VELOCITY OF 40 MPH



LUMINAIRE AND STEEL SUPPORT POST ASSEMBLY

LUMINAIRE AND ALUMINUM SUPPORT POST ASSEMBLY

TWIN LUMINAIRE AND STEEL SUPPORT POST ASSEMBLY

THE TYPES OF LUMINAIRE SUPPORT POST ASSEMBLIES CONSIDERED IN THE PARAMETER STUDY

FIGURE 21

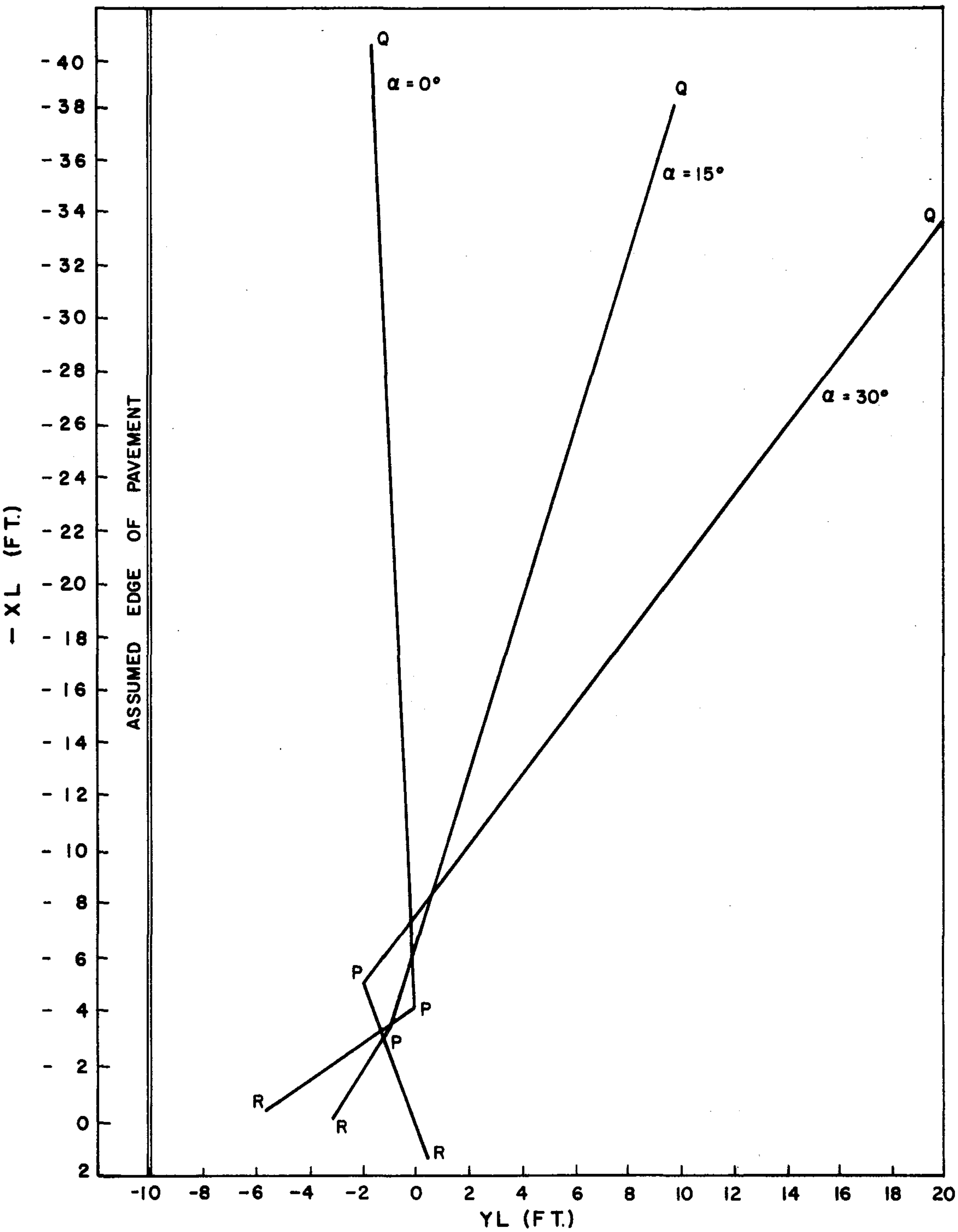


FIGURE 22 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 40 MPH

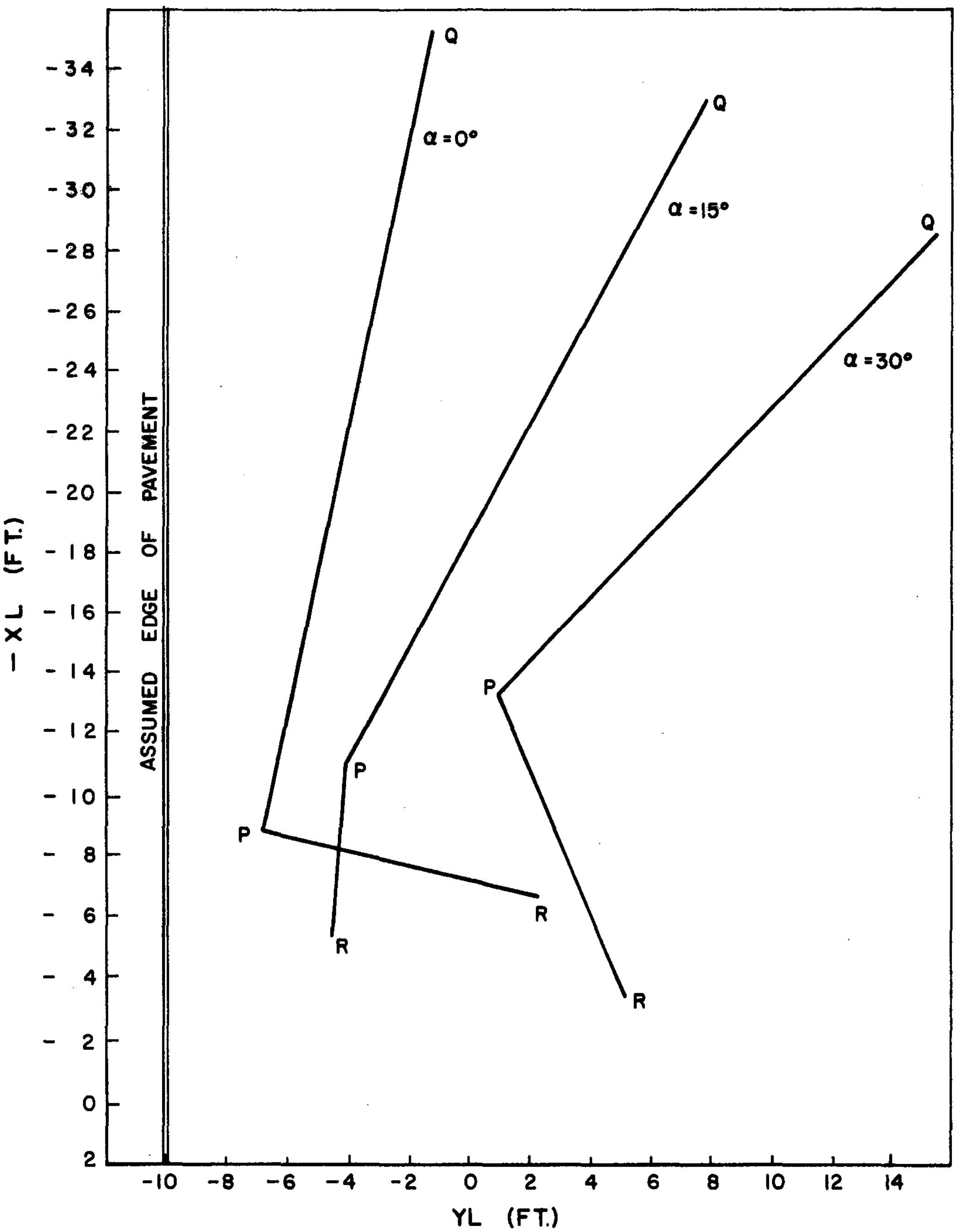


FIGURE 23 THE TERMINAL POSITION OF THE ALUMINUM SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 40 MPH

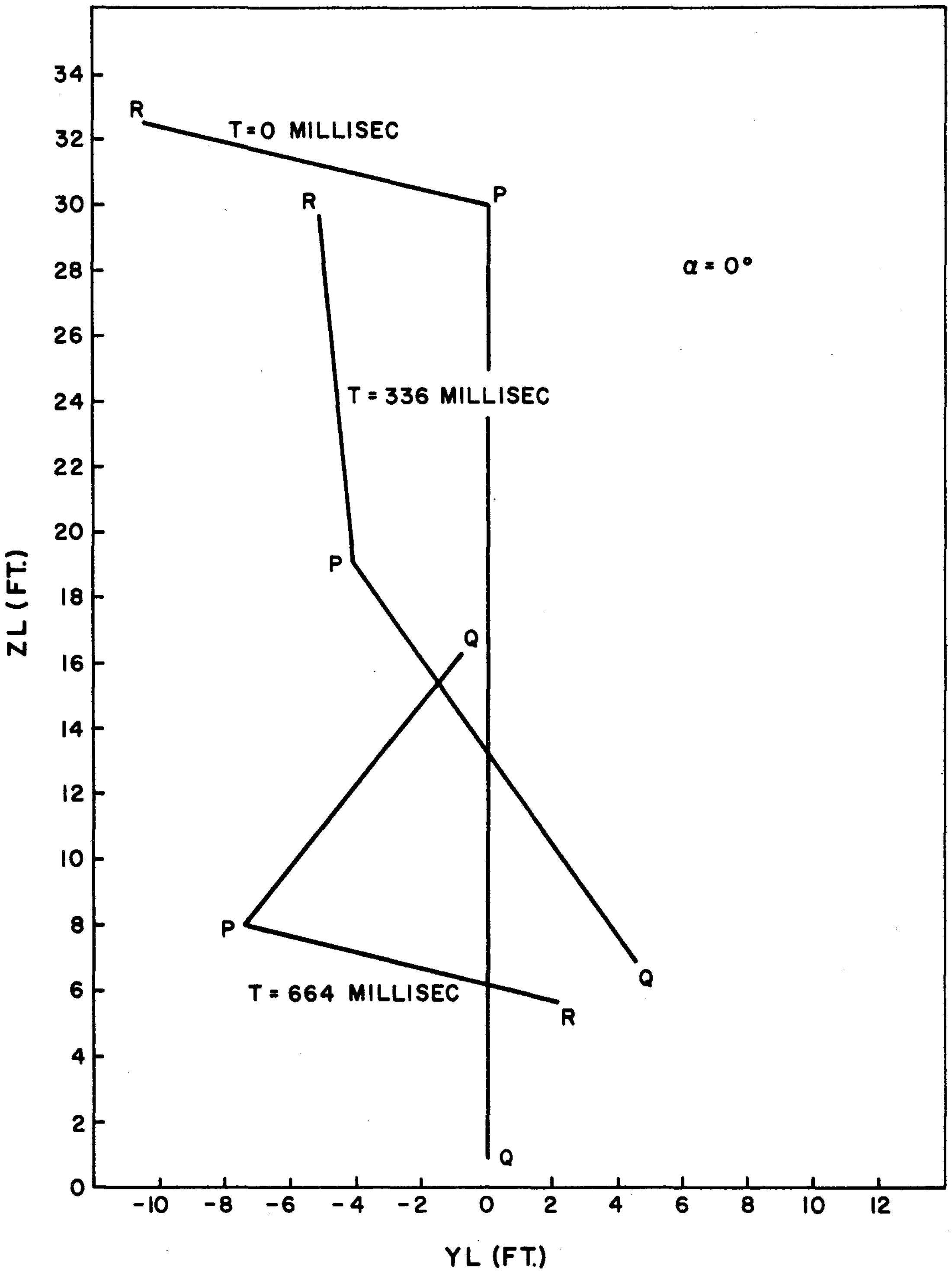


FIGURE 24 THE TRAJECTORY OF THE ALUMINUM SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 40 MPH

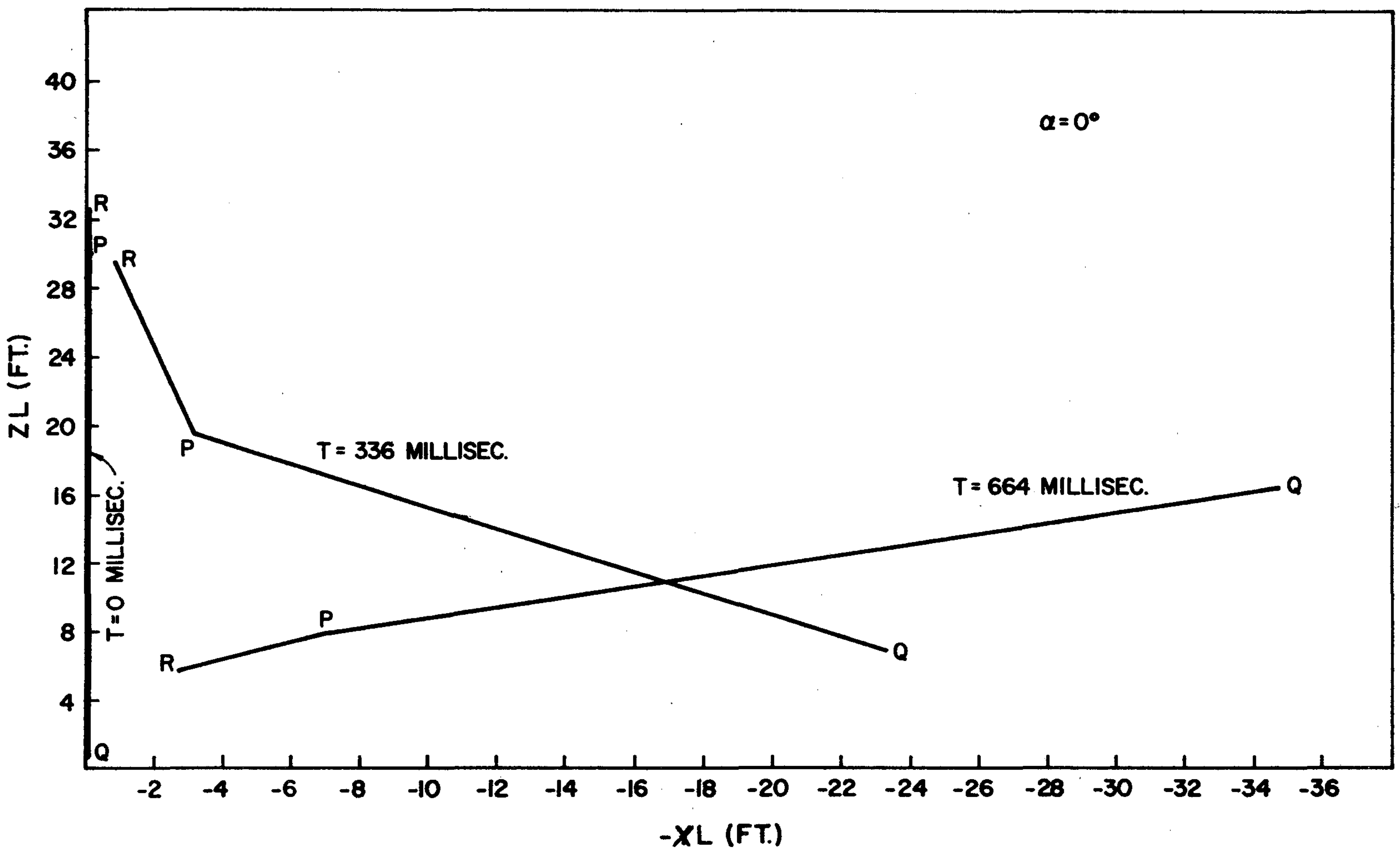


FIGURE 24 THE TRAJECTORY OF THE ALUMINUM SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 40 MPH

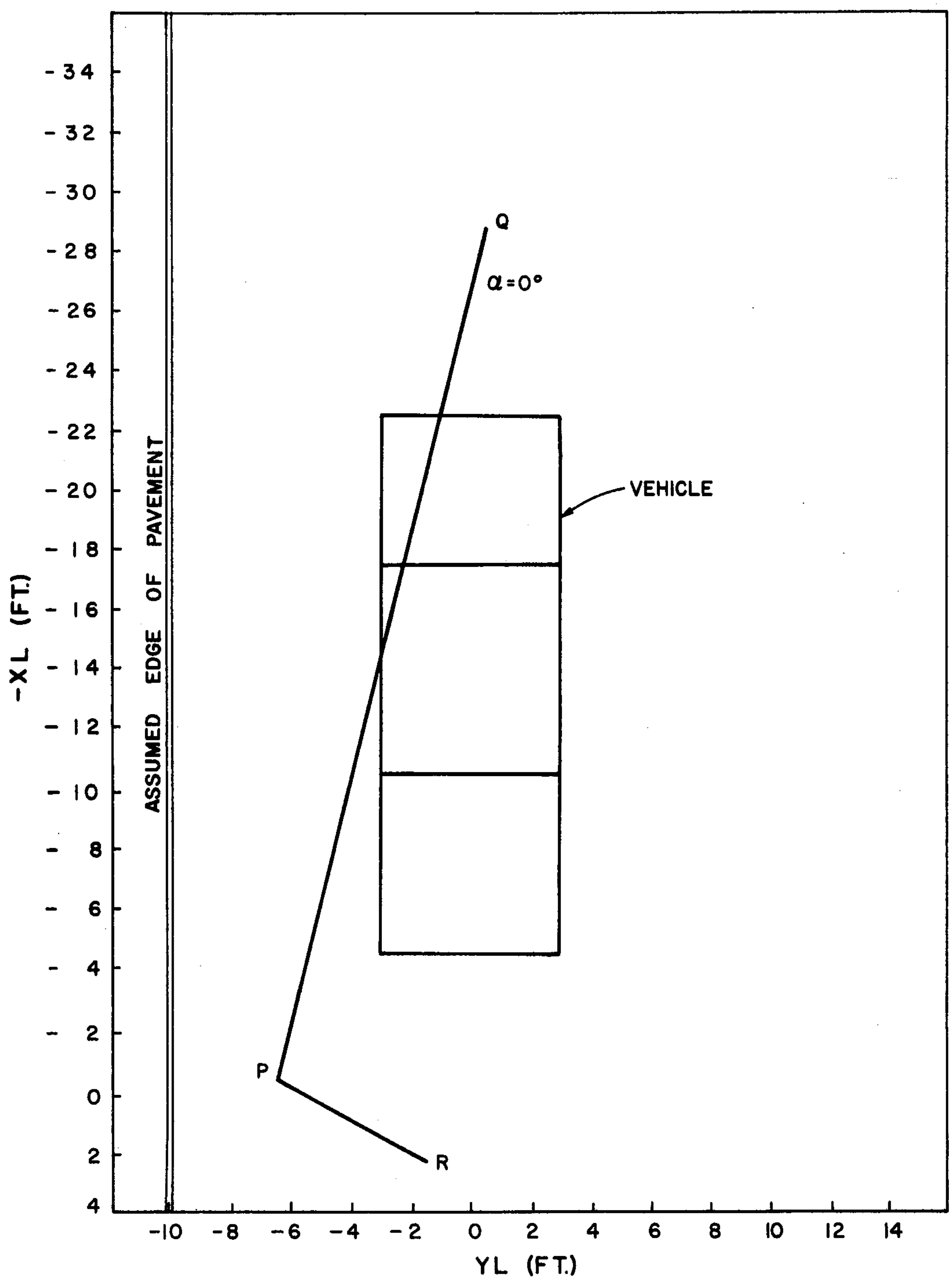


FIGURE 25 THE TERMINAL POSITION OF THE ALUMINUM SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 20 MPH

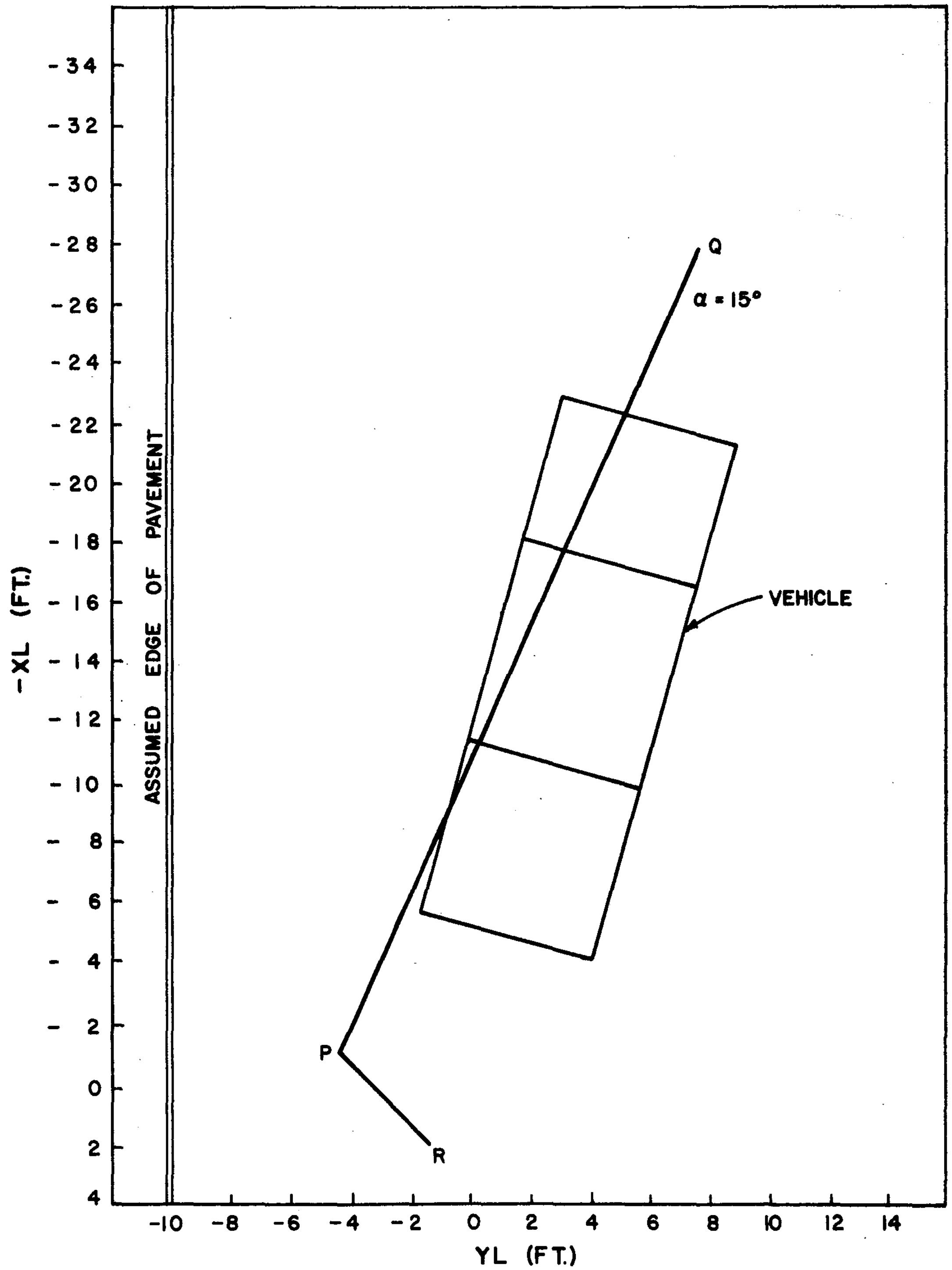


FIGURE 25 THE TERMINAL POSITION OF THE ALUMINUM SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 20 MPH

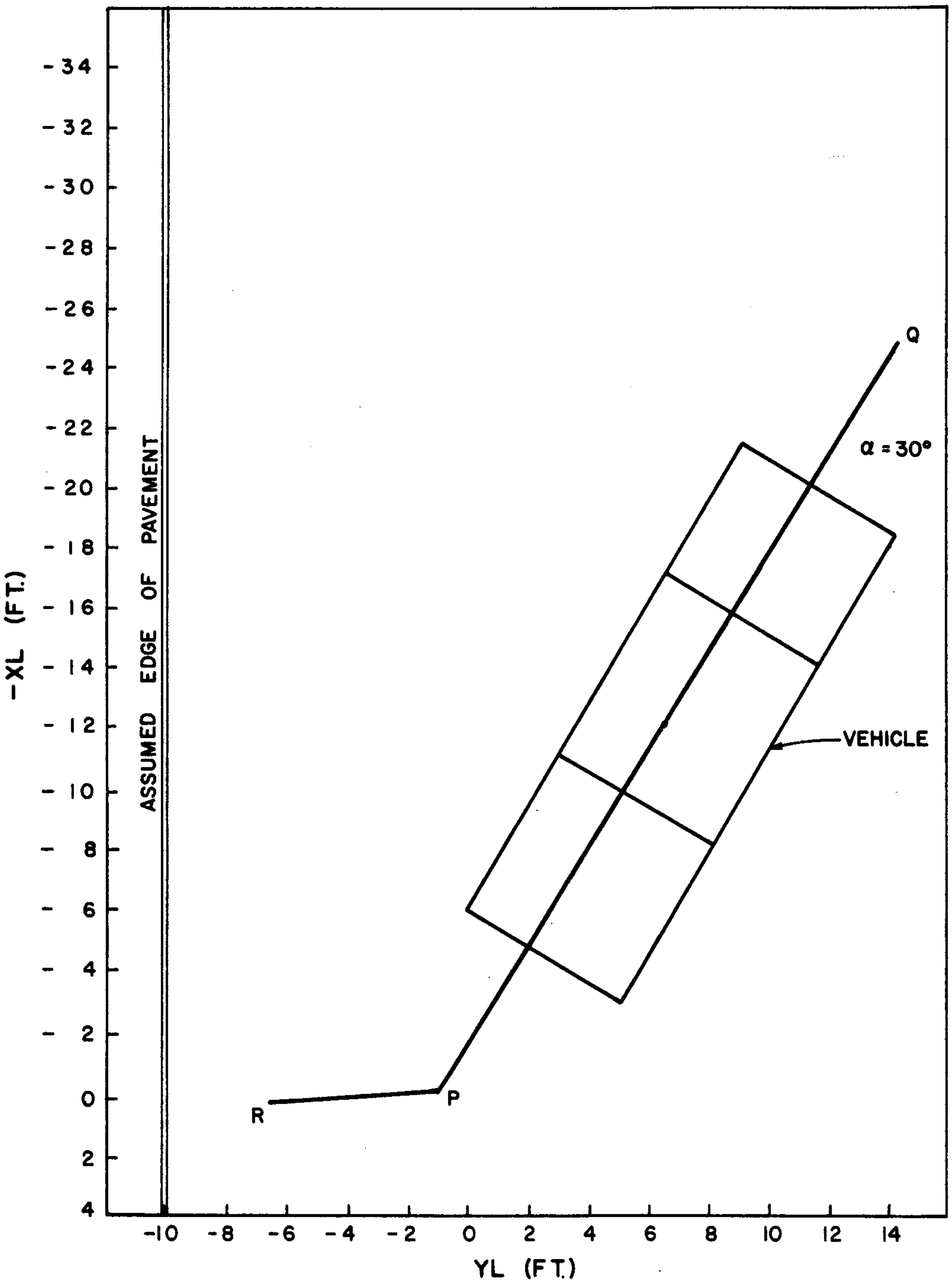


FIGURE 25 THE TERMINAL POSITION OF THE ALUMINUM SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 20 MPH

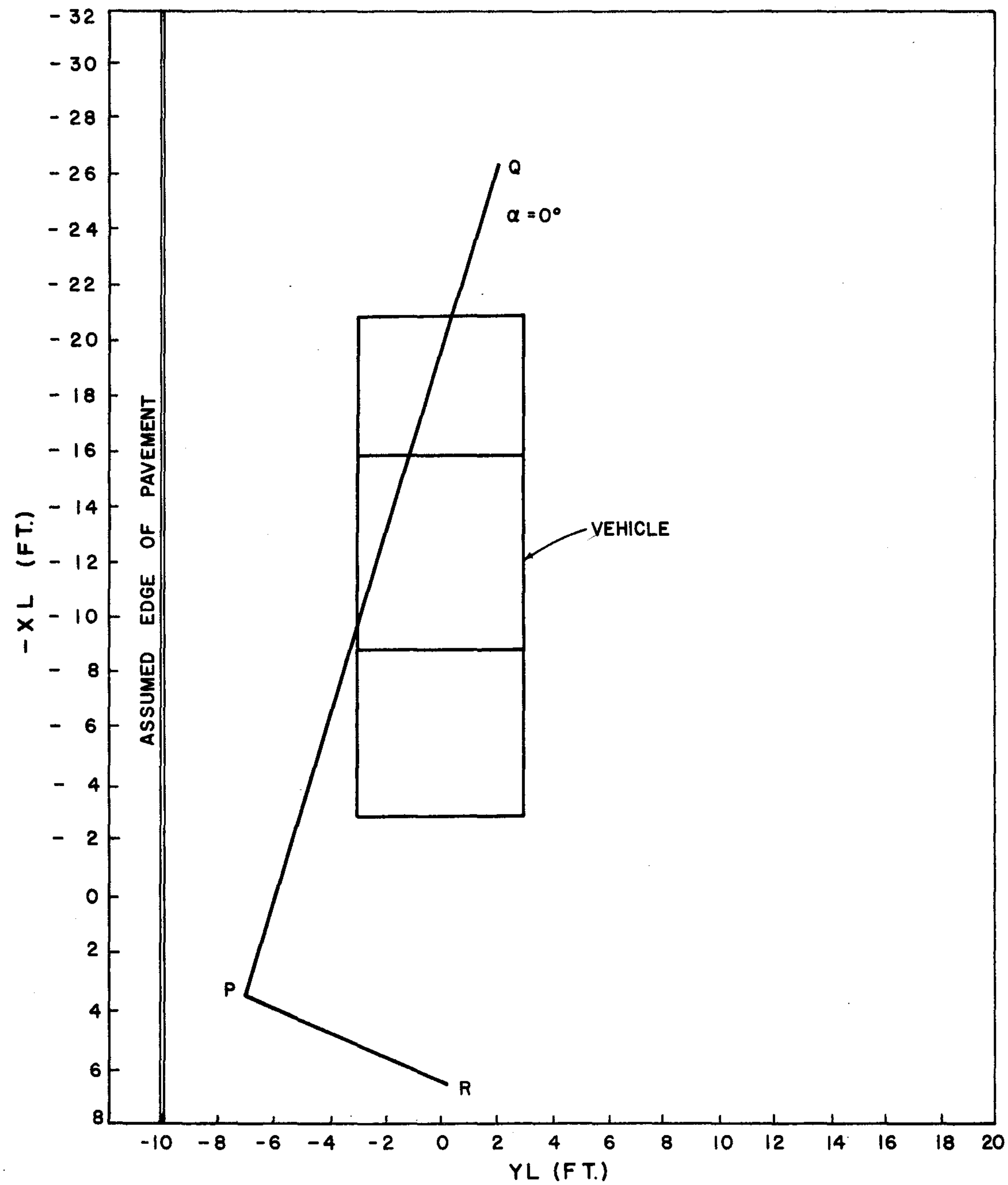


FIGURE 26 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 20 MPH

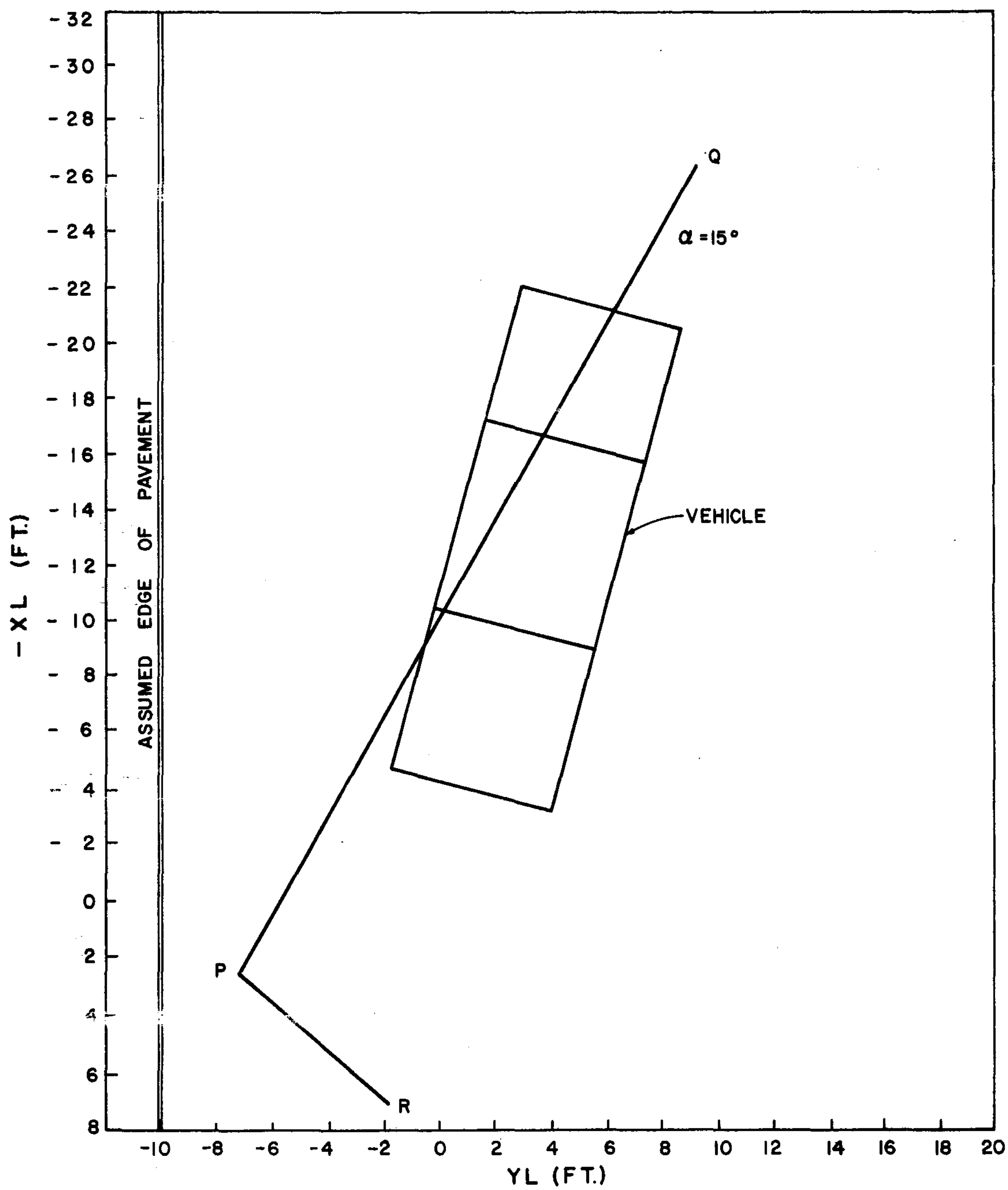


FIGURE 26 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 20 MPH

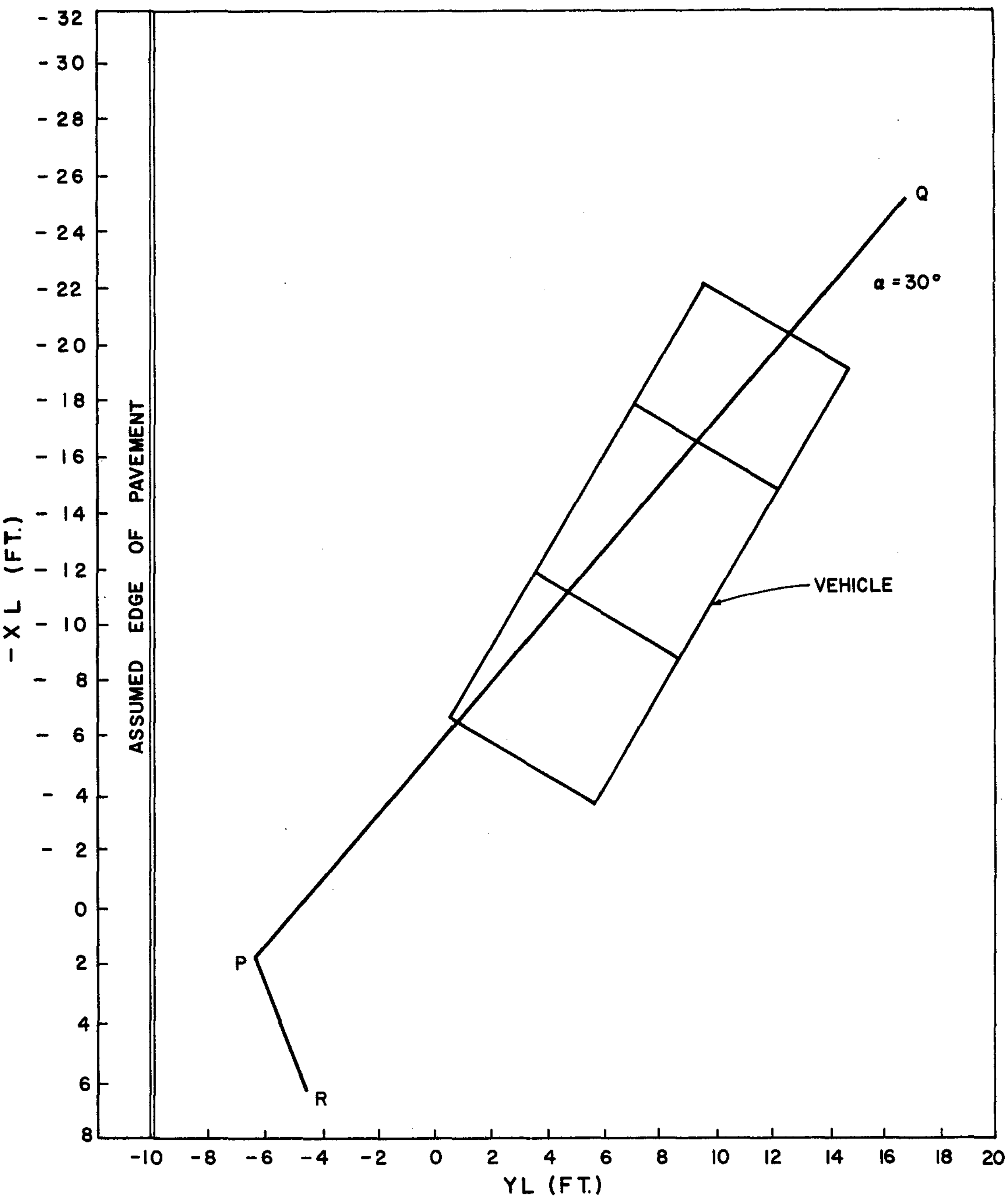


FIGURE 26 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY FOR A VEHICLE VELOCITY OF 20 MPH

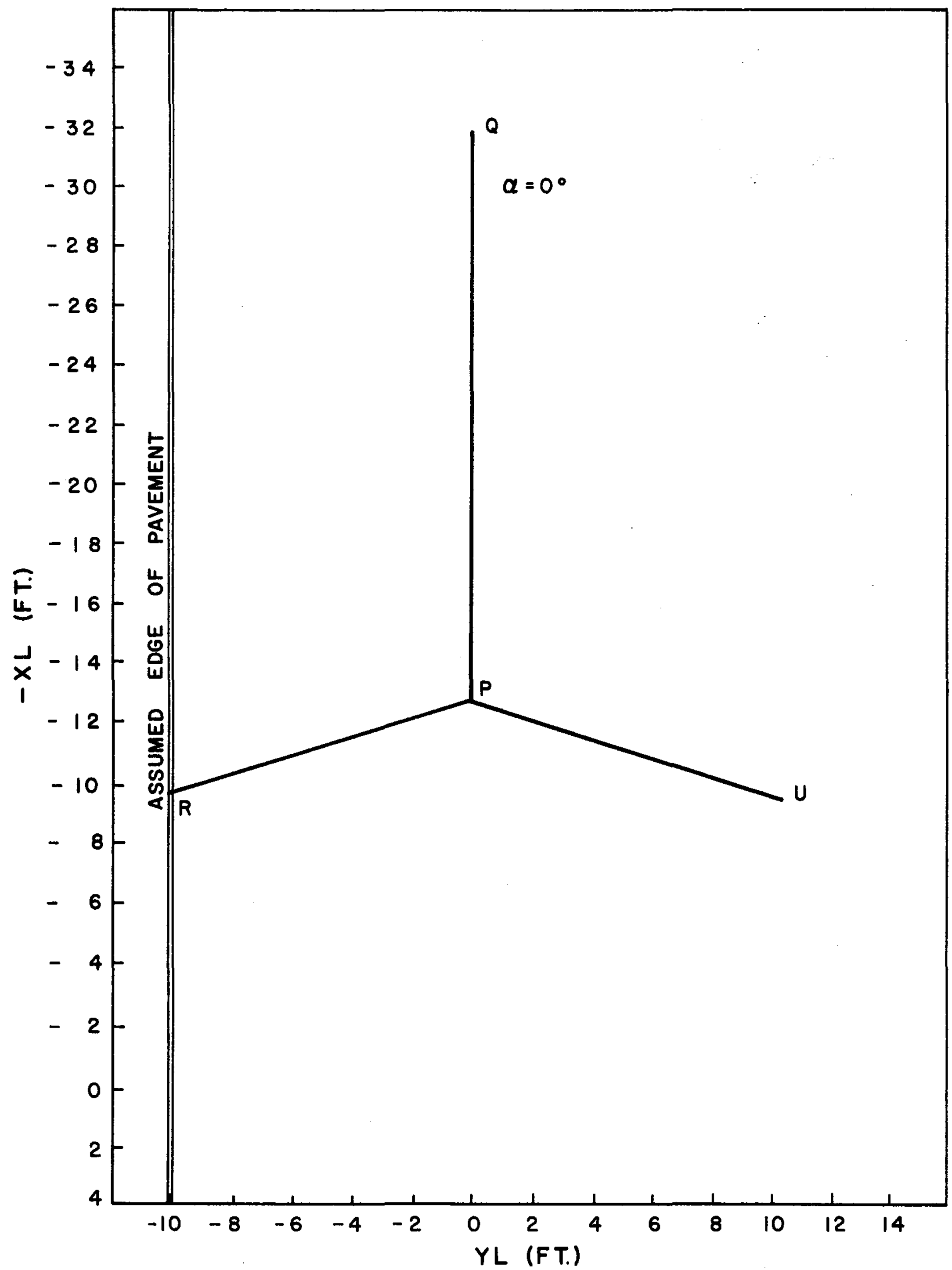


FIGURE 27 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 40 MPH

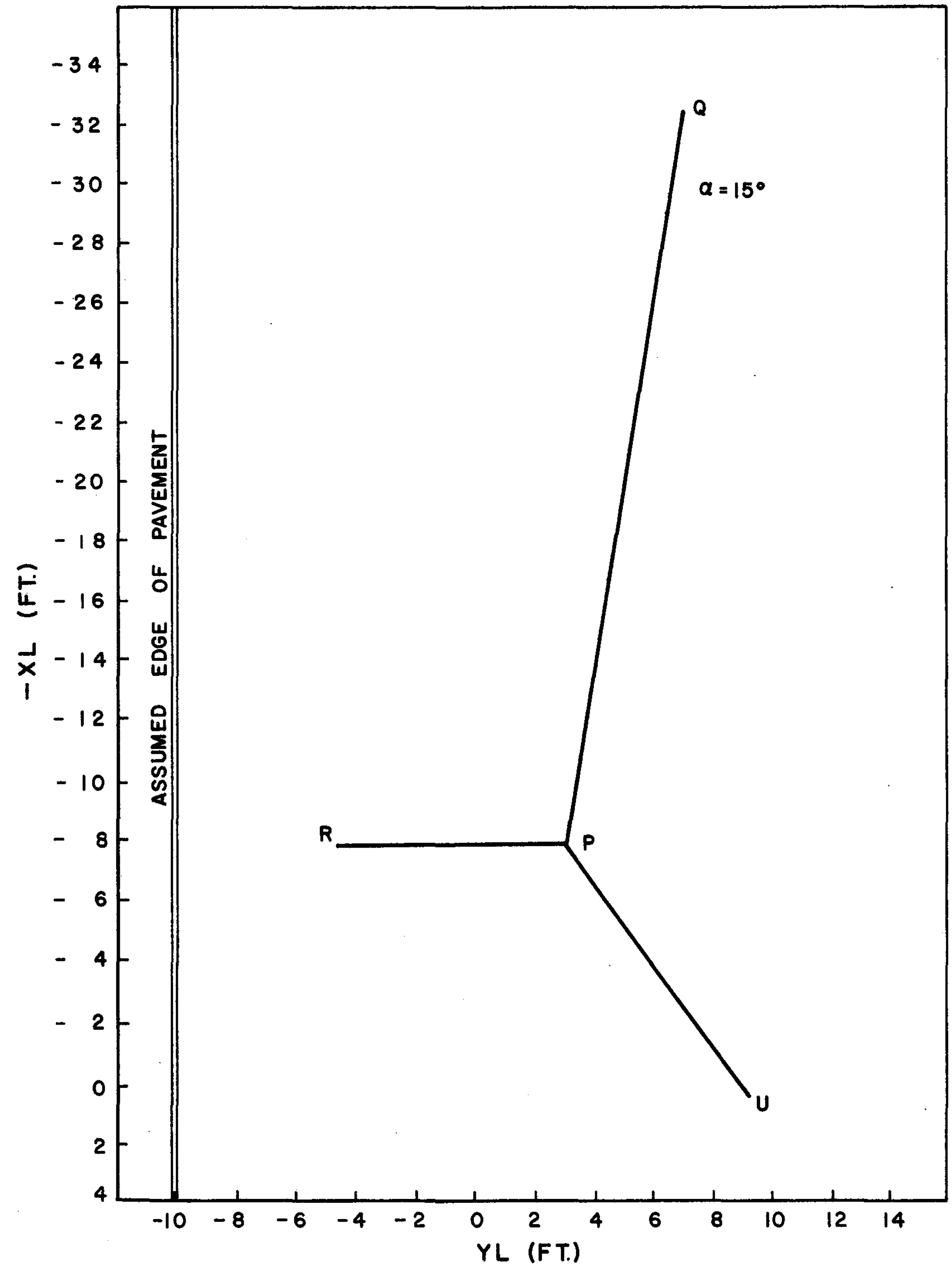


FIGURE 27 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 40 MPH

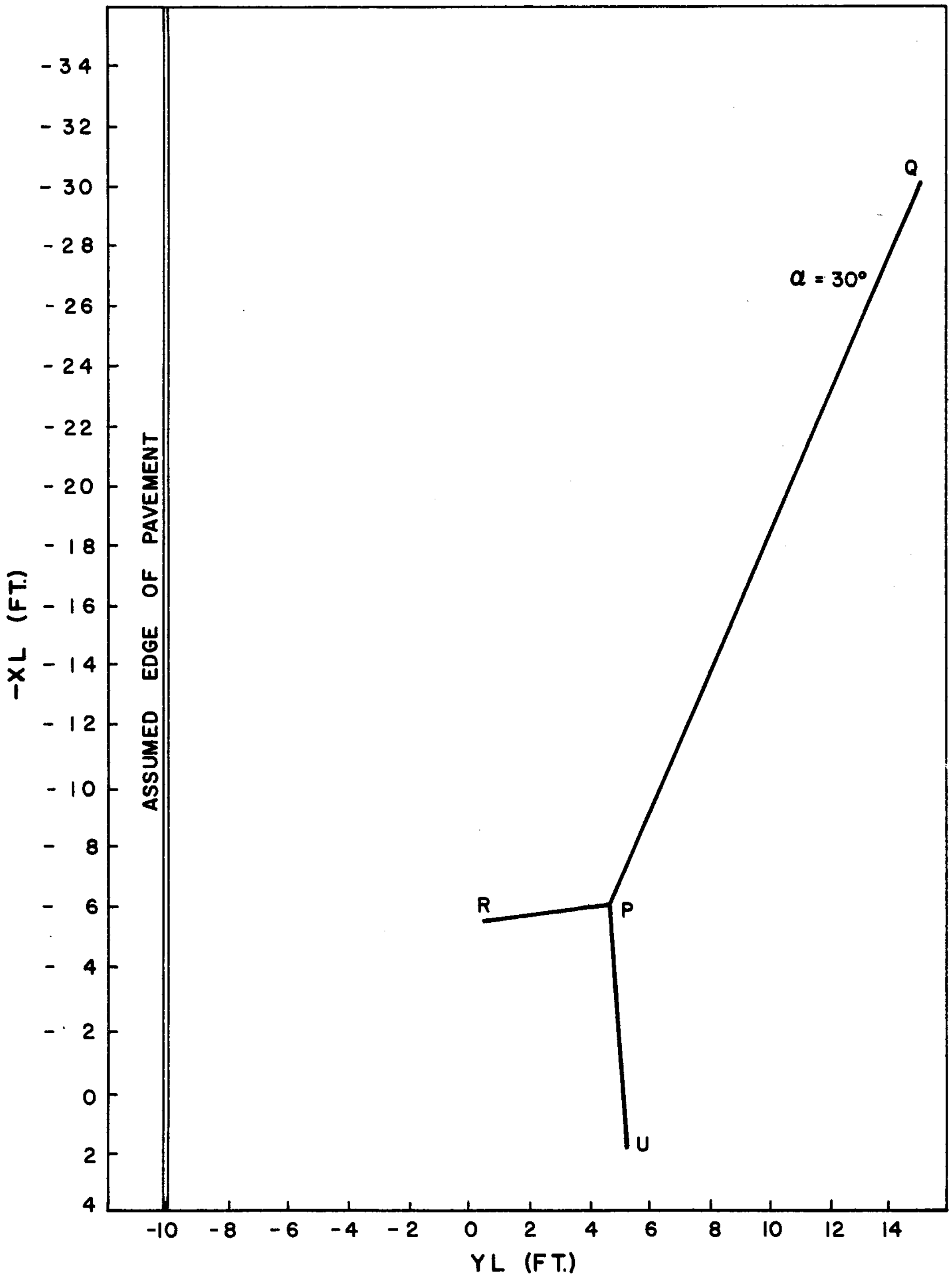


FIGURE 27 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 40 MPH

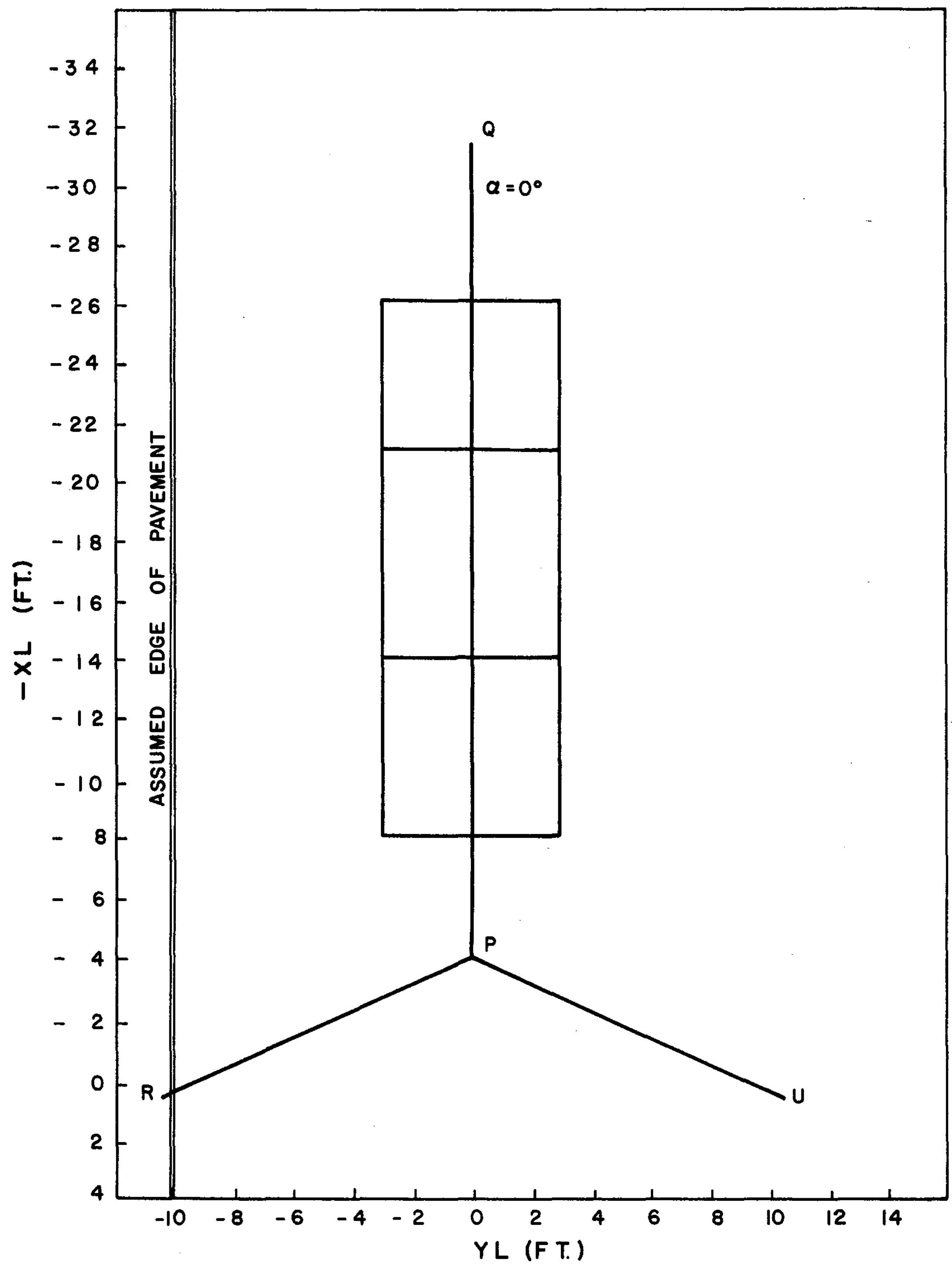


FIGURE 28 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 20 MPH

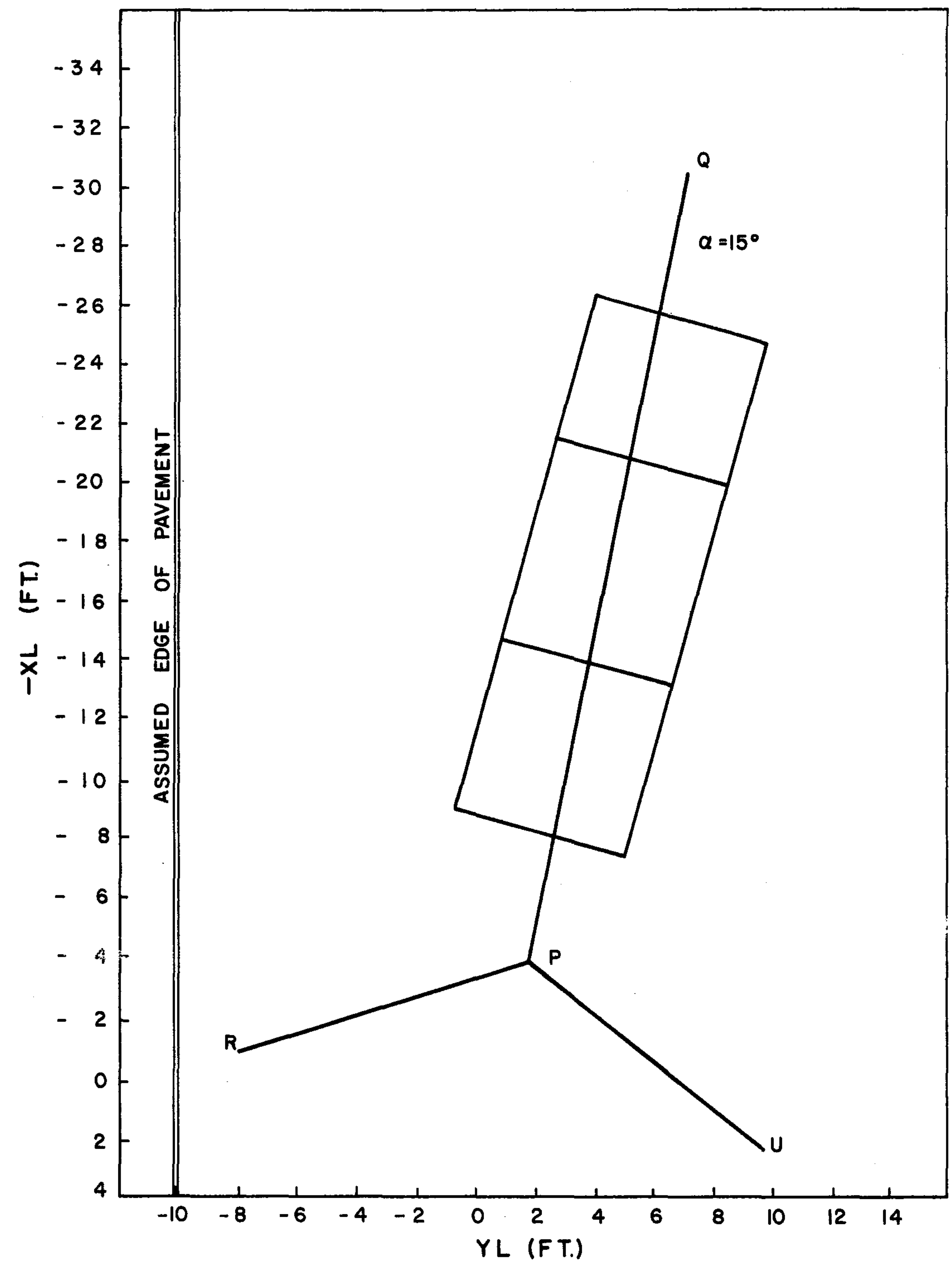


FIGURE 28 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 20 MPH

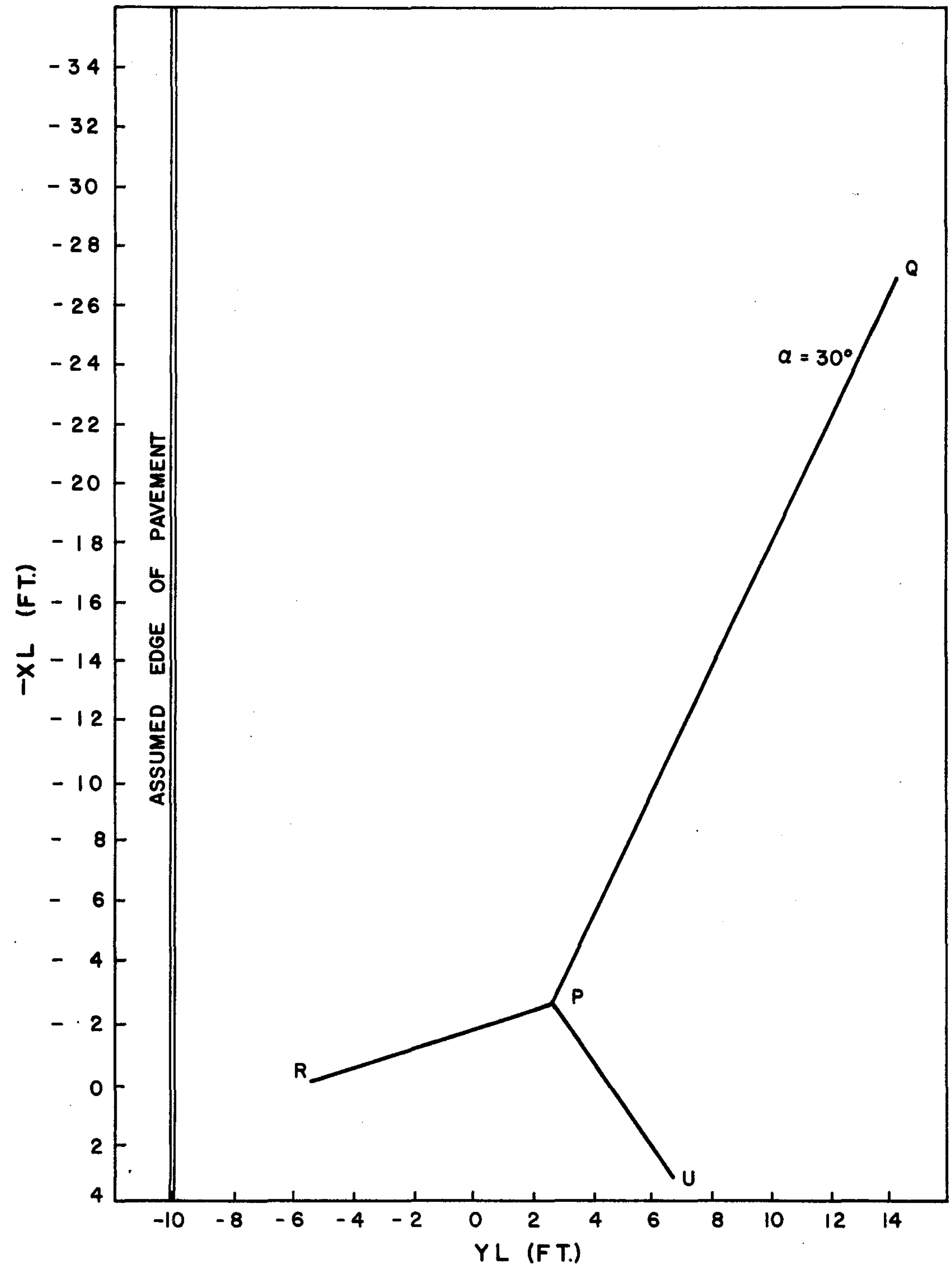


FIGURE 28 THE TERMINAL POSITION OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 20 MPH

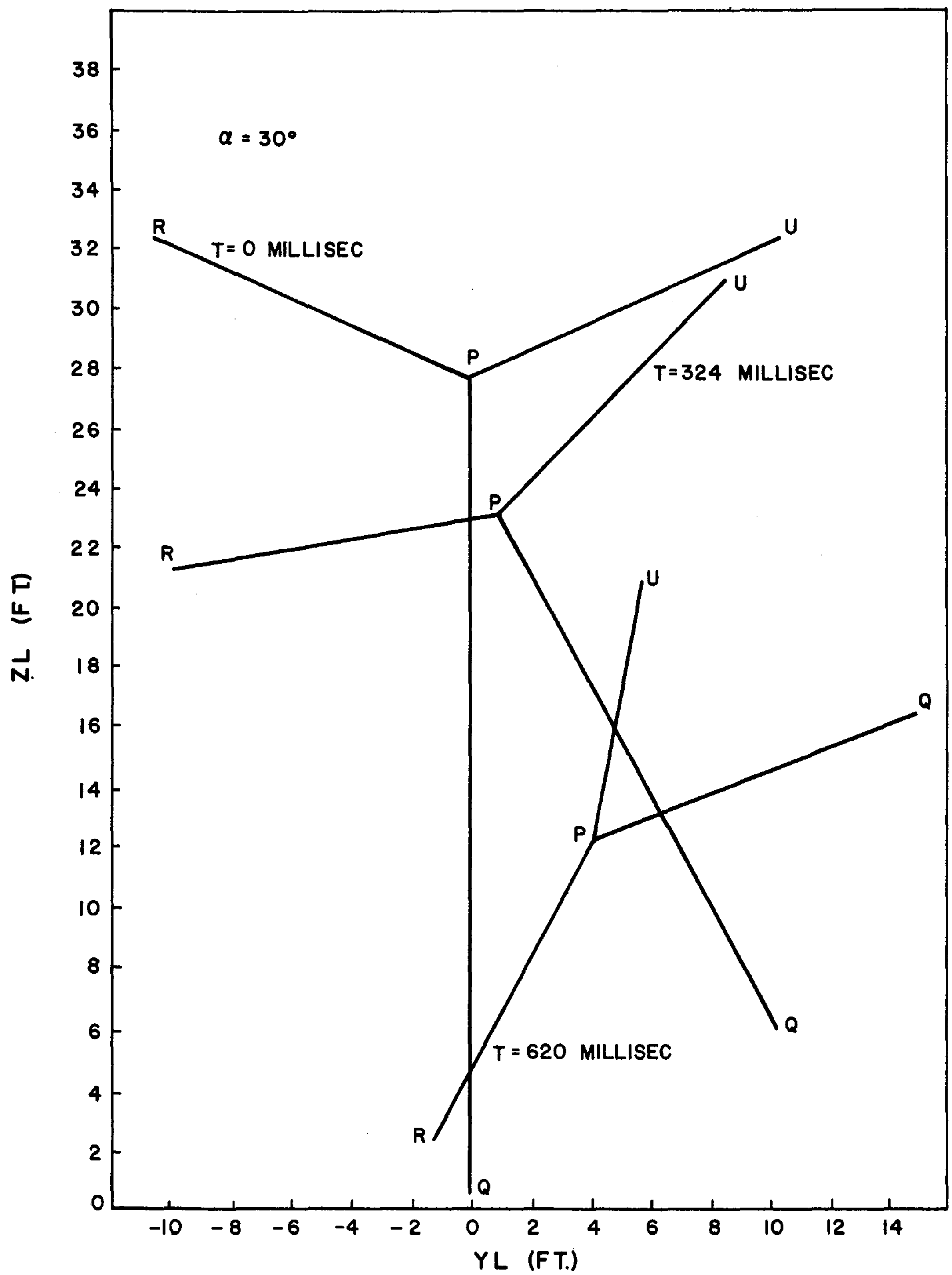


FIGURE 29 THE TRAJECTORY OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 40 MPH

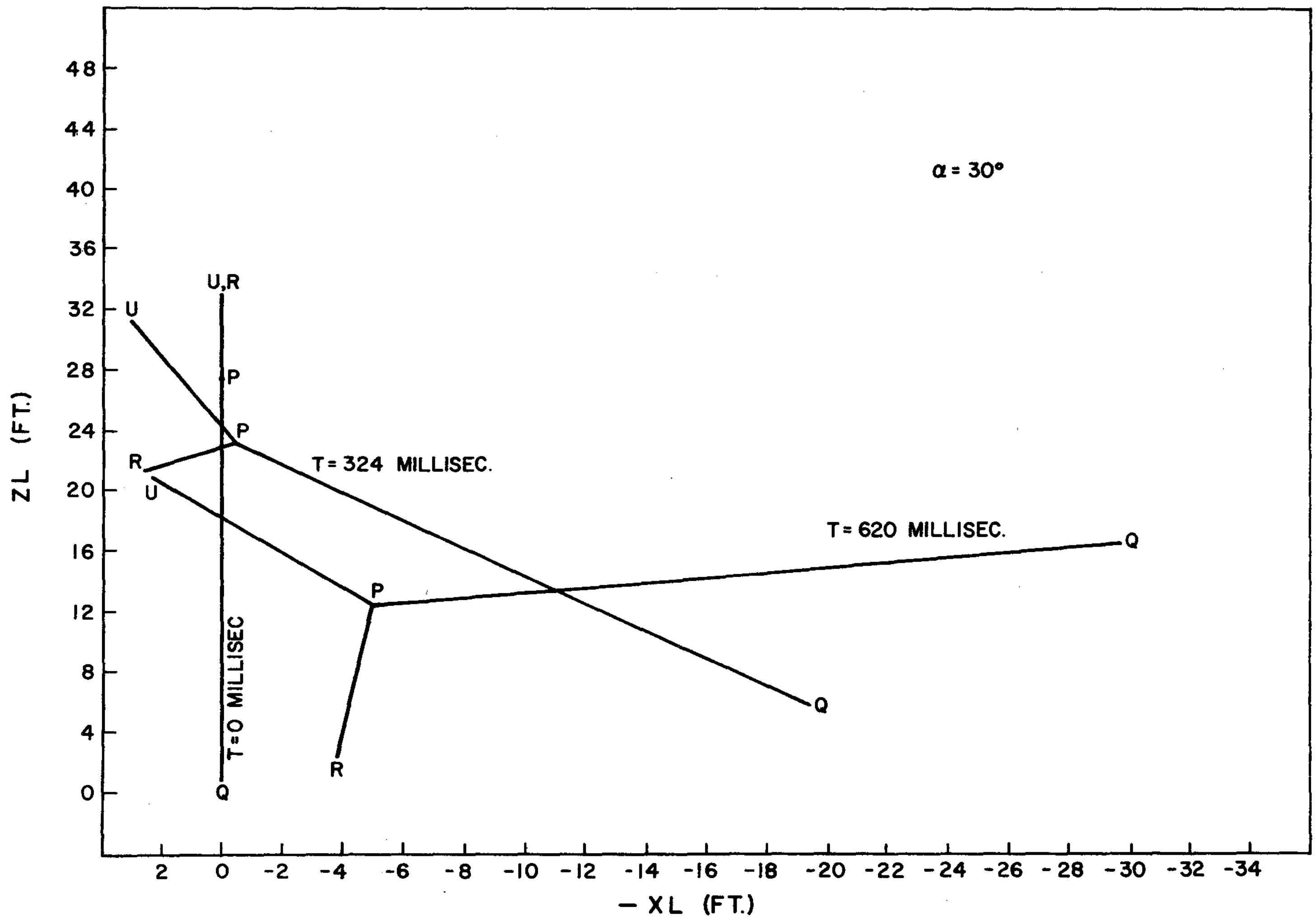


FIGURE 29 THE TRAJECTORY OF THE STEEL SUPPORT POST ASSEMBLY WITH TWIN LUMINAIRES FOR A VEHICLE VELOCITY OF 40 MPH

CHAPTER VII

CONCLUSIONS

The correlation of the mathematical model with data obtained from the full-scale crash test demonstrates the feasibility of the application of the model to the luminaire support pole problem. Since some of the significant parameters are not known precisely, and since the model vehicle is highly idealized, the correlation can only be termed approximate. When more experimental data on the unknown parameters becomes available, a closer correlation can be expected. In any case, the general behavior of the real system can be simulated with this model.

Even though a case exhibiting planer motion was chosen for the correlation, the model has also verified the phenomenological behavior for the non-planer motion case of a vehicle striking a luminaire support. With the model simulating the true physical situation, studies can be conducted to evaluate the hazard potential of existing and proposed designs. Parameter studies can be made of promising designs, and these designs can be investigated to determine the response of the post to a variety of conditions. The effect of such variables as pole weight, pole weight per unit length, length of luminaire arm, weight of luminaire, weight of base assembly, weight and speed of impacting vehicle and angle of attack of impacting vehicle can be investigated; and this information may be utilized to establish basic design criteria and to establish critical limitations on such things as pole weight, height, and base connections.

The model can prove invaluable in reducing the number of full-scale crash tests required to develop and evaluate a particular design. The testing program would reduce to the interpretation of the results obtained from the mathematical model and the testing of the most promising designs obtained from the model study.

CHAPTER VIII

RECOMMENDATIONS FOR FURTHER RESEARCH

Due to the difficulty encountered in the correlation, it is recommended that a testing program be initiated to investigate certain areas.

Static and dynamic tests of the break-away base should be conducted to determine information that would enable one to obtain reasonable values of the frictional resisting forces for various conditions. These values would then be used as input information to the computer program for the mathematical model.

A further investigation into the energy-absorption characteristics of representative vehicles should also be made. Force-deformation characteristics obtained from such a study would be of great value to the researcher. These data would produce a more accurate value of the spring force and bring about a better prediction of the motion of the support assembly.

It is further recommended that the present mathematical model be verified by conducting a fully instrumented crash test of a luminaire support assembly. Such a test would determine whether the rigid body motions assumed in the model are in harmony with the motions of a deformable body.

REFERENCES

1. Samson, C. H., Jr., Rowan, N. J., Olson, R. M. and Tidwell, D. R. "Impact Behavior of Sign Supports," Research Report 68-1, Texas Transportation Institute, College Station, Texas, March 1965, p. 44.

2. Edwards, T. C. "An Analytical Solution of the Impact Behavior of Sign Posts," Unpublished Doctoral Dissertation, Texas A&M University, College Station, Texas, May 1966.

3. Synge, John L. and Griffith, Byron A. Principles of Mechanics, McGraw-Hill Book Company, Inc., New York, N. Y., 1959.

4. Whittaker, E. T. A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, Cambridge University Press, New York, N. Y., 1961.

5. Goldstein, Herbert. Classical Mechanics, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1950.

6. Pars, L. A. A Treatise on Analytical Dynamics, John Wiley and Sons, Inc., New York, N. Y., 1965.

7. Langhaar, Henry L. Energy Methods in Applied Mechanics, John Wiley and Sons, Inc., New York, N. Y., 1962.

8. McCuskey, S. W. Introduction to Advanced Dynamics, Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1959.

9. Timoshenko, S. and Young, D. H. Advanced Dynamics, McGraw-Hill Book Company, Inc., New York, N. Y., 1948.

10. Karman, Theodore V. and Biot, Maurice A. Mathematical Methods in Engineering, McGraw-Hill Book Company, Inc., New York, N. Y., 1940.

11. Salvadori, Mario G. and Baron, Melvin L. Numerical Methods in Engineering, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961

12. Johnson, Walter C. Mathematical and Physical Principles of Engineering Analysis, McGraw-Hill Book Company, Inc., New York, N. Y., 1944.

A P P E N D I X

FORTRAN PROGRAM AND FLOW DIAGRAM

DEFINITION OF FORTRAN STATEMENT NAMES

ALPHA	=	Angle the XX, YY, ZZ coordinate system is rotated about the negative ZZ axis to obtain the XV, YV and ZV system.
A,B,C	=	Principal mass moments of inertia at the mass center of the post.
BL	=	Length of the base of the post.
BOL	=	Length of the bolts holding the post to the base.
CARLEN	=	The absolute value of the displacement of the vehicle while it is in contact with the post.
CFA	=	Coefficient of friction.
CK	=	Spring constant of the vehicle.
DELTA	=	Angle the XX, YY, ZZ coordinate system is rotated about the XX axis to obtain the X, Y, Z system.
DIFF	=	The difference between Poslen and Carlen.
DIRC1X DIRC1Y DIRC1Z	=	Direction cosines between the 1 axis and the X, Y, Z axes respectively.
DIRC2X DIRC2Y DIRC2Z	=	Direction cosines between the 2 axis and the X, Y, Z axes respectively.
DIRC3X DIRC3Y DIRC3Z	=	Direction cosines between the 3 axis and the X, Y, Z axes respectively.
E	=	Print interval.
FFXA, FFYA	=	Frictional forces in the XX and YY directions respectively.
FNA	=	The normal force.
FSA	=	The spring force.

FSXA, FSYA	=	The components of the spring force in the XX and YY directions respectively.
F1A, F2A, F3A, F1B, F2B, F3B	=	Forces in the 1, 2 and 3 directions for the times under consideration and a time increment behind, respectively, while the post and the vehicle are in contact.
F1FB, F2FB, F3FB	=	Forces in the 1, 2 and 3 directions respectively for a time increment behind the time under consideration when the post and the vehicle are no longer in contact.
G	=	Gravity.
H, HI	=	Time increments.
HEIGHT	=	The initial ZZ coordinate of the ground.
HLEN	=	Length of the hood of the vehicle.
HHV, HTRV, HTV	=	Coordinates of the hood, trunk, and top of the vehicle respectively.
I, M	=	Counting indices.
POSLEN	=	The absolute value of the translation of the post at the point of impact while the post and the vehicle are in contact.
Q	=	The value of the time while the post and the vehicle are in contact.
SQUIGA, ETAA, ZETAA, CHIA	=	The rotation parameters while the post and the vehicle are in contact.
SQUIFA, ETAFA, ZETAFA, CHIFA	=	The rotation parameters after the post and the vehicle lose contact.
S1A, S2A, S3A	=	Translations of the post center of mass in the 1, 2 and 3 directions respectively, while the post and the vehicle are in contact.
S1Fa, S2FA, S3FA	=	Translations of the post center of mass in the 1, 2 and 3 directions respectively, after the post and the vehicle have lost contact.
SVA, SVB	=	Displacements of the vehicle for the times under consideration and a time increment behind,

		respectively, while the post and the vehicle are in contact.
SVFA, SVFB	=	Displacements of the vehicle for the times under consideration and a time increment behind, respectively, after the post and the vehicle have lost contact.
T	=	The value of the time after the post and the vehicle have lost contact.
THA, PHIA, PSIA, THB, PHIB, PSIB	=	The Eulerian Angles for the times under consideration and a time increment behind, respectively, while the post and the vehicle are in contact.
THDA, PHIDA, PSIDA, THDB, PHIDB, PSIDB	=	The time-rate of change of the Eulerian angles for the times under consideration and a time increment behind, respectively, while the post and the vehicle are in contact.
THFA, PHIFA, PSIFA, THFB, PHIFB, PSIFA	=	The Eulerian angles for the times under consideration and a time increment behind, respectively, after the post and the vehicle have lost contact.
THDFA, PHIDFA, PSIDFA, THDFB, PHIDFB, PSIDFB	=	The time-rate of change of the Eulerian angles for the times under consideration and a time increment behind, respectively, after the post and the vehicle have lost contact.
TXA, TYA, TZA	=	The torques about the X, Y and Z axes respectively.
T1A, T]A, T3A, T1B, T2B, T3B	=	The torques about the 1, 2 and 3 axes at the times under consideration and a time increment behind, respectively.
V1A, V2A, V3A, V1B, V2B, V3B	=	The linear velocities of the post center of mass in the 1, 2 and 3 directions for the times under consideration and a time increment behind, respectively, while the post and the vehicle are in contact.
VVA, VVB	=	The linear velocities of the vehicle while it is in contact with the post for times under consideration and a time increment behind, respectively.

V1FA, V2FA, V3FA, V1FB, V2FB, V3FB	=	The linear velocities of the post center of mass in the 1, 2 and 3 directions for the times under consideration and a time increment behind, respectively, after the post and the vehicle have lost contact.
VVFA, VVFB	=	The linear velocities of the vehicle after it has lost contact with the post for times under consideration and a time increment behind, respectively.
W1A, W2A, W3A, W2B, W2B, W3B	=	The angular velocities of the post about the 1, 2 and 3 axes for the times under consideration and a time increment behind, respectively, while the post and the vehicle are in contact.
W1FA, W2FA, W3FA, W1FB, W2FB, W3FB	=	The angular velocities of the post about the 1, 2 and 3 axes for the times under consideration and a time increment behind, respectively, after the post and the vehicle have lost contact.
XCMA, YCMA, ZCMA, XCMB, YCMB, ZCMB	=	The translations of the post center of mass in the X, Y and Z directions for the times under consideration and a time increment behind, respectively, while the post and the vehicle are in contact.
XCMFA, YCMFA, ZCMFA, XCMFB, YCMFB, ZCMFB	=	The translations of the post center of mass in the X, Y and Z directions for times under consideration and a time increment behind, respectively, after the post and the vehicle have lost contact.
YYCMA, ZZCMA	=	YCMA and XCMA resolved to the YY and ZZ directions.
XXCMFA, YYCMFA, ZZCMFA	=	XCMFA, YCMFA and ZCMFA resolved to the XX, YY and ZZ directions.
XPA, YPA, ZPA, XQA, YQA, ZQA, XSA, YSA, ZSA	=	The translations of the points P, Q and S in the X, Y and Z directions respectively, while the post and the vehicle are in contact.
YYPA, ZZPA, YYQA, ZZQA, YYSA, ZZSA	=	YPA, ZPA, YQA, ZQA, ZSA resolved to the YY and XX directions.

XVSA, YVSA	=	XSA and YSA resolved to the XV and YV directions.
XPFA, YPFA, ZPFA, XQFA, YQFA, ZQFA, XRFA, YRFA, ZRFA	=	The translations of the points P, Q and R in the X, Y and Z directions respectively, when the post and the vehicle are no longer in contact.
YYPFA, ZZPFA, YYQFA, ZZQFA, YYRFA, ZZRFA	=	YPFA, ZPFA, YQFA, ZQFA, YRFA and ZRFA resolved to the YY and ZZ directions.
XVPFA, YVPFA, XVRFA, YVRFA	=	XPFA, YPFA, XRFA and YRFA resolved to the XV and YV directions.
XPO, YPO, ZPO, XQO, YQO, ZQO, XRO, YRO, ZRO, XSO, YSO, ZSO	=	The initial coordinates of the points P, Q, R and S measured from the post center of mass with respect to the X, Y, Z coordinate system.
XXPO, YYPO, ZZPO, XXQO, YYQO, ZZQO, XXRO, YYRO, ZZRO, XXSO, YYSO, ZZSO	=	The initial coordinates of the points P, Q, R and S measured from the post center of mass with respect to the XX, YY and ZZ coordinate system.

```
$EXECUTE        IBJOB
$IBJOB
$IBFTC MAIN
      READ(5,1)V1B,V2B,V3B,VVB,F1B,F2B,F3B,PSIDB
    1 FORMAT(8F10.5)
      READ(5,2)THDB,PHIDB,W,WV,A,B,C,G
    2 FORMAT(8F10.5)
      READ(5,3)XCMB,YCMB,ZCMB,CK,W1B,W2B,W3B,HEIGHT
    3 FORMAT(8F10.5)
      READ(5,4)PSIB,THB,PHIB,SVB,XSO,YYSO,ZZSO,FNA
    4 FORMAT(8F10.5)
      READ(5,5)XPO,YYPO,ZZPO,XQO,YYQO,ZZQO,SVO,ALPHA
    5 FORMAT(8F10.5)
      READ(5,89)T1B,T2B,T3B,HLEN,TLEN,DELTA,YLFEN,YRFEN
   89 FORMAT(8F10.5)
      READ(5,90)TRLEN,HTRV,FSB,HTV,HHV,XRO,YYRO,ZZRO
   90 FORMAT(8F10.5)
      READ(5,130)XCO,YYCO,ZZCO,XTO,YYTO,ZZTO,CT,FCONST
  130 FORMAT(8F10.5)
      HI=0.0004
      E=0.0
      I=0
      M=0
      H=0.0004
      Q=0.0
      LL=0
      N=0
      K=0
      YPO=YYPO*COS(DELTA)+ZZPO*SIN(DELTA)
      ZPO=-YYPO*SIN(DELTA)+ZZPO*COS(DELTA)
      YQO=YYQO*COS(DELTA)+ZZQO*SIN(DELTA)
      ZQO=-YYQO*SIN(DELTA)+ZZQO*COS(DELTA)
      YSO=YYSO*COS(DELTA)+ZZSO*SIN(DELTA)
      ZSO=-YYSO*SIN(DELTA)+ZZSO*COS(DELTA)
      YTO=YYTO*COS(DELTA)+ZZTO*SIN(DELTA)
```

```
      ZTO=-YYTO*SIN(DELTA)+ZZTO*COS(DELTA)
      YRO=YYRO*COS(DELTA)+ZZRO*SIN(DELTA)
      ZRO=-YYRO*SIN(DELTA)+ZZRO*COS(DELTA)
      YCO=YYCO*COS(DELTA)+ZZCO*SIN(DELTA)
      ZCO=-YYCO*SIN(DELTA)+ZZCO*COS(DELTA)
      XVSO=XSO*COS(ALPHA)-YYSO*SIN(ALPHA)
      XVTO=XTO*COS(ALPHA)-YYTO*SIN(ALPHA)
  125 CONTINUE
      IF(Q-0.0004)123,44,44
  123 CONTINUE
      W1A=0.0
      W2A=0.0545
      W3A=0.0
      FSB=10000.0
      F1B=-10000.0
      THDA=0.0545
      PHIDA=0.0
      PSIDA=0.0
      THA=0.000022
      PHIA=0.0
      PSIA=0.0
      GO TO 124
C
C     ANGULAR VELOCITY CALCULATIONS 1,2,3
C
   44 CONTINUE
      W1A=((H*T1B)/A)+H*(((B-C)/A)*W2B*W3B)+W1B
      W2A=(H/B)*T2B+H*(((C-A)/B)*W1B*W3B)+W2B
      W3A=(H/C)*T3B+H*(((A-B)/C)*W1B*W2B)+W3B
C
C     EULERIAN ANGLE CALCULATIONS
C
      CALL ED(W1A,W2A,W3A,THDB,PHIDB,PSIDB,HI,D,PSIB,THB)
      PSIA=D
      PSIAD=PSIA*57.3
```

```
      PFORK=(W1A*COS(PSIA))-(W2A*SIN(PSIA))
      THA=(H/2.)*(W2A*COS(PSIA)+W1A*SIN(PSIA)+THDB)+THB
      THAD=THA*57.3
      PHIA=(H/2.)*((((SIN(THA))*PFORK)/((COS(THA))*COS(THA)-1.0))+PHIDB)
     X+PHIB
      PHIAD=PHIA*57.3
C
C     ANGULAR VELOCITY CALCULATIONS T,P,P
C
      THDA=(W2A/COS(PSIA))+((SIN(PSIA))/COS(PSIA))*PFORK
      PHIDA=(-1.0/SIN(THA))*PFORK
      PSIDA=W3A-(COS(THA))*PHIDA
  124 CONTINUE
C
C     LINEAR VELOCITY CALCULATIONS
C
      V1A=H*((F1B/W)+V2B*W3B-V3B*W2B)+V1B
      V2A=H*((F2B/W)+V3B*W1B-V1B*W3B)+V2B
      V3A=H*((F3B/W)+V1B*W2B-V2B*W1B)+V3B
      VVA=VVB+(H/WV)*FSB
C
C     PRINCIPAL TRANSLATIONS OF MASS CENTER
C
      S1A=(H/2.)*(V1A+V1B)
      S2A=(H/2.)*(V2A+V2B)
      S3A=(H/2.)*(V3A+V3B)
      SVA=(H/2.)*(VVA+VVB)+SVB
C
C     CALCULATION OF DIRECTION COSINES
C
      DIRC1X=(((-SIN(PHIA))*SIN(PSIA))+((COS(THA))*COS(PHIA))*COS(PSIA))
      DIRC1Y=(((COS(PHIA))*SIN(PSIA))+((COS(THA))*SIN(PHIA))*COS(PSIA))
      DIRC1Z=((-SIN(THA))*COS(PSIA))
      DIRC2X=(((-SIN(PHIA))*COS(PSIA))-((COS(THA))*COS(PHIA))*SIN(PSIA))
      DIRC2Y=(((COS(PHIA))*COS(PSIA))-((COS(THA))*SIN(PHIA))*SIN(PSIA))
```

```
      DIRC2Z=((SIN(THA))*SIN(PSIA))
      DIRC3X=((SIN(THA))*COS(PHIA))
      DIRC3Y=((SIN(THA))*SIN(PHIA))
      DIRC3Z=COS(THA)
C
C     X,Y,Z, TRANSLATIONS OF MASS CENTER
C
      XCMA=S1A*DIRC1X+S2A*DIRC2X+S3A*DIRC3X+XCMB
      YCMA=S1A*DIRC1Y+S2A*DIRC2Y+S3A*DIRC3Y+YCMB
      ZCMA=S1A*DIRC1Z+S2A*DIRC2Z+S3A*DIRC3Z+ZCMB
      YYCMA=YCMA*COS(DELTA)-ZCMA*SIN(DELTA)
      ZZCMA=YCMA*SIN(DELTA)+ZCMA*COS(DELTA)
C
C     CALCULATION OF ROTATION PARAMETERS
C
      SQUIGA=SIN(THA/2.)*SIN((PSIA-PHIA)/2.)
      ETAA=SIN(THA/2.)*COS((PSIA-PHIA)/2.)
      ZETAA=COS(THA/2.)*SIN((PSIA+PHIA)/2.)
      CHIA=COS(THA/2.)*COS((PSIA+PHIA)/2.)
C
C     TRANSLATIONS OF A POINT P
C
      XPA=XCMA+(XPO)*(SQUIGA**2-ETAA**2-ZETAA**2+CHIA**2)+(2.*YPO)*((SQU
     XIGA*ETAA)-(ZETAA*CHIA))+(2.*ZPO)*((SQUIGA*ZETAA)+(ETAA*CHIA))
      YPA=YCMA+(2.*XPO)*((SQUIGA*ETAA)+(ZETAA*CHIA))+(YPO)*(-SQUIGA**2+E
     XTAA**2-ZETAA**2+CHIA**2)+(2.*ZPO)*((ETAA*ZETAA)-(SQUIGA*CHIA))
      ZPA=ZCMA+(2.*XPO)*((SQUIGA*ZETAA)-(ETAA*CHIA))+(2.*YPO)*((ETAA*ZET
     XAA)+(SQUIGA*CHIA))+(ZPO)*(-SQUIGA**2-ETAA**2+ZETAA**2+CHIA**2)
      YYPA=YPA*COS(DELTA)-ZPA*SIN(DELTA)
      ZZPA=YPA*SIN(DELTA)+ZPA*COS(DELTA)
C
C     TRANSLATIONS OF A POINT Q
C
      XQA=XCMA+(XQO)*(SQUIGA**2-ETAA**2-ZETAA**2+CHIA**2)+(2.*YQO)*((SQU
     XIGA*ETAA)-(ZETAA*CHIA))+(2.*ZQO)*((SQUIGA*ZETAA)+(ETAA*CHIA))
```

```
      YQA=YCMA+(2.*XQO)*((SQUIGA*ETAA)+(ZETAA*CHIA))+(YQO)*(-SQUIGA**2+E
     XTAA**2-ZETAA**2+CHIA**2)+(2.*ZQO)*((ETAA*ZETAA)-(SQUIGA*CHIA))
      ZQA=ZCMA+(2.*XQO)*((SQUIGA*ZETAA)-(ETAA*CHIA))+(2.*YQO)*((ETAA*ZET
     XAA)+(SQUIGA*CHIA))+(ZQO)*(-SQUIGA**2-ETAA**2+ZETAA**2+CHIA**2)
      YYQA=YQA*COS(DELTA)-ZQA*SIN(DELTA)
      ZZQA=YQA*SIN(DELTA)+ZQA*COS(DELTA)
C
C     TRANSLATIONS OF A POINT S
C
      XSA=XCMA+(XSO)*(SQUIGA**2-ETAA**2-ZETAA**2+CHIA**2)+(2.*YSO)*((SQU
     XIGA*ETAA)-(ZETAA*CHIA))+(2.*ZSO)*((SQUIGA*ZETAA)+(ETAA*CHIA))
      YSA=YCMA+(2.*XSO)*((SQUIGA*ETAA)+(ZETAA*CHIA))+(YSO)*(-SQUIGA**2+E
     XTAA**2-ZETAA**2+CHIA**2)+(2.*ZSO)*((ETAA*ZETAA)-(SQUIGA*CHIA))
      ZSA=ZCMA+(2.*XSO)*((SQUIGA*ZETAA)-(ETAA*CHIA))+(2.*YSO)*((ETAA*ZET
     XAA)+(SQUIGA*CHIA))+(ZSO)*(-SQUIGA**2-ETAA**2+ZETAA**2+CHIA**2)
      YYSA=YSA*COS(DELTA)-ZSA*SIN(DELTA)
      ZZSA=YSA*SIN(DELTA)+ZSA*COS(DELTA)
      XVSA=XSA*COS(ALPHA)-YYSA*SIN(ALPHA)
      YVSA=XSA*SIN(ALPHA)+YYSA*COS(ALPHA)
C
C     TRANSLATIONS OF A POINT R
C
      XRA=XCMA+(XRO)*(SQUIGA**2-ETAA**2-ZETAA**2+CHIA**2)+(2.*YRO)*((SQU
     XIGA*ETAA)-(ZETAA*CHIA))+(2.*ZRO)*((SQUIGA*ZETAA)+(ETAA*CHIA))
      YRA=YCMA+(2.*XRO)*((SQUIGA*ETAA)+(ZETAA*CHIA))+(YRO)*(-SQUIGA**2+E
     XTAA**2-ZETAA**2+CHIA**2)+(2.*ZRO)*((ETAA*ZETAA)-(SQUIGA*CHIA))
      ZRA=ZCMA+(2.*XRO)*((SQUIGA*ZETAA)-(ETAA*CHIA))+(2.*YRO)*((ETAA*ZET
     XAA)+(SQUIGA*CHIA))+(ZRO)*(-SQUIGA**2-ETAA**2+ZETAA**2+CHIA**2)
      YYRA=YRA*COS(DELTA)-ZRA*SIN(DELTA)
      ZZRA=YRA*SIN(DELTA)+ZRA*COS(DELTA)
      XVRA=XRA*COS(ALPHA)-YYRA*SIN(ALPHA)
      YVRA=XRA*SIN(ALPHA)+YYRA*COS(ALPHA)
C
C     TRANSLATIONS OF A POINT T
C
```

```
      XTA=XCMA+(XTO)*(SQUIGA**2-ETAA**2-ZETAA**2+CHIA**2)+(2.*YTO)*((SQU
     XIGA*ETAA)-(ZETAA*CHIA))+(2.*ZTO)*((SQUIGA*ZETAA)+(ETAA*CHIA))
      YTA=YCMA+(2.*XTO)*((SQUIGA*ETAA)+(ZETAA*CHIA))+(YTO)*(-SQUIGA**2+E
     XTAA**2-ZETAA**2+CHIA**2)+(2.*ZTO)*((ETAA*ZETAA)-(SQUIGA*CHIA))
      ZTA=ZCMA+(2.*XTO)*((SQUIGA*ZETAA)-(ETAA*CHIA))+(2.*YTO)*((ETAA*ZET
     XAA)+(SQUIGA*CHIA))+(ZTO)*(-SQUIGA**2-ETAA**2+ZETAA**2+CHIA**2)
      YYTA=YTA*COS(DELTA)-ZTA*SIN(DELTA)
      ZZTA=YTA*SIN(DELTA)+ZTA*COS(DELTA)
      XVTA=XTA*COS(ALPHA)-YYTA*SIN(ALPHA)
      YVTA=XTA*SIN(ALPHA)+YYTA*COS(ALPHA)
C
C     DIFFERENCE CALCULATIONS
C
      POSLEN=ABS(XVSA-XVSO)
      TDISP=ABS(XVTA-XVTO)
      CARLEN=ABS(SVA-SVO)
      DIFF=POSLEN-CARLEN
      DIFFT=TDISP-CARLEN
C
C     CALCULATION OF FORCES
C
      IF(Q-.0004)83,84,84
   83 FSA=1320.0
      GO TO 85
   84 CONTINUE
      IF(DIFF-0.01)356,356,357
  356 CONTINUE
      FSA=ABS(CK*((SVO-SVA)-(XVSO-XVSA)))
      FSA=0.8*FSA
      GO TO 358
  357 CONTINUE
      FSA=ABS(CT*((SVO-SVA)-(XVTO-XVTA)))
      FSA=0.8*FSA
      ZSA=ZTA
      XSA=XTA
```

```
      YSA=YTA
      GO TO 358
  358 CONTINUE
      IF(LL-20)600,501,501
  501 IF(DIFF-DIFFB)600,502,502
  502 IF(FSA-FCONST)600,504,504
  504 FSA=FCONST
  600 CONTINUE
      IF(XQA+0.083333)101,101,102
  102 IF(XQA+0.083333-0.004)101,101,104
  101 CONTINUE
      FFXA=0.0
      GO TO 105
  104 CONTINUE
   85 CFA=0.25
      FNAC=635.0
      T=300.0
      FFXA=3.1*T
  105 CONTINUE
      IF(XQA+0.4166)107,107,106
  107 FNA=0.0
  106 CONTINUE
      FFYA=0.0
      FSXA=FSA*COS(ALPHA)
      FSYA=FSA*SIN(ALPHA)
      F1A=(FFXA-FSXA)*(DIRC1X)+(FSYA*COS(DELTA)+FNA*SIN(DELTA)-W*G*SIN(D
     XELTA)-FFYA*COS(DELTA))*(DIRC1Y)+(FFYA*SIN(DELTA)+FNA*COS(DELTA)-W*
     XG*COS(DELTA)-FSYA*SIN(DELTA))*(DIRC1Z)
      F2A=(FFXA-FSXA)*(DIRC2X)+(FSYA*COS(DELTA)+FNA*SIN(DELTA)-W*G*SIN(D
     XELTA)-FFYA*COS(DELTA))*(DIRC2Y)+(FFYA*SIN(DELTA)+FNA*COS(DELTA)-W*
     XG*COS(DELTA)-FSYA*SIN(DELTA))*(DIRC2Z)
      F3A=(FFXA-FSXA)*(DIRC3X)+(FSYA*COS(DELTA)+FNA*SIN(DELTA)-W*G*SIN(D
     XELTA)-FFYA*COS(DELTA))*(DIRC3Y)+(FFYA*SIN(DELTA)+FNA*COS(DELTA)-W*
     XG*COS(DELTA)-FSYA*SIN(DELTA))*(DIRC3Z)
C
```

```
C      CALCULATION OF TORQUES
C
       TXA=(FFYA*COS(DELTA))*(ZQA-ZCMA)+(FNA*COS(DELTA))*(YQA-YCMA)-(FNA*
      XSIN(DELTA))*(ZQA-ZCMA)+(FFYA*SIN(DELTA))*(YQA-YCMA)-(FSYA*COS(DELT
      XA))*(ZSA-ZCMA)-(FSYA*SIN(DELTA))*(YSA-YCMA)
       TYA=FFXA*(ZQA-ZCMA)-(FNA*COS(DELTA))*(XQA-XCMA)-FSXA*(ZSA-ZCMA)-(F
      XFYA*SIN(DELTA))*(XQA-XCMA)+(FSYA*SIN(DELTA))*(XSA-XCMA)
       TZA=-FFXA*(YQA-YCMA)-(FFYA*COS(DELTA))*(XQA-XCMA)+FSXA*(YSA-YCMA)+
      X(FSYA*COS(DELTA))*(XSA-XCMA)+(FNA*SIN(DELTA))*(XQA-XCMA)
       T1A=(TXA*DIRC1X)+(TYA*DIRC1Y)+(TZA*DIRC1Z)
       T2A=(TXA*DIRC2X)+(TYA*DIRC2Y)+(TZA*DIRC2Z)
       T3A=(TXA*DIRC3X)+(TYA*DIRC3Y)+(TZA*DIRC3Z)
C
C      TIME CALCULATION
C
       LL=LL+1
       Q=Q+H
       E=E+.0004
       IF(E-0.0016)54,87,87
   87  E=0.0
   88  CONTINUE
       WRITE(6,15)VVA
   15  FORMAT(1H1/5X,20HTHE VALUE OF VVA IS ,F15.5)
       WRITE(6,19)SVA
   19  FORMAT(//,5X,20HTHE VALUE OF SVA IS ,F15.5)
       WRITE(6,120)THAD
  120  FORMAT(//,5X,20HTHE VALUE OF THA IS ,F15.5)
       WRITE(6,121)PHIAD
  121  FORMAT(//,5X,21HTHE VALUE OF PHIA IS ,F15.5)
       WRITE(6,122)PSIAD
  122  FORMAT(//,5X,21HTHE VALUE OF PSIA IS ,F15.5)
       WRITE(6,20)XCMA
   20  FORMAT(//,5X,21HTHE VALUE OF XCMA IS ,F15.5)
       WRITE(6,27)XPA
   27  FORMAT(//,5X,20HTHE VALUE OF XPA IS ,F15.5)
```

```
      WRITE(6,33)XSA
   33 FORMAT(//,5X,20HTHE VALUE OF XSA IS ,F15.5)
      WRITE(6,30)XQA
   30 FORMAT(//,5X,20HTHE VALUE OF XQA IS ,F15.5)
      WRITE(6,206)YYCMA
  206 FORMAT(//,5X,22HTHE VALUE OF YYCMA IS ,F15.5)
      WRITE(6,200)YYPA,XRA
  200 FORMAT(//,5X,21HTHE VALUE OF YYPA IS ,F15.5,30X,20HTHE VALUE OF XR
     1A IS ,F15.5)
      WRITE(6,202)YYSA,YYRA
  202 FORMAT(//,5X,21HTHE VALUE OF YYSA IS ,F15.5,30X,21HTHE VALUE OF YY
     3RA IS ,F15.5)
      WRITE(6,204)YYQA,ZZRA
  204 FORMAT(//,5X,21HTHE VALUE OF YYQA IS ,F15.5,30X,21HTHE VALUE OF ZZ
     2RA IS ,F15.5)
      WRITE(6,207)ZZCMA
  207 FORMAT(//,5X,22HTHE VALUE OF ZZCMA IS ,F15.5)
      WRITE(6,201)ZZPA
  201 FORMAT(//,5X,21HTHE VALUE OF ZZPA IS ,F15.5)
      WRITE(6,203)ZZSA
  203 FORMAT(//,5X,21HTHE VALUE OF ZZSA IS ,F15.5)
      WRITE(6,205)ZZQA
  205 FORMAT(//,5X,21HTHE VALUE OF ZZQA IS ,F15.5)
      WRITE(6,39)FSA
   39 FORMAT(//,5X,20HTHE VALUE OF FSA IS ,F15.5)
      WRITE(6,53)DIFF
   53 FORMAT(//,5X,21HTHE VALUE OF DIFF IS ,F15.5)
      WRITE(6,40)Q
   40 FORMAT(//,5X,21HTHE VALUE OF TIME IS ,F10.6)
   54 CONTINUE
      IF(M-1)55,354,350
   55 CONTINUE
      IF(DIFF)41,354,43
   43 IF(DIFF-0.01)56,56,108
   56 M=1
```

```
      GO TO 88
  350 IF(N-1)351,42,351
  351 CONTINUE
      IF(DIFFT)41,42,352
  352 IF(DIFFT-0.01)353,353,108
  353 N=1
      GO TO 88
  108 Q=Q-H
      H=H/2.
      HI=HI/2.
      GO TO 44
   41 V1B=V1A
      V2B=V2A
      V3B=V3A
      W1B=W1A
      W2B=W2A
      W3B=W3A
      T1B=T1A
      T2B=T2A
      T3B=T3A
      F1B=F1A
      F2B=F2A
      F3B=F3A
      DIFFB=DIFF
      VVB=VVA
      THB=THA
      PHIB=PHIA
      PSIB=PSIA
      THDB=THDA
      PHIDB=PHIDA
      PSIDB=PSIDA
      FSB=FSA
      SVB=SVA
      XCMB=XCMA
      YCMB=YCMA
```

```
      ZCMB=ZCMA
      GO TO 125
C
C     POST LOSES CONTACT WITH VEHICLE
C
  354 CONTINUE
      M=2
      H=0.0004
      HI=0.0004
      WRITE(6,355)
  355 FORMAT(//,5X,43HTHE POST HAS LOST CONTACT WITH FIRST SPRING)
      GO TO 41
   42 CONTINUE
      WRITE(6,100)
  100 FORMAT(//,5X,42HTHE POST HAS LOST CONTACT WITH THE VEHICLE)
      H=0.0004
      HI=0.0004
      E=0.0
      XCMFB=XCMA
      YCMFB=YCMA
      ZCMFB=ZCMA
      THDFB=THDA
      PHIDFB=PHIDA
      PSIDFB=PSIDA
      THFB=THA
      PHIFB=PHIA
      PSIFB=PSIA
      W1FB=W1A
      W2FB=W2A
      W3FB=W3A
      V1FB=V1A
      V2FB=V2A
      V3FB=V3A
      VVFB=VVA
      TF=Q
```

```
      SVFB=SVA
   57 CONTINUE
   59 T=TF+H
C
C     FREE ANGULAR VELOCITY CALCULATIONS
C
      W1FA=H*(((B-C)/A)*W2FB*W3FB)+W1FB
      W2FA=H*(((C-A)/B)*W1FB*W3FB)+W2FB
      W3FA=H*(((A-B)/C)*W1FB*W2FB)+W3FB
C
C     EULERIAN ANGLE CALCULATIONS (FREE)
C
      CALL ED(W1FA,W2FA,W3FA,THDFB,PHIDFB,PSIDFB,HI,D,PSIFB,THFB)
      PSIFA=D
      PSIFAD=PSIFA*57.3
      PFORKF=(W1FA*COS(PSIFA))-(W2FA*SIN(PSIFA))
      THFA=(H/2.)*(W2FA*COS(PSIFA)+W1FA*SIN(PSIFA)+THDFB)+THFB
      THFAD=THFA*57.3
      PHIFA=(H/2.)*((((SIN(THFA))*PFORKF)/((COS(THFA))*COS(THFA)-1.0))+P
     XHIDFB)+PHIFB
      PHIFAD=PHIFA*57.3
C
C     ANGULAR VELOCITY CALCULATIONS T,P,P (FREE)
C
      THDFA=(W2FA/COS(PSIFA))+((SIN(PSIFA))/COS(PSIFA))*PFORKF
      PHIDFA=(-1.0/SIN(THFA))*PFORKF
      PSIDFA=W3FA-(COS(THFA))*PHIDFA
C
C     FORCE CALCULATIONS (FREE)
C
      F1FB=W*G*COS(DELTA)*SIN(THFB)*COS(PSIFB)-W*G*SIN(DELTA)*(COS(PHIFB
     X)*SIN(PSIFB)+COS(THFB)*SIN(PHIFB)*COS(PSIFB))
      F2FB=-W*G*COS(DELTA)*SIN(THFB)*SIN(PSIFB)-W*G*SIN(DELTA)*(COS(PHIF
     XB)*COS(PSIFB)-COS(THFB)*SIN(PHIFB)*SIN(PSIFB))
      F3FB=-W*G*COS(DELTA)*COS(THFB)-W*G*SIN(DELTA)*SIN(THFB)*SIN(PHIFB)
```

```
C
C     LINEAR VELOCITY CALCUALTIONS (FREE)
C
      V1FA=H*((F1FB/W)+V2FB*W3FB-V3FB*W2FB)+V1FB
      V2FA=H*((F2FB/W)+V3FB*W1FB-V1FB*W3FB)+V2FB
      V3FA=H*((F3FB/W)+V1FB*W2FB-V2FB*W1FB)+V3FB
      VVFA=VVFB
C
C     PRINCIPAL TRANSLATIONS OF MASS CENTER (FREE)
C
      S1FA=(H/2.)*(V1FA+V1FB)
      S2FA=(H/2.)*(V2FA+V2FB)
      S3FA=(H/2.)*(V3FA+V3FB)
      SVFA=(H/2.)*(VVFA+VVFB)+SVFB
      XBUMP=SVFA-HLEN
      XENTOP=SVFA+TLEN
      XTAIL=XENTOP+TRLEN
C
C     X,Y,Z TRANSLATIONS OF MASS CENTER (FREE)
C
      XCMFA=(S1FA)*((-SIN(PHIFA))*(SIN(PSIFA))+((COS(THFA))*(COS(PHIFA))
     X*COS(PSIFA)))+(S2FA)*((-SIN(PHIFA))*(COS(PSIFA))-((COS(THFA))*(COS
     X(PHIFA))*SIN(PSIFA)))+(S3FA)*((SIN(THFA))*COS(PHIFA))+XCMFB
      YCMFA=(S1FA)*((COS(PHIFA))*(SIN(PSIFA))+((COS(THFA))*(SIN(PHIFA))*
     XCOS(PSIFA)))+(S2FA)*((COS(PHIFA))*(COS(PSIFA))-((COS(THFA))*(SIN(P
     XHIFA))*SIN(PSIFA)))+(S3FA)*((SIN(THFA))*SIN(PHIFA))+YCMFB
      ZCMFA=(S1FA)*(-SIN(THFA))*(COS(PSIFA))+(S2FA)*(SIN(THFA))*(SIN(PSI
     XFA))+(S3FA)*(COS(THFA))+ZCMFB
      YYCMFA=YCMFA*COS(DELTA)-ZCMFA*SIN(DELTA)
      ZZCMFA=YCMFA*SIN(DELTA)+ZCMFA*COS(DELTA)
C
C     CALCULATION OF ROTATION PARAMETERS (FREE)
C
      SQUIFA=SIN(THFA/2.)*SIN((PSIFA-PHIFA)/2.)
      ETAFA=SIN(THFA/2.)*COS((PSIFA-PHIFA)/2.)
      ZETAFA=COS(THFA/2.)*SIN((PSIFA+PHIFA)/2.)
```

```
      CHIFA=COS(THFA/2.)*COS((PSIFA+PHIFA)/2.)
C
C     TRANSLATIONS OF A POINT P (FREE)
C
      XPFA=XCMFA+(XPO)*(SQUIFA**2-ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*YPO)*
     X((SQUIFA*ETAFA)-(ZETAFA*CHIFA))+(2.*ZPO)*((SQUIFA*ZETAFA)+(ETAFA*C
     XHIFA))
      YPFA=YCMFA+(2.*XPO)*((SQUIFA*ETAFA)+(ZETAFA*CHIFA))+(YPO)*(-SQUIFA
     X**2+ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*ZPO)*((ETAFA*ZETAFA)-(SQUIFA*
     XCHIFA))
      ZPFA=ZCMFA+(2.*XPO)*((SQUIFA*ZETAFA)-(ETAFA*CHIFA))+(2.*YPO)*((ETA
     XFA*ZETAFA)+(SQUIFA*CHIFA))+(ZPO)*(-SQUIFA**2-ETAFA**2+ZETAFA**2+CH
     XIFA**2)
      YYPFA=YPFA*COS(DELTA)-ZPFA*SIN(DELTA)
      ZZPFA=YPFA*SIN(DELTA)+ZPFA*COS(DELTA)
      XVPFA=XPFA*COS(ALPHA)-YYPFA*SIN(ALPHA)
      YVPFA=XPFA*SIN(ALPHA)+YYPFA*COS(ALPHA)
C
C     TRANSLATIONS OF A POINT Q (FREE)
C
      XQFA=XCMFA+(XQO)*(SQUIFA**2-ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*YQO)*
     X((SQUIFA*ETAFA)-(ZETAFA*CHIFA))+(2.*ZQO)*((SQUIFA*ZETAFA)+(ETAFA*C
     XHIFA))
      YQFA=YCMFA+(2.*XQO)*((SQUIFA*ETAFA)+(ZETAFA*CHIFA))+(YQO)*(-SQUIFA
     X**2+ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*ZQO)*((ETAFA*ZETAFA)-(SQUIFA*
     XCHIFA))
      ZQFA=ZCMFA+(2.*XQO)*((SQUIFA*ZETAFA)-(ETAFA*CHIFA))+(2.*YQO)*((ETA
     XFA*ZETAFA)+(SQUIFA*CHIFA))+(ZQO)*(-SQUIFA**2-ETAFA**2+ZETAFA**2+CH
     XIFA**2)
      YYQFA=YQFA*COS(DELTA)-ZQFA*SIN(DELTA)
      ZZQFA=YQFA*SIN(DELTA)+ZQFA*COS(DELTA)
C
C     TRANSLATIONS OF A POINT R (FREE)
C
      XRFA=XCMFA+(XRO)*(SQUIFA**2-ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*YRO)*
```

```
     X((SQUIFA*ETAFA)-(ZETAFA*CHIFA))+(2.*ZRO)*((SQUIFA*ZETAFA)+(ETAFA*C
     XHIFA))
      YRFA=YCMFA+(2.*XRO)*((SQUIFA*ETAFA)+(ZETAFA*CHIFA))+(YRO)*(-SQUIFA
     X**2+ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*ZRO)*((ETAFA*ZETAFA)-(SQUIFA*
     XCHIFA))
      ZRFA=ZCMFA+(2.*XRO)*((SQUIFA*ZETAFA)-(ETAFA*CHIFA))+(2.*YRO)*((ETA
     XFA*ZETAFA)+(SQUIFA*CHIFA))+(ZRO)*(-SQUIFA**2-ETAFA**2+ZETAFA**2+CH
     XIFA**2)
      YYRFA=YRFA*COS(DELTA)-ZRFA*SIN(DELTA)
      ZZRFA=YRFA*SIN(DELTA)+ZRFA*COS(DELTA)
      XVRFA=XRFA*COS(ALPHA)-YYRFA*SIN(ALPHA)
      YVRFA=XRFA*SIN(ALPHA)+YYRFA*COS(ALPHA)
C
C     TRANSLATIONS OF A POINT C (FREE)
C
      XCFA=XCMFA+(XCO)*(SQUIFA**2-ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*YCO)*
     X((SQUIFA*ETAFA)-(ZETAFA*CHIFA))+(2.*ZCO)*((SQUIFA*ZETAFA)+(ETAFA*C
     XHIFA))
      YCFA=YCMFA+(2.*XCO)*((SQUIFA*ETAFA)+(ZETAFA*CHIFA))+(YCO)*(-SQUIFA
     X**2+ETAFA**2-ZETAFA**2+CHIFA**2)+(2.*ZCO)*((ETAFA*ZETAFA)-(SQUIFA*
     XCHIFA))
      ZCFA=ZCMFA+(2.*XCO)*((SQUIFA*ZETAFA)-(ETAFA*CHIFA))+(2.*YCO)*((ETA
     XFA*ZETAFA)+(SQUIFA*CHIFA))+(ZCO)*(-SQUIFA**2-ETAFA**2+ZETAFA**2+CH
     XIFA**2)
      YYCFA=YCFA*COS(DELTA)-ZCFA*SIN(DELTA)
      ZZCFA=YCFA*SIN(DELTA)+ZCFA*COS(DELTA)
      XVCFA=XCFA*COS(ALPHA)-YYCFA*SIN(ALPHA)
      YVCFA=XCFA*SIN(ALPHA)+YYCFA*COS(ALPHA)
      E=E+0.0002
      IF(E-0.008)77,80,80
   80 CONTINUE
      E=0.0
   98 CONTINUE
      WRITE(6,60)THFAD
   60 FORMAT(1H1/5X,21HTHE VALUE OF THFA IS ,F10.5)
```

```
      WRITE(6,61)PHIFAD
   61 FORMAT(//,5X,22HTHE VALUE OF PHIFA IS ,F10.5)
      WRITE(6,62)PSIFAD
   62 FORMAT(//,5X,22HTHE VALUE OF PSIFA IS ,F10.5)
      WRITE(6,63)XCMFA
   63 FORMAT(//,5X,22HTHE VALUE OF XCMFA IS ,F10.5)
      WRITE(6,64)XPFA
   64 FORMAT(//,5X,21HTHE VALUE OF XPFA IS ,F10.5)
      WRITE(6,66)XQFA
   66 FORMAT(//,5X,21HTHE VALUE OF XQFA IS ,F10.5)
      WRITE(6,113)XRFA
  113 FORMAT(//,5X,21HTHE VALUE OF XRFA IS ,F10.5)
      WRITE(6,131)XCFA
  131 FORMAT(//,5X,21HTHE VALUE OF XCFA IS ,F15.5)
      WRITE(6,208)YYCMFA
  208 FORMAT(//,5X,23HTHE VALUE OF YYCMFA IS ,F15.5)
      WRITE(6,210)YYPFA
  210 FORMAT(//,5X,22HTHE VALUE OF YYPFA IS ,F15.5)
      WRITE(6,212)YYQFA
  212 FORMAT(//,5X,22HTHE VALUE OF YYQFA IS ,F15.5)
      WRITE(6,216)YYRFA
  216 FORMAT(//,5X,22HTHE VALUE OF YYRFA IS ,F15.5)
      WRITE(6,132)YYCFA
  132 FORMAT(//,5X,22HTHE VALUE OF YYCFA IS ,F15.5)
      WRITE(6,209)ZZCMFA
  209 FORMAT(//,5X,23HTHE VALUE OF ZZCMFA IS ,F15.5)
      WRITE(6,211)ZZPFA
  211 FORMAT(//,5X,22HTHE VALUE OF ZZPFA IS ,F15.5)
      WRITE(6,213)ZZQFA
  213 FORMAT(//,5X,22HTHE VALUE OF ZZQFA IS ,F15.5)
      WRITE(6,217)ZZRFA
  217 FORMAT(//,5X,22HTHE VALUE OF ZZRFA IS ,F15.5)
      WRITE(6,133)ZZCFA
  133 FORMAT(//,5X,22HTHE VALUE OF ZZCFA IS ,F15.5)
      WRITE(6,75)SVFA
```

```
   75 FORMAT(//,5X,21HTHE VALUE OF SVFA IS ,F10.5)
      WRITE(6,76)T
   76 FORMAT(//,5X,21HTHE VALUE OF TIME IS ,F10.5)
   77 CONTINUE
      IF(I-1)81,82,81
   81 CONTINUE
      IF(ZZPFA+HEIGHT)91,93,91
   91 IF(ABS(ZZPFA+HEIGHT)-0.1)93,93,94
   94 IF(ZZQFA+HEIGHT)95,96,95
   95 IF(ABS(ZZQFA+HEIGHT)-0.01)96,96,111
  111 CONTINUE
      IF(ZZRFA+HEIGHT)103,109,103
  103 IF(ABS(ZZRFA+HEIGHT)-0.1)109,109,230
  230 CONTINUE
      IF(ZZCFA+HEIGHT)141,140,141
  141 IF(ABS(ZZCFA+HEIGHT)-0.1)140,140,500
  500 CONTINUE
C
C     CHECK ON THE POINT P
C
      IF(YVPFA-YLFEN)112,250,249
  249 IF(YVPFA-YRFEN)250,250,112
  250 IF(XVPFA-XBUMP)112,231,232
  232 IF(XVPFA-SVFA)231,233,234
  231 IF(ZZPFA-HHV)235,236,235
  235 IF(ABS(ZZPFA-HHV)-0.1)236,236,112
  233 IF(ZZPFA-HTV)238,240,241
  241 IF(ABS(ZZPFA-HTV)-0.1)240,240,112
  234 IF(XVPFA-SVFA-0.1)233,233,243
  243 IF(XVPFA-XENTOP)233,240,244
  244 IF(XVPFA-XTAIL)245,245,112
  245 IF(ZZPFA-HTRV)246,247,246
  246 IF(ABS(ZZPFA-HTRV)-0.1)247,247,112
  112 CONTINUE
C
```

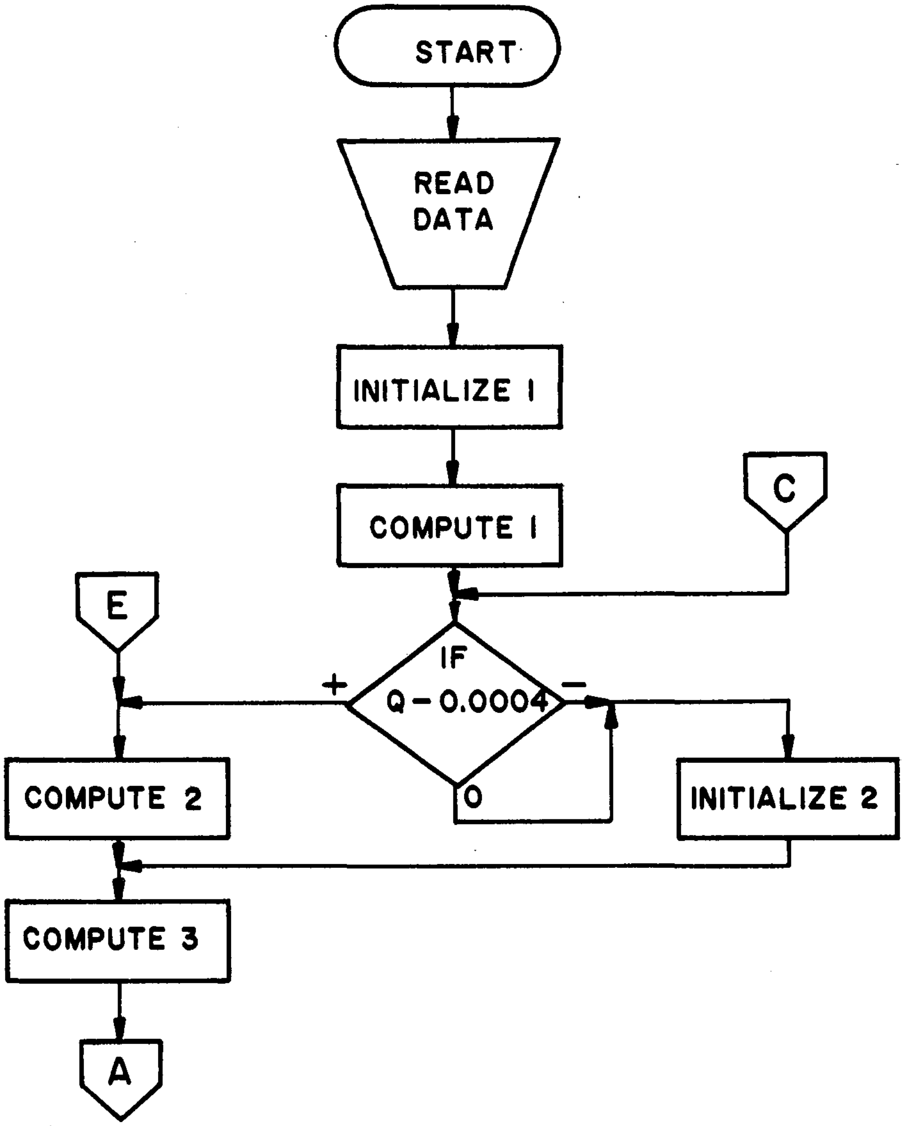
```
C     CHECK ON THE POINT R
C
      IF(YVRFA-YLFEN)300,251,252
  252 IF(YVRFA-YRFEN)251,251,300
  251 IF(XVRFA-XBUMP)300,253,254
  254 IF(XVRFA-SVFA)253,255,256
  253 IF(ZZRFA-HHV)257,236,257
  257 IF(ABS(ZZRFA-HHV)-0.1)236,236,300
  255 IF(ZZRFA-HTV)238,240,258
  258 IF(ABS(ZZRFA-HTV)-0.1)240,240,300
  256 IF(XVRFA-SVFA-0.1)255,255,259
  259 IF(XVRFA-XENTOP)255,240,260
  260 IF(XVRFA-XTAIL)261,261,300
  261 IF(ZZRFA-HTRV)262,247,262
  262 IF(ABS(ZZRFA-HTRV)-0.1)247,247,300
  300 CONTINUE
C
C     CHECK ON THE POINT C
C
      IF(YVCFA-YLFEN)400,451,452
  452 IF(YVCFA-YRFEN)451,451,400
  451 IF(XVCFA-XBUMP)400,453,454
  454 IF(XVCFA-SVFA)453,455,456
  453 IF(ZZCFA-HHV)457,236,457
  457 IF(ABS(ZZCFA-HHV)-0.1)236,236,400
  455 IF(ZZCFA-HTV)238,240,458
  458 IF(ABS(ZZCFA-HTV)-0.1)240,240,400
  456 IF(XVCFA-SVFA-0.1)455,455,459
  459 IF(XVCFA-XENTOP)455,240,460
  460 IF(XVCFA-XTAIL)461,461,400
  461 IF(ZZCFA-HTRV)462,247,462
  462 IF(ABS(ZZCFA-HTRV)-0.1)247,247,400
  140 I=1
      WRITE(6,134)
  134 FORMAT(//,5X,40HTHE POST HAS HIT THE GROUND WITH POINT C)
```

```
      GO TO 98
   96 I=1
      WRITE(6,97)
   97 FORMAT(//,5X,40HTHE POST HAS HIT THE GROUND WITH POINT Q)
      GO TO 98
   93 I=1
      WRITE(6,99)
   99 FORMAT(//,5X,40HTHE POST HAS HIT THE GROUND WITH POINT P)
      GO TO 98
  109 I=1
      WRITE(6,110)
  110 FORMAT(//,5X,40HTHE POST HAS HIT THE GROUND WITH POINT R)
      GO TO 98
  236 I=1
      WRITE(6,237)
  237 FORMAT(//,5X,40HTHE POST HAS HIT THE HOOD OF THE VEHICLE)
      GO TO 98
  238 I=1
      WRITE(6,239)
  239 FORMAT(//,5X,46HTHE POST HAS HIT THE WINDSHIELD OF THE VEHICLE)
      GO TO 98
  240 I=1
      WRITE(6,242)
  242 FORMAT(//,5X,39HTHE POST HAS HIT THE TOP OF THE VEHICLE)
      GO TO 98
  247 I=1
      WRITE(6,248)
  248 FORMAT(//,5X,41HTHE POST HAS HIT THE TRUNK OF THE VEHICLE)
      GO TO 98
  400 CONTINUE
      TF=T
      THFB=THFA
      PHIFB=PHIFA
      PSIFB=PSIFA
      VVFB=VVFA
```

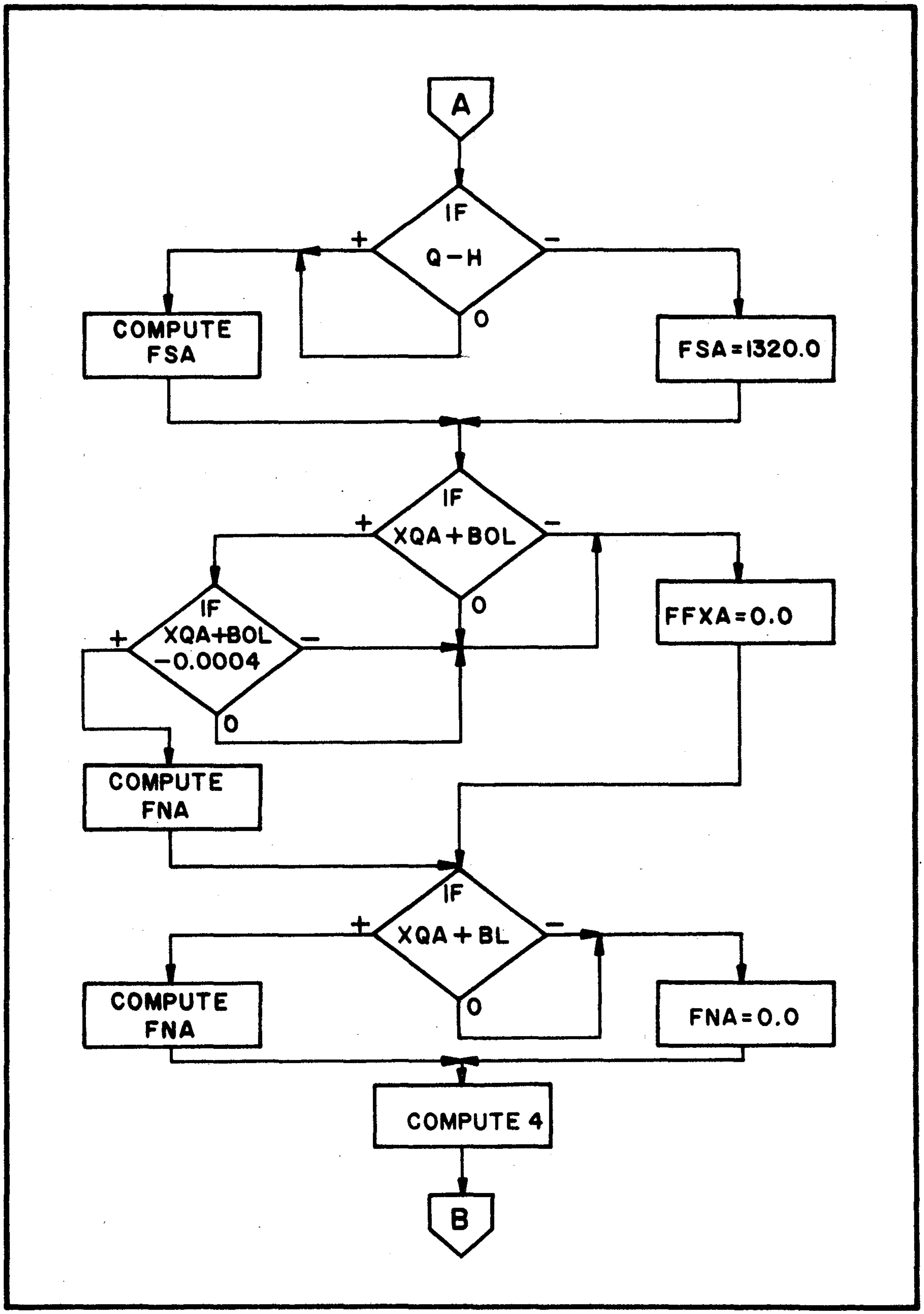
```
      V1FB=V1FA
      V2FB=V2FA
      V3FB=V3FA
      W1FB=W1FA
      W2FB=W2FA
      W3FB=W3FA
      THDFB=THDFA
      PHIDFB=PHIDFA
      PSIDFB=PSIDFA
      XCMFB=XCMFA
      YCMFB=YCMFA
      ZCMFB=ZCMFA
      SVFB=SVFA
      GO TO 57
   82 CONTINUE
      STOP
      END
$IBFTC SR1
      SUBROUTINE ED(W1A,W2A,W3A,THDB,PHIDB,PSIDB,HI,D,PSIB,THB)
      DF(D)=D-((HI/2.)*(W3A+(COS((HI/2.)*(W2A*COS(D)+W1A*SIN(D)+THDB)+TH
     XB)/SIN((HI/2.)*(W2A*COS(D)+W1A*SIN(D)+THDB)+THB))*(W1A*COS(D)-W2A*
     XSIN(D))+PSIDB)+PSIB)
      IF(L-25)21,21,22
   21 D=.15
      H=-0.01
      IF(DF(D))1,20,10
   22 D=PSIB
      H=-0.01
      IF(DF(D))1,20,10
C
C     ROOT GOES NEGATIVE TO POSITIVE
C
    1 DH=D+H
      IF(DF(DH))2,20,15
    2 D=DH
```

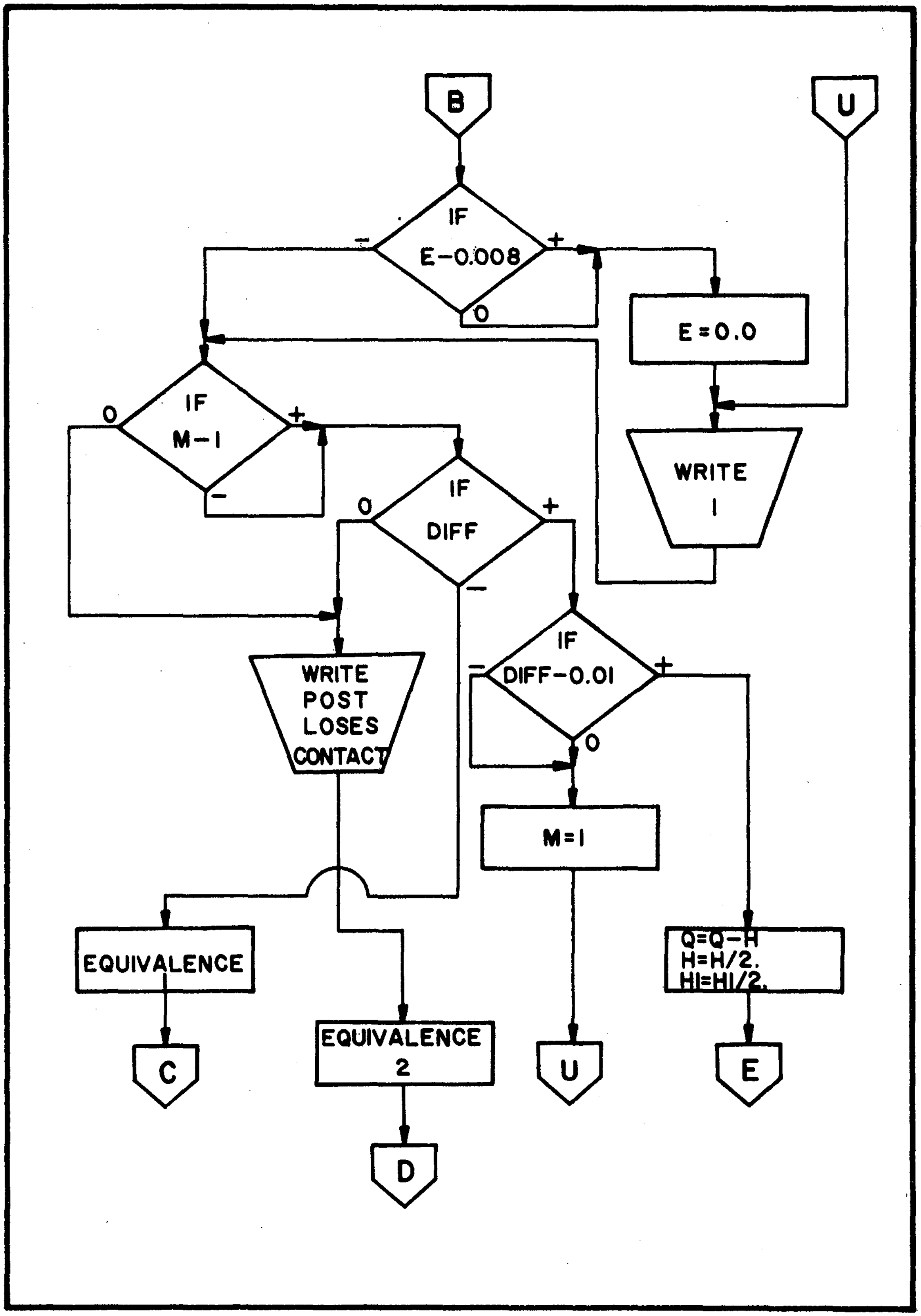
```
      GO TO 1
   15 DH=D+H/2.
      IF(ABS(DF(DH))-.0001)20,20,4
    4 IF(DF(DH))5,20,3
    3 H=H/2.
      GO TO 15
    5 D=DH
      H=H/2.
      GO TO 15
C
C     ROOT GOES POSITIVE TO NEGATIVE
C
   10 DH=D+H
      IF(DF(DH))16,20,11
   11 D=DH
      GO TO 10
   16 DH=D+H/2.
      IF(ABS(DF(DH))-.0001)20,20,13
   13 IF(DF(DH))12,20,14
   12 H=H/2.
      GO TO 16
   14 D=DH
      H=H/2.
      GO TO 16
   20 D=DH
      L=L+1
      RETURN
      END
$DATA
0.0       0.0       0.0       -58.66    0.0       0.0       0.0       0.0
0.0       0.0       15.1      100.0     3351.41   3282.6    69.68     32.174
0.0       0.0       0.0       33000.0   0.0       0.0       0.0       20.27
0.0       0.0       0.0       5.0       0.0       1.53      -18.77    485.07
0.0       1.53      16.938    0.0       1.53      -19.77    5.0       0.0
0.0       0.0       0.0       5.0       7.0       0.1463    -1.47     4.53
```

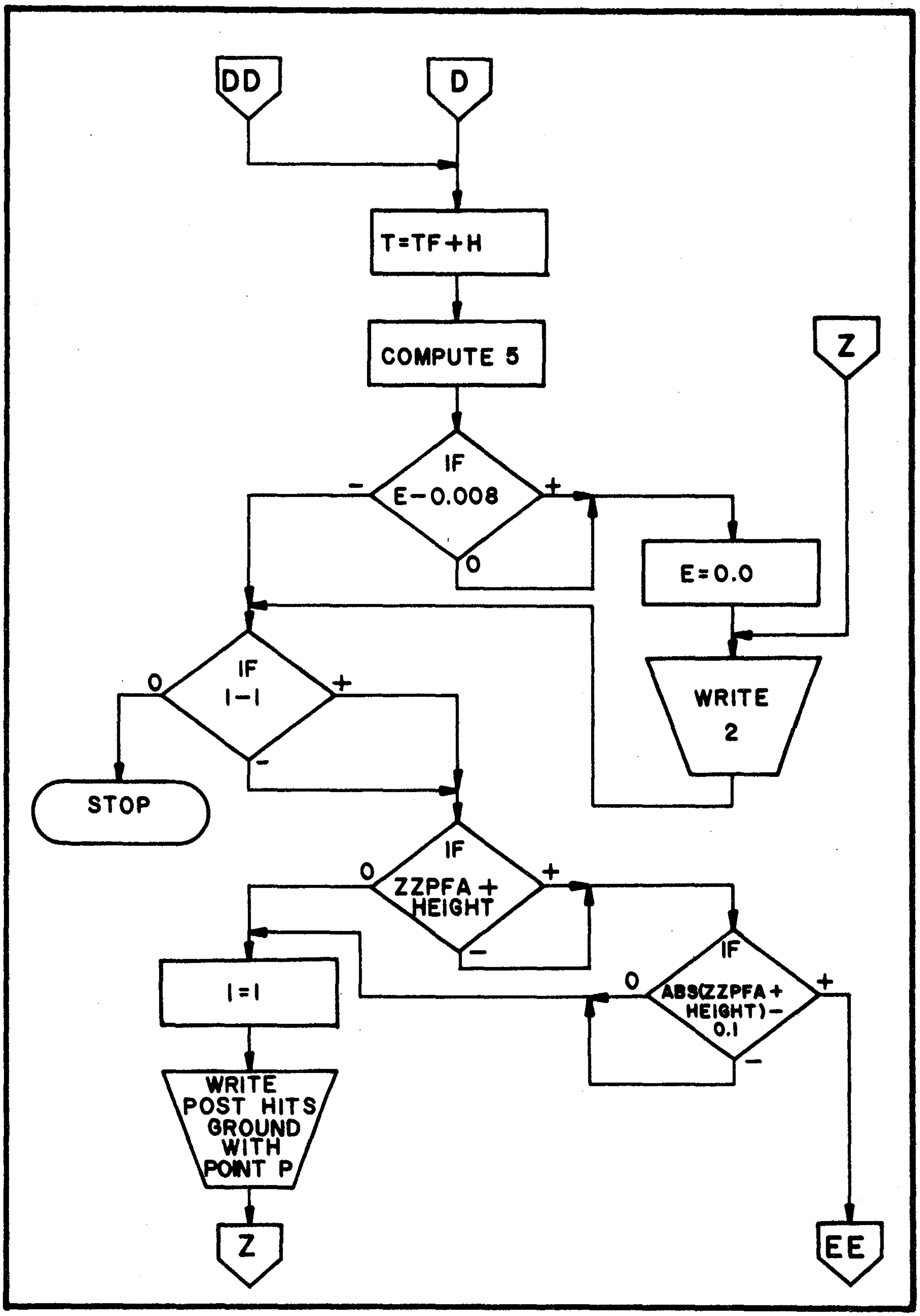
6.0	-16.77	0.0	-15.27	-16.27	0.0	-8.47	21.938
0.0	1.53	-19.77	0.0	1.53	-17.5	20000.0	1200.0

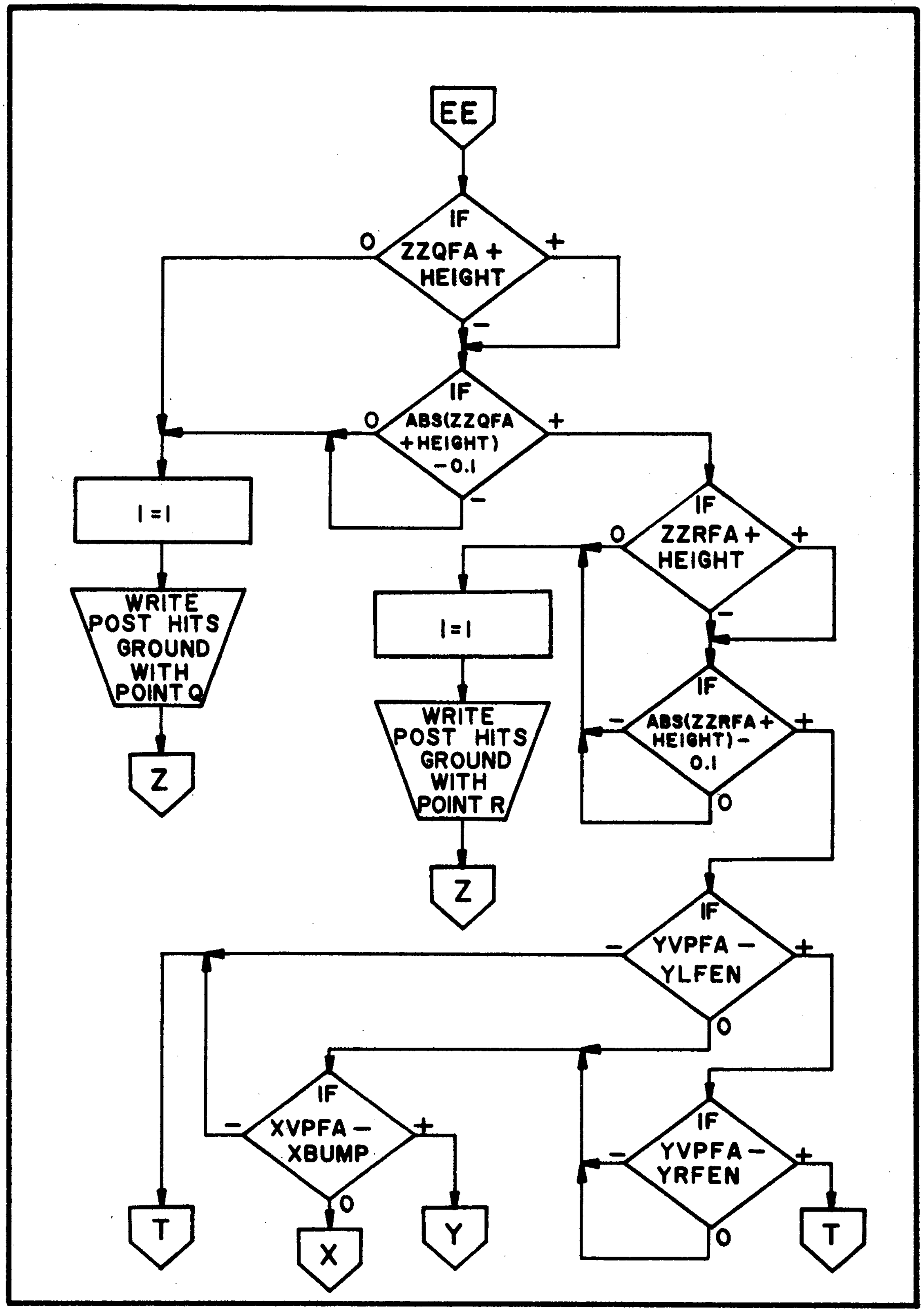


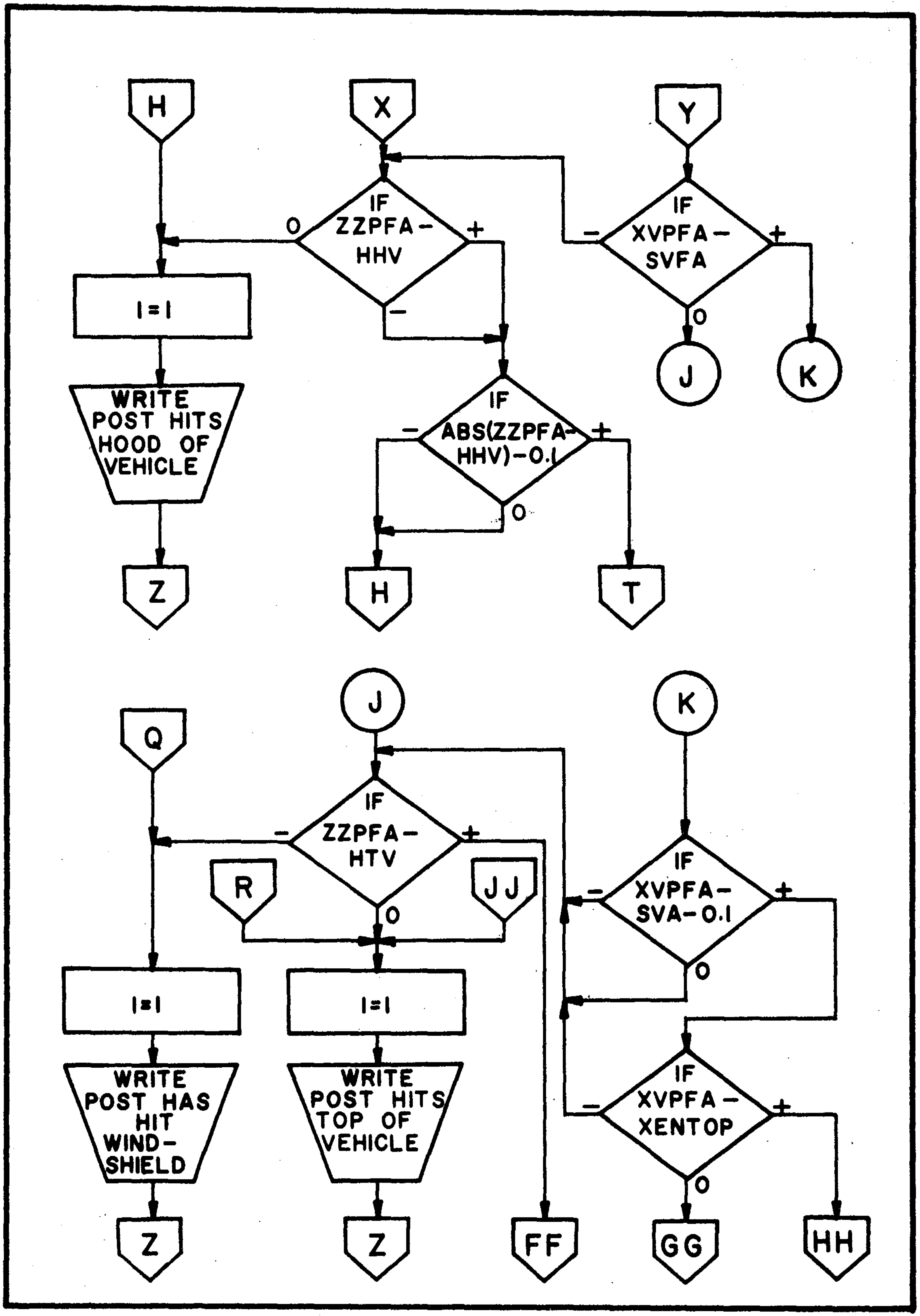
DEFINITIONS OF INITIALIZE, COMPUTE, EQUIVALENCE, AND WRITE ARE GIVEN IN THE LAST PAGE OF THIS SECTION.

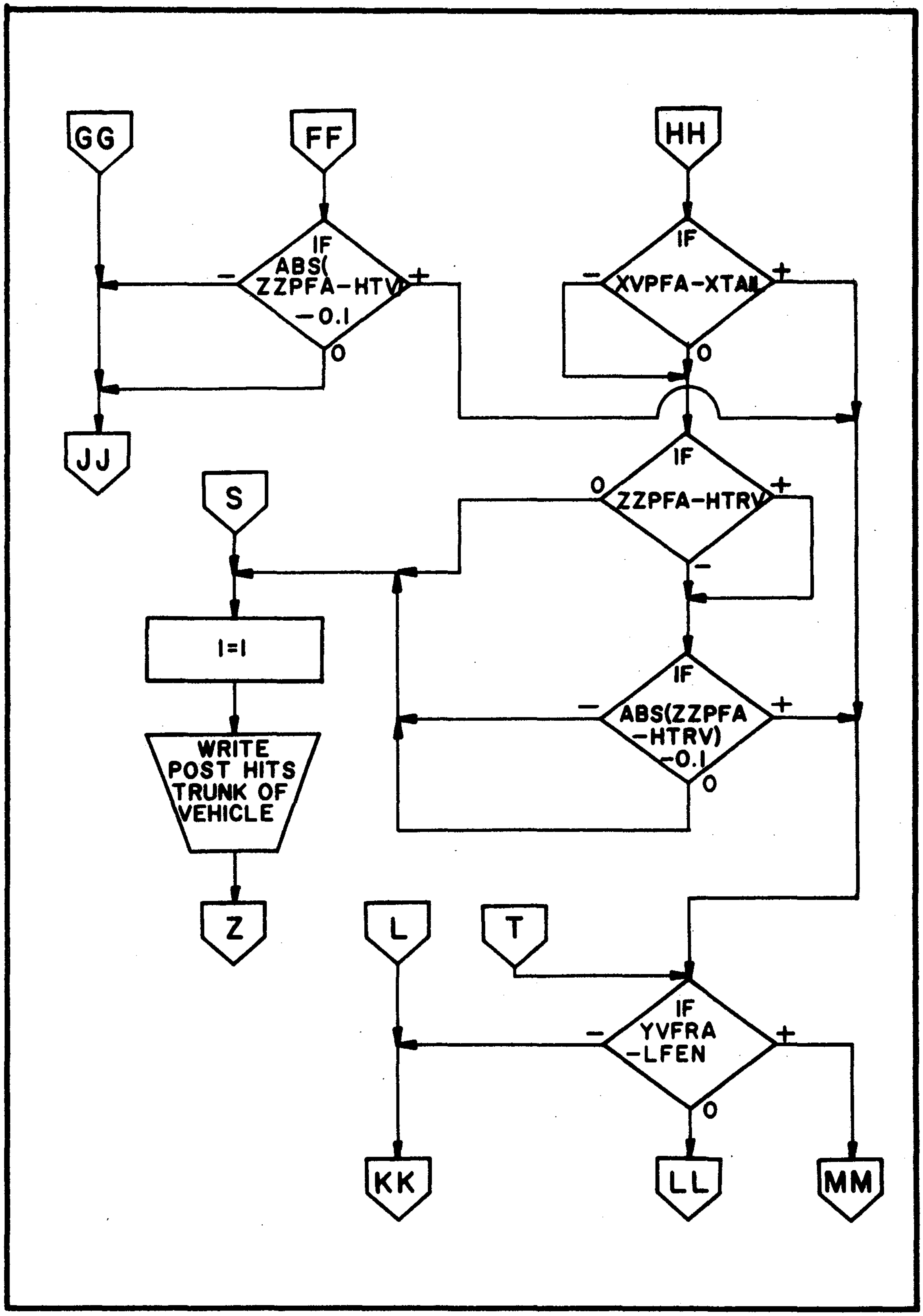

A
IF
Q−H
+
−
0
COMPUTE FSA
FSA=1320.0
IF
XQA+BOL
+
−
0
IF
XQA+BOL
−0.0004
+
−
0
FFXA=0.0
COMPUTE FNA
IF
XQA+BL
+
−
0
COMPUTE FNA
FNA=0.0
COMPUTE 4
B

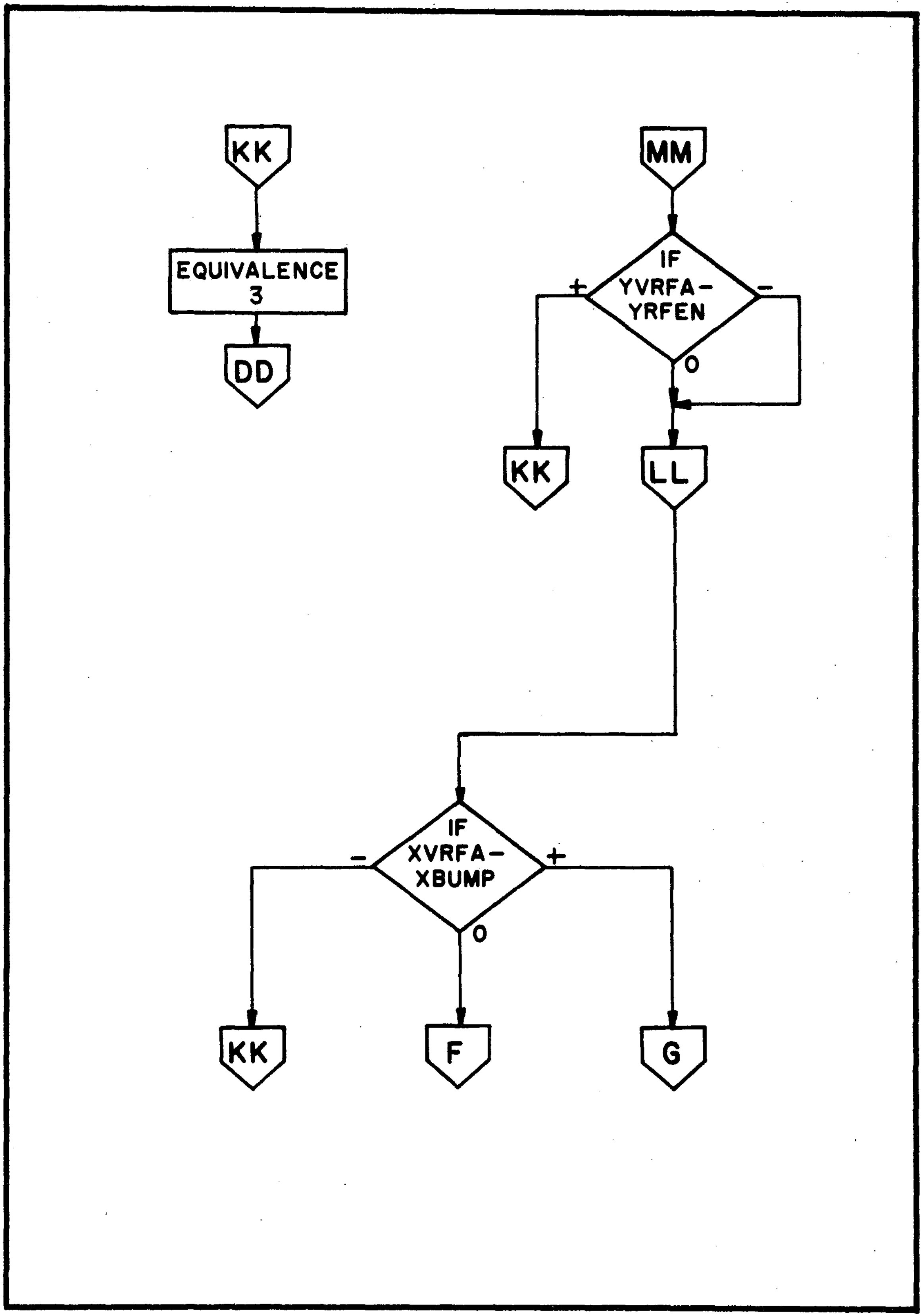

B
U
IF
E-0.008
E=0.0
IF
M-1
WRITE
1
IF
DIFF
WRITE
POST
LOSES
CONTACT
IF
DIFF-0.01
M=1
EQUIVALENCE
Q=Q-H
H=H/2.
HI=HI/2.
C
EQUIVALENCE
2
U
E
D

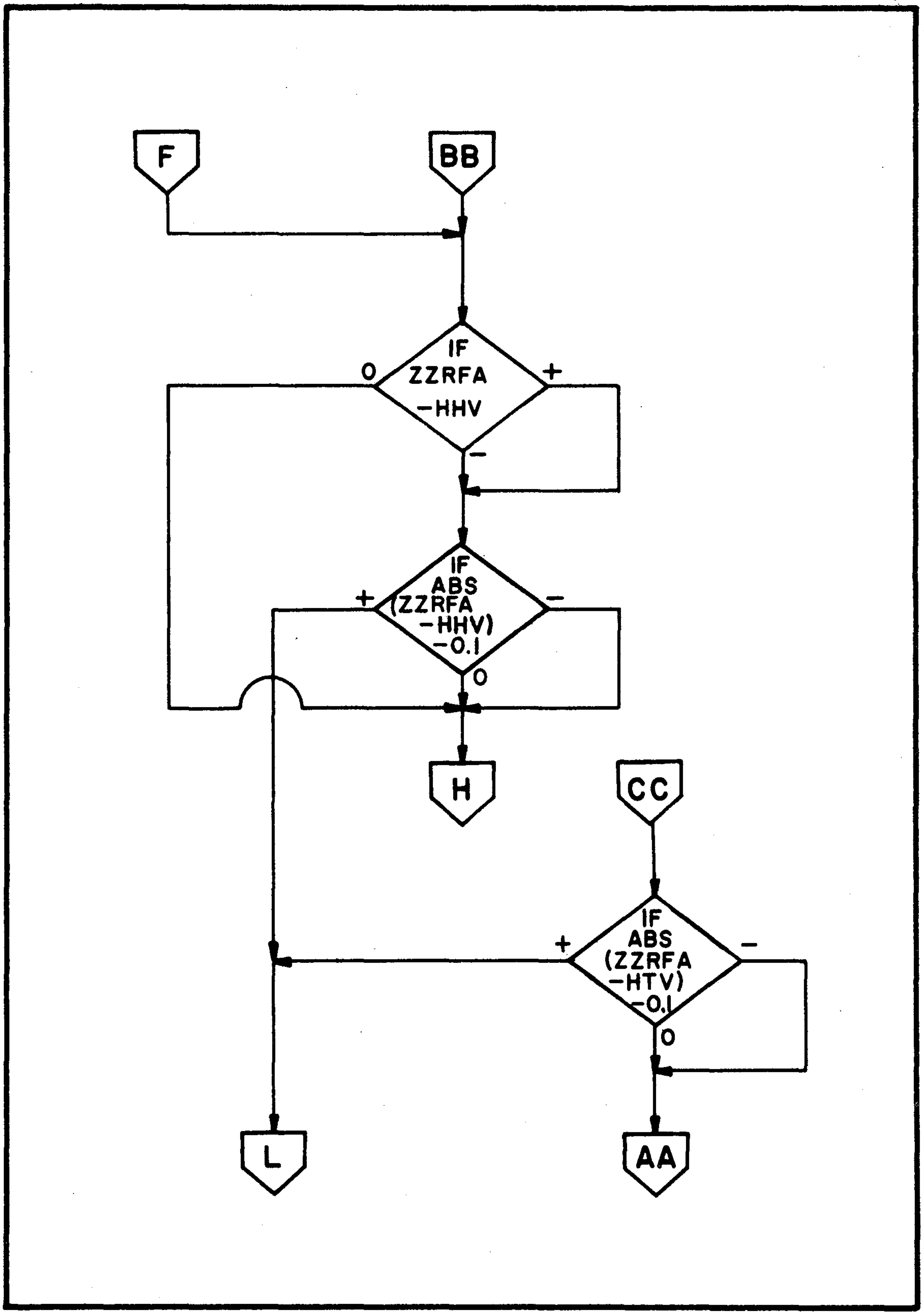

DD
D
T=TF+H
COMPUTE 5
Z
IF
E−0.008
−
+
0
E=0.0
WRITE
2
IF
I−I
0
+
−
STOP
IF
ZZPFA +
HEIGHT
0
+
−
I=I
IF
ABS(ZZPFA +
HEIGHT)−
0.I
0
+
−
WRITE
POST HITS
GROUND
WITH
POINT P
Z
EE

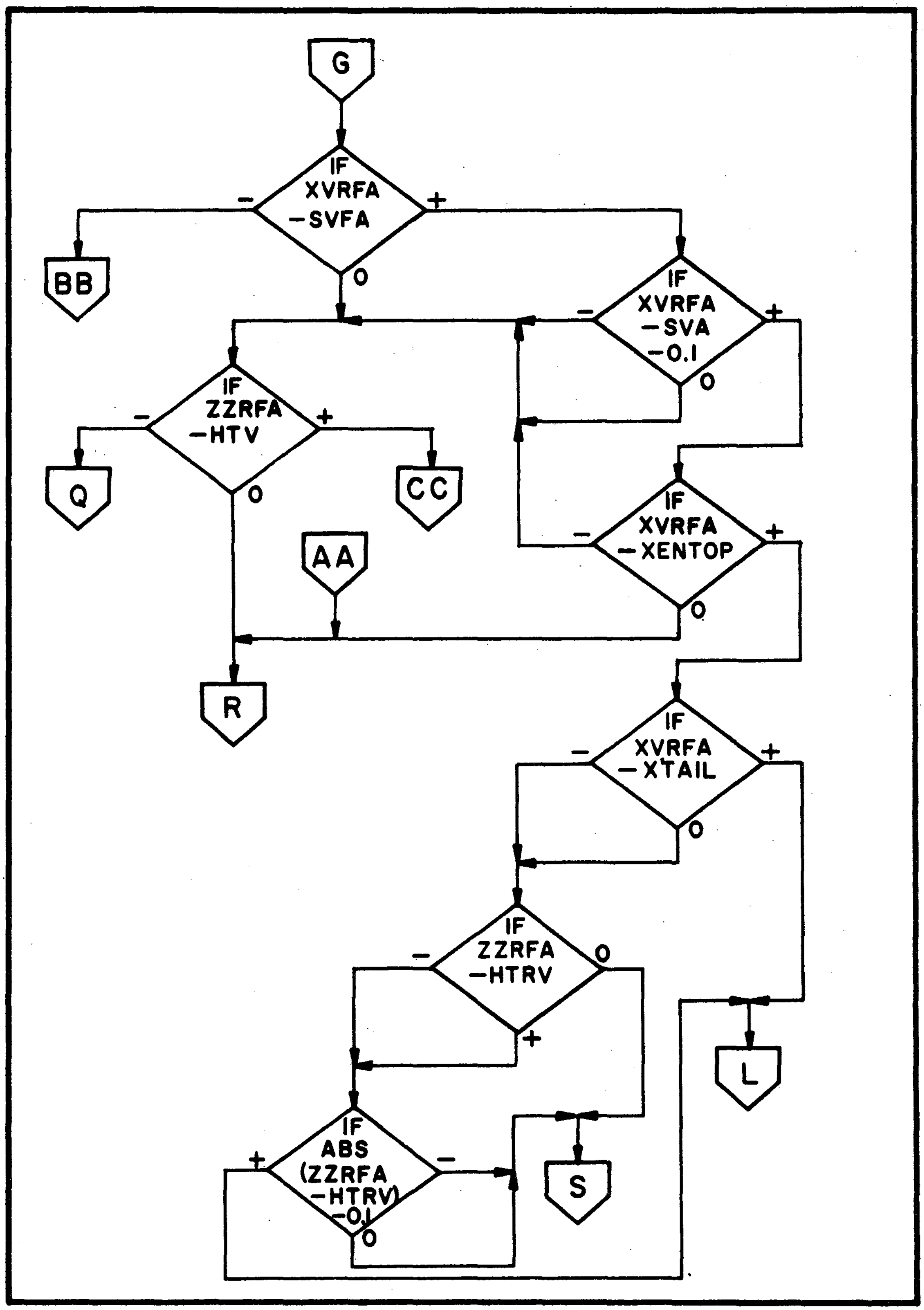

EE
IF
ZZQFA +
HEIGHT
O
+
−
IF
ABS(ZZQFA
+HEIGHT)
−0.1
O
+
−
I=I
WRITE
POST HITS
GROUND
WITH
POINT Q
Z
IF
ZZRFA +
HEIGHT
O
+
−
I=I
WRITE
POST HITS
GROUND
WITH
POINT R
Z
IF
ABS(ZZRFA +
HEIGHT) −
0.1
−
+
O
IF
YVPFA −
YLFEN
−
+
O
IF
XVPFA −
XBUMP
−
+
O
T
X
Y
IF
YVPFA −
YRFEN
−
+
O
T


H
X
Y
IF ZZPFA-HHV
0
+
-
IF XVPFA-SVFA
-
+
0
I=1
WRITE POST HITS HOOD OF VEHICLE
IF ABS(ZZPFA-HHV)-0.1
-
+
0
J
K
Z
H
T
J
K
Q
IF ZZPFA-HTV
-
+
0
R
JJ
IF XVPFA-SVA-0.1
-
+
0
I=1
I=1
WRITE POST HAS HIT WIND-SHIELD
WRITE POST HITS TOP OF VEHICLE
IF XVPFA-XENTOP
-
+
0
Z
Z
FF
GG
HH


GG
FF
HH
IF ABS(ZZPFA-HTV) -0.1
IF XVPFA-XTAIL
JJ
S
IF ZZPFA-HTRV
I=I
IF ABS(ZZPFA -HTRV) -0.1
WRITE POST HITS TRUNK OF VEHICLE
Z
L
T
IF YVFRA -LFEN
KK
LL
MM


KK
EQUIVALENCE
3
DD
MM
IF
YVRFA-
YRFEN
+
-
0
KK
LL
IF
XVRFA-
XBUMP
-
+
0
KK
F
G


F
BB
IF
ZZRFA
-HHV
0
+
-
IF
ABS
(ZZRFA
-HHV)
-0.1
+
-
0
H
CC
IF
ABS
(ZZRFA
-HTV)
-0.1
+
-
0
L
AA



G
IF XVRFA −SVFA
BB
IF XVRFA −SVA −0.1
IF ZZRFA −HTV
Q
CC
AA
IF XVRFA −XENTOP
R
IF XVRFA −XTAIL
IF ZZRFA −HTRV
L
IF ABS (ZZRFA −HTRV) −0.1
S

INITIALIZE 1

H1=0.0002
E=0.0
I=0
M=0
H=0.0002
Q=0.0
N=0
K=0

INITIALIZE 2

W1A=0.0
W2A=0.0
W3A=0.0
FSB=1000.0
THDA=0.0545
PHIDA=0.0
PSIDA=0.0
THA=0.000022
PHIA=0.0
PSIA=0.0

COMPUTE 1

YPO, ZPO
YQO, ZQO
YSO, ZSO
YRO, ZRO, XVSO

COMPUTE 2

W1A, W2A, W3A
THA, PHIA, PSIA
THDA, PHIDA, PSIDA

COMPUTE 3

V1A, V2A, V3A, VVA, S1A, S2A, S3A, SVA
DIRC1X, DIRC1Y, DIRC1Z, DIRC2X, DIRC2Y, DIRC2Z,
DIRC3X, DIRC3Y, DIRC3Z, XCMA, YCMA, ZCMA, YYCMA,
ZZCMA, SQUIGA, ETAA, ZETAA, CHIA, XPA, YPA,
ZPA, YYPA, ZZPA, XQA, YQA, ZQA, YYQA, ZZQA,
XSA, YSA, ZSA, YYSA, ZZSA, XVSA, YVSA

COMPUTE 4

FFYA, FSXA, FSYA, FIA
F2A, F3A, TXA, TYA, TZA,
TIA, T2A, T3A, POSLEN,
CARLEN, DIFF, Q, E

COMPUTE 5

WIFA, W2FA, W3FA, THFA, PHIFA, PSIFA
THDFA, PHIDFA, PSIDFA, FIFB, F2FB, F3FB,
VIFA, V2FA, V3FA, VVFA, SIFA, S2FA, S3FA,
SVFA, XBUMP, XENTOP, XTAIL, XCMFA, YCMFA,
ZCMFA, YYCMFA, ZZCMFA, SQUIFA, ETAFA,
ZETAFA, CHIFA, XPFA, YPFA, ZPFA, YYPFA,
ZZPFA, XVPFA, YVPFA, XQFA, YQFA, ZQFA,
YYQFA, ZZQFA, XRFA, YRFA, ZRFA, YYRFA,
ZZRFA, XVRFA, YVRFA, E, T

EQUIVALENCE I

VIB=VIA, V2B=V2A, V3B=V3A, WIB=WIA, W2B=W2A,
W3B=W3A, TIB=TIA, T2B=T2A, T3B=T3A, FIB=FIA,
F2B=F2A, F3B=F3A, VVB=VVA, THB=THA, PHIB=PHIA,
PSIB=PSIA, THDB=THDA, PHIDB=PHIDA, PSIDB=PSIDA,
FSB=FSA, SVB=SVA, XCMB=XCMA, YCMB=YCMA,
ZCMB=ZCMA

EQUIVALENCE 2

VIFB=VIA, V2FB=V2A, V3FB=V3A, SVFB=SVA, H=0.0002, HI=0.0002, E=0.0, XCMFB=XCMA, YCMFB=YCMA, ZCMFB=ZCMA, THDFB=THDA, PHIDFB=PHIDA, PSIDFB=PSIDA, THFB=THA, PHIFB=PHIA, PSIFB=PSIA, WIFB=WIA, W2FB=W2A, W3FB=W3A, TF=Q

EQUIVALENCE 3

VIFB=VIFA, V2FB=V2FA, V3FB=V3FA, SVFB=SVFA, TF=T, THFB=THFA, PHIFB=PHIFA, PSIFB=PSIFA, VVFB=VVFA, WIFB=WIFA, W2FB=W2FA, W3FB=W3FA, THDFB=THDFA, PHIDFB=PHIDFA, PSIDFB=PSIDFA, XCMFB=XCMFA, YCMFB=YCMFA, ZCMFB=ZCMFA, SVFB=SVFA

WRITE I

ANY OF THE QUANTITIES IN COMPUTE I THROUGH COMPUTE 4

WRITE 2

ANY OF THE QUANTITIES IN COMPUTE 5