# A TENTATIVE FLEXIBLE PAVEMENT DESIGN FORMULA AND ITS RESEARCH BACKGROUND

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# A TENTATIVE FLEXIBLE PAVEMENT DESIGN FORMULA AND ITS RESEARCH BACKGROUND

## 1. Introduction

This report, one of a series stemming from the Research Project, "Application of the AASHO Road Test Results to Texas Conditions," was written to satisfy, at least in part, the following objectives quoted from the Project Proposal:

> 1. In general, to correlate the average level of pavement performance determined from a two-year controlled traffic test (the AASHO Road Test) with performance of Texas pavements under normal mixed traffic, and to study the effect of weather and the so-called regional effect throughout the state.
>
> 2. For flexible pavements in Texas, to determine approximate values (or a range of values) of coefficients for representing Texas materials to replace the layer coefficients determined at the AASHO Road Test for the materials used there, and to develop relationships between these coefficients and materials tests.

This is a progress report. A large portion of the flexible pavement data accumulated in the project is not treated in this report. Its principal purpose is to elicit a reaction from the engineers of the Texas Highway Department, not necessarily to the research itself, but rather to the proposed design method based on the research.

The comments of Texas Highway Department engineers--particularly their reaction to the division of the state into regions of similar pavement performance--could benefit the project considerably.

## 2. Data Selected for Analysis

The data utilized in this report were gathered from a portion of the test sections comprising the project's Flexible Pavement Experiment. Every test section consisted of two sub-sections, each a traffic lane in width and 1200 feet long. A transition reserved for sampling operations, usually 100 feet long, separated the two sub-sections. All sections were located on existing highways, and none was constructed especially for this experiment.

The sections were chosen (in 1962-63) to conform, as nearly as possible, to the following experiment design involving five variables:

TABLE 1

| Variable No. | Variable | No. Levels | Levels |
|---|---|---|---|
| 1 | Region | 3 | Eastern, Central, Western |
| 2 | Surfacing thickness | 3 | 0-1", 1' -2.5", 2.5" + |
| 3 | Base strength | 3 | Low, Medium, High |
| 4 | Subbase strength | 3 | Low, High |
| 5 | Subgrade strength | 3 | Low, Medium, High |

Surfacings less than one inch thick were surface treatments; thicker surfacings were hot-mix asphaltic concrete. Materials classified as medium strength were those approximately equal in strength to the AASHO Road Test materials.

According to Table 1, it can be seen that the minimum number of sections required for a complete factional experiment would be $2 \times 3^4 = 162$. Actually, more than this number were selected, and after 44 had been eliminated because of excessive irregularities discovered when the sections were drilled, a total of 323 remained in the experiment. Of this total, 188 were surfaced with asphaltic concrete ranging in thickness from 1 to 8 inches, and it is these sections that are discussed in the remainder of this report.

From each of the 188 sections surfaced with asphaltic concrete, the following data were utilized directly or indirectly in the analyses:

1. Section location (used in regional analysis).

2. Serviceability index.

3. Layer thicknesses. (Average of drill hole measurements.)

4. Triaxial class of the materials. (Furnished by District personnel.)

5. Equivalent number of 18-kip single axle load applications made from the time of construction (or from the time of the last overlay with asphaltic concrete) to the time the first measurement of serviceability index was made. (Data furnished by the Texas Highway Department's Planning Survey.)

6. Dynamic deflections measured in 1964 by the Lane-Wells Dynaflect, converted to estimates of the static deflection that would be caused by a 9000 lb. dual wheel load. (The basis for this conversion is described in Research Report 32-4.)

## 3. General Relationship Sought from Analysis of Data

As a first step in preparation for the analysis, a statement of the desired relationship between pavement performance, pavement design, and region was formulated, as follows:

Pavement Performance = A function of layer thicknesses, layer strengths and regional effects (if any).

## 4. Pavement Performance Defined

As the second step, a definition of pavement performance was developed. It is given below:

If a pavement is subjected to $W_L$ applications of a single axle load, L, in a time period during which the serviceability index drops from a value $P_o$ to a value P, then the performance $Q_L$ of the pavement is given by the equation

$$Q_L \equiv \log \left[ \frac{W_L}{\log P_o - \log P} \right] \qquad (1)$$

where the symbol $\equiv$ means "is defined as" (as opposed to $=$, read as "equal to").

The expression for pavement performance was suggested by the work done by Painter* for the Asphalt Institute in his analysis of data from the AASHO Road Test. If the quantity, $\log P_o - \log P$, is interpreted as the pavement damage accruing while $W_L$ applications of the axle load L are made, then the term in the brackets represents axle applications per unit of damage. A relatively high value of $Q_L$ corresponds to a relatively large number of axle applications required to produce a unit of damage, and thus represents a relatively high level of pavement performance. Conversely, a relatively low value of $Q_L$ corresponds to a relatively low level of pavement performance.

If the axle loads in the traffic on each of a group of highways are reduced to the equivalent number of axles of load L, then Equation 1 can be used to compare the performance of any highway in the group with that of any other.

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*Painter, L. J., "Analysis of AASHO Road Test Asphalt Pavement Data by the Asphalt Institute," Highway Research Record, Number 71, Highway Research Board, Washington, D.C., 1965.

A more concise form for Equation 1 is shown below:

$$Q_L \equiv \log W_L - \log\log (P_o / P)$$

## 5. Estimating Pavement Performance from Deflections

As the next step in preparation for the analysis, an alternate equation for performance was developed for the reasons given below.

It will be noted that $P_o$ occurs in Equation 1; hence, $P_o$ must be known if Equation 1 is to be used to compute $Q_L$. But $P_o$ is not known for the test sections used in this analysis, since all had been in service from a few months to several years when the first determination of serviceability index was made in 1962 and 1963. Subsequent determinations made in 1964 and 1965 revealed that the changes occurring in the interval between the first and second measurements were generally too small to permit estimates of trends. Had well-defined trends been found in these data, the 1962-63 measurements could have been used for $P_o$ and the later measurements as P in Equation 1.

Furthermore, a study of the serviceability index of a group of relatively new test sections revealed that the variation in $P_o$ for the new sections was nearly as great as the variation in P among relatively old sections. It thus became apparent that an average value of $P_o$ could not be used to compute $Q_L$ from Equation 1 for individual sections with any degree of reliability, although this scheme worked well at the AASHO Road Test.

Some attempts were made to perform an analysis by averaging large groups of sections, for which an average value of $P_o$ could be assumed with some confidence. But such a procedure practically rules out the possibility of discovering regional effects, the existence of which can best be proved if the performance of individual sections scattered throughout the state can be compared.

This state of affairs led to the formulation of an additional hypothesis as an alternate method for estimating $Q_L$. The hypothesis was tested against data from the AASHO Road Test. The new hypothesis follows:

If a series of equal deflections, $U_L$, of a pavement surface are produced by $W_L$ applications of the single axle load L, while the serviceability index drops from $P_o$ to P, then there is a functional relationship between $U_L$, $W_L$, $P_o$ and P.

The specific relationship assumed is the following:

$$\log \left[ \frac{W_L}{\log P_o - \log P} \right] = A_o + A_1 \log U_L$$

where $A_o$ and $A_1$ are constants to be determined from certain AASHO Road Test deflection data.

With $U_L$ expressed in thousandths of an inch, the constants $A_o$ and $A_1$ were found to be 9 and -3/2, respectively, so that the above equation becomes

$$\log \frac{W_L}{\log P_o - \log P} = 9 - 3.2 \log U_L$$

The term to the left of the equal sign is the definition of $Q_L$ (see Equation 1). Therefore, we now have the desired alternative method of estimating $Q_L$:

$$Q_L = 9 - 3.2 \log U_L \qquad (2)$$

The reader, while perhaps accepting Equation 2 as representing AASHO Road Test data, may question its application to Texas highways. To this we can only reply that we have assumed that the deflection, $U_L$, is a quantity which in itself is responsive to regional effects and can therefore be applied anywhere. This assumption is supported by the fact that $U_L$ responded to seasonal variations in pavement strength at the AASHO Road Test.

Details of the derivation of Equation 2 are given in Appendix A.

### 6. Mathematical Statement of Relationship of Performance to Design and Region

In the foregoing, alternate methods were given for computing the pavement performance term in the basic equation to be derived from the data. We now turn to the remaining terms. For convenience the basic equation is restated below:

Pavement performance = A function of layer thicknesses, layer strengths, and regional effects (if any).

The next step taken toward the analysis was the adoption of the specific statement that follows:

$$Q = B \times TDI + Cr \qquad (3)$$

(In Equation 3 and throughout the report except in Appendix A, it will be understood that then the subscript L is deleted from the symbols Q, W and U, then L = 18, meaning an 18-kip single axle load, $Q = Q_{18}$, $W = W_{18}$ and $U = U_{18}$.)

The term, Q, in Equation 3 is given by

$$Q \equiv \log W - \log\log (P_o / P), \qquad (1)$$

or, alternatively,

$$Q = 9 - 3/2 \log U. \qquad (2)$$

B is a constant to be determined from the Texas data, TDI is a special function of layer strengths and layer thicknesses to be discussed in the next section, and Cr is a quantity that is constant anywhere within a region but varies between regions. The term "region" means an area within which pavements of like design exhibit similar performance. Regions are numbered, and the variable Cr has the value $C_1$ for any pavement in Region 1, $C_2$ for Region 2, etc.

## 7. Texas Design Index - Basis of Derivation

A further step that had to be taken before the analysis of the data could begin was the choice of a formula, referred to as a "design index," from which a number could be computed to represent the design of a pavement. For example, the formula chosen at the AASHO Road Test was the following:

$$\text{Road Test design index} = S_1D_1 + S_2D_2 + S_3D_3 + 1$$

where $D_1$, $D_2$, $D_3$ are thickness of surface, base and subbase, respectively, while $S_1$, $S_2$, $S_3$ are constants that were evaluated by analysis of the pavement performance data. Some engineers have regarded these constants as measures of the relative strength of the materials used at the Road Test for surfacing, base, and subbase. Their values were found to be 0.44, 0.14, and 0.11.

In the Road Test index we may regard the term $S_1D_1$ as the contribution of the surfacing layer to the index, $S_2D_2$ as the contribution of the base, $S_3D_3$ as the contribution of the subbase, and 1.0 as the contribution of the foundation. And this index can be said to satisfy the condition, imposed by the Road Test Staff, that it should be the sum of the individual layer contributions.

Four conditions were imposed on the Texas Design Index. The first was borrowed directly from the Road Test Index, and is stated below:

1. Corresponding to each structural layer in the system there will be a term in the index called the "contribution" of the layer, and the index will be the sum of the individual layer contributions.

The second condition, suggested by the Road Test Index but only partially satisfied by it, follows:

2. The contribution of a layer will increase if either of its strength or its thickness is increased.

The third condition, also partially satisfied by the Road Test Index, was imposed to insure that the Texas version could be applied to any number of layers:

3. (Corollary) The contribution of any layer will decrease if the thickness of the overlying material is increased (for otherwise the index would become infinite).

The fourth and last condition follows from the reflection that if increasing the thickness of a pavement will decrease the contribution of underlying layers (as required by the corollary to Condition 3), then increasing the strength of the pavement should have a similar effect. This notion stems in part from certain effects predicted by the theory of elasticity. As the top layers become stiffer in layered elastic systems the stresses in the lower layers decrease. These considerations led to the formulation of the fourth condition, which is equivalent to a broadening of the corollary to Condition 3:

4. The contribution of a layer will decrease if either the thickness or the strength of an overlying layer is increased.

The Road Test Index does not fully satisfy conditions 2 and 3 because the contribution of the subgrade was---for the reasons given in the next paragraph---considered to be a constant, 1.0. It does not satisfy Condition 4 because changing either the thickness or the strength of any layer has no effect on the contribution of any other layer.

The use of the constant, 1.0, to represent the contribution of a subgrade in the Road Test Index can probably be explained as follows. Since subgrade strength was not a variable in the experiment, no means were available for estimating the effect of subgrade strength on subgrade contribution. With this important effect unknown it would have been difficult--perhaps impossible--to estimate the effect of any of the subgrade strength on subgrade contribution. As a result, the contribution of the subgrade was treated as a constant. The value assigned to that constant was immaterial; the results of the analysis would have been essentially the same for any value. In the interest of simplicity, the value chosen was unity.

## 8. Derivation of Texas Design Index

It is realized that many functions satisfying the stated conditions could be constructed. The function selected was, at least in its differential form, the simplest that the authors could devise.

To simplify the problem, the real system of structural layers composing a pavement is replaced by an idealized system of structural layers each of uniform strength and thickness. The system extends downward to infinity. Horizontal dimensions are not considered in the problem. Such a system is illustrated in Figure 1 which represents a pavement structure of q layers. Structural layers are numbered consecutively from the top layer downward, with q being the number assigned to the foundation layer.

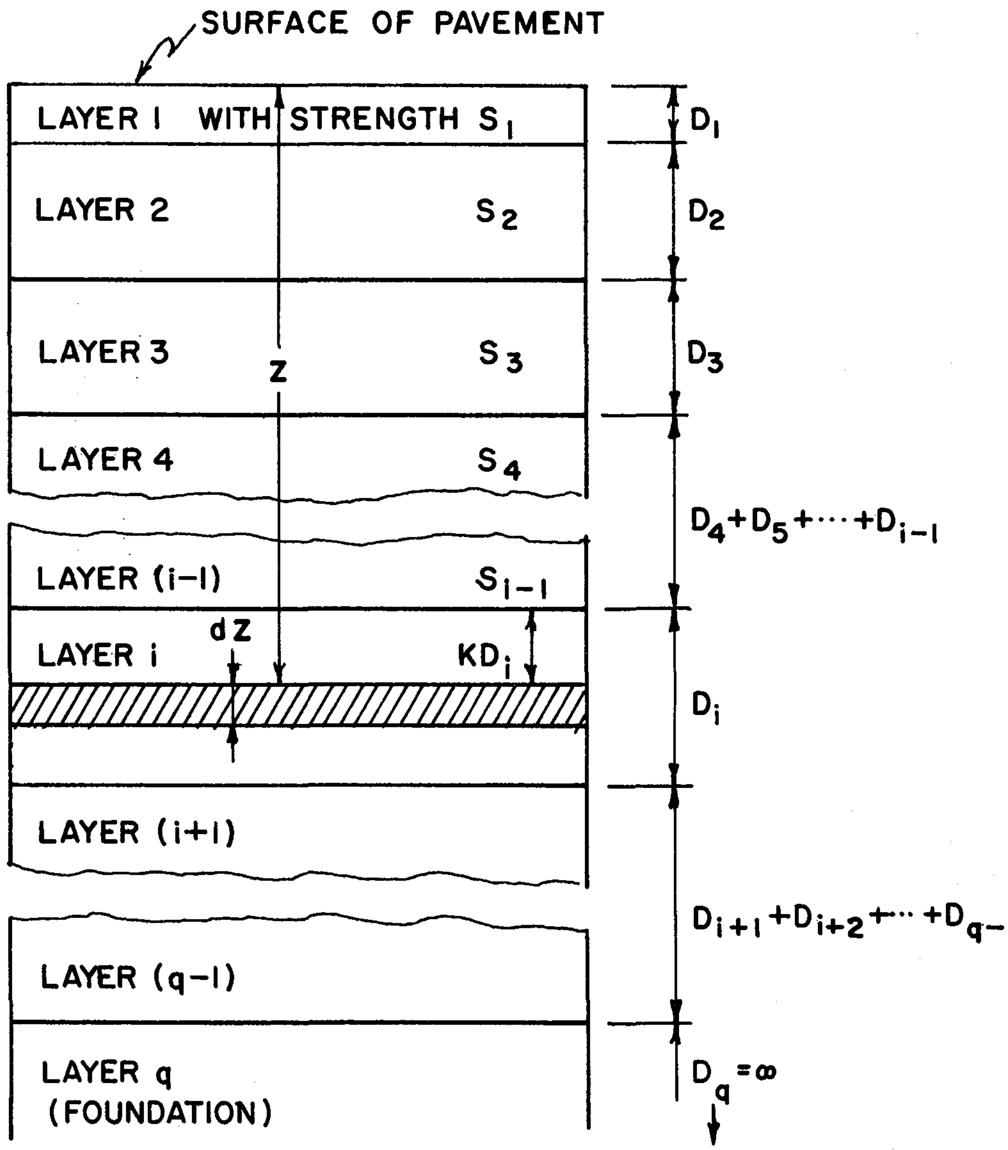


FIGURE 1: IDEALIZED PAVEMENT STRUCTURE OF q LAYERS.

The symbols S for strength and D for thickness are subscripted by i, which is the number of the corresponding structural layer. The depth below the surface of any point in the structure is represented by the symbol Z. Thus, the depth, Z, of an elemental layer in the $i^{th}$ structural layer is

$$Z = D_1 + D_2 + \ldots + D_{i-1} + KD_i \qquad \text{(A)}$$

where $0 \leq K \leq 1.0$.

Transformation to "Effective" Depth: We begin the derivation of the index by transforming every depth, Z, to an effective depth, X, defined by the following equation:

$$X = S_1D_1 + S_2D_2 + \ldots + S_{i-1}D_{i-1} + KS_iD_i \qquad \text{(B)}$$

where $0 \leq K \leq 1.0$.

Thus the effective depth of the upper boundary of the $i^{th}$ structural layer, shown in Figure 2, is $H_{i-1} = S_1D_1 + S_2D_2 + \ldots + S_{i-1}D_{i-1}$, while the depth of the lower boundary is $H_i = H_{i-1} + S_iD_i$, and the effective thickness of the layer is $S_iD_i$.

The remaining steps of the derivation will be taken in the X coordinate system. The reason for making the transformation from true depths, Z, to effective depths, X, is explained as follows. The corollary to Condition 3 requires that a layer contribution shall decrease if the thickness of an overlying layer is increased. Condition 4 requires further that the layer contribution shall decrease if the strength of the overlying layer is increased. The most ready means for satisfying both these conditions appeared to be to make the transformation to effective depth, X, since effective depth increases if either the thickness or the strength of an overlying layer is increased. Thus, a layer contribution term that decreases as X increases will decrease if either the strength or the thickness of an overlying layer is increased.

Application of First Condition: Let Y be the desired index, and $\Delta Yi$ be the contribution of the $i^{th}$ structural layer. Then Condition 1 required that

$$Y = \Delta Y_1 + \Delta Y_2 + \ldots + \Delta Y_q \qquad \text{(C)}$$

Application of Second Condition: Condition 2 required that the form of the expression for $\Delta Y_i$ will be the same for every i. To insure that this condition is satisfied we specify that $\Delta Y_i$ is the sum of the contributions of all layers of infinitesimal effective thickness making up the $i^{th}$ structural layer, and that the form of the contribution function will be the same for all elemental layers. That is,

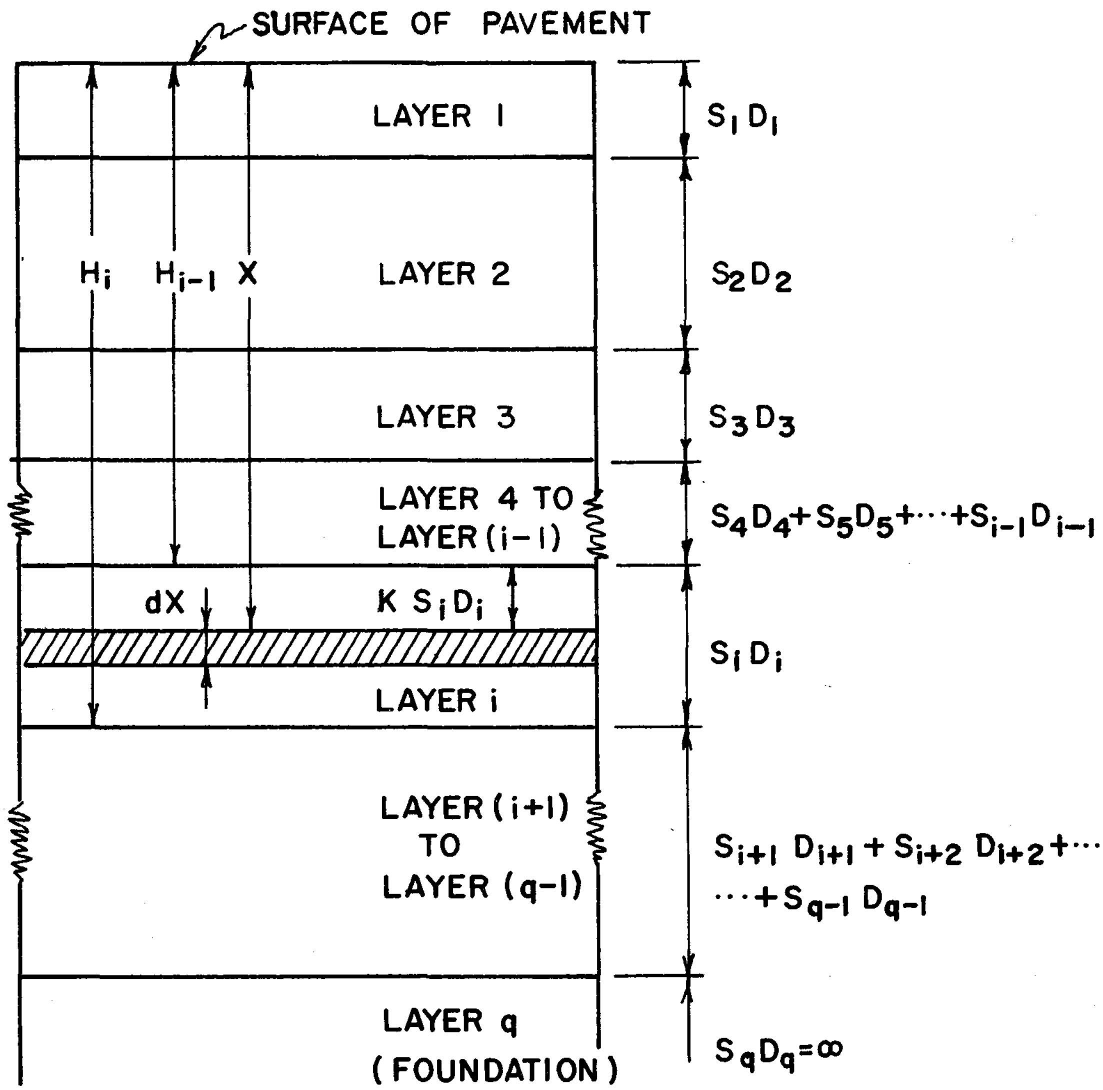


FIGURE 2: PAVEMENT STRUCTURE OF FIGURE 1 WITH "EFFECTIVE" DIMENSION REPLACING REAL DIMENSION.

$$\Delta Y_i = \int_{X = H_{i-1}}^{X = H_i} dY \qquad (i = 1, 2, \ldots, q) \quad \text{(D)}$$

where Y is a function of X that is sectionally continuous and finite in the interval $0 \leq X < \infty$.

Functional Form of dY: The functional form chosen for dY is given below:

$$dY = \frac{F(S)\,dX}{(a + X)^b} \qquad \text{(E)}$$

where a and b are constants greater than zero, and F(S) is a function of strength, S.

General Form of Layer Contribution Term: It has been pointed out that the use of Equations C and D insures that the index will satisfy Conditions 1 and 3. It remains to show that the expression for $\Delta Y_i$ resulting from the use of Equation E will satisfy Conditions 2 and 4.

The expression for $\Delta Y_i$ in terms of effective depths is found by substituting Equation E in Equation D and integrating over the interval $X = H_{i-1}$ to $X = H_i$, with the following result:

$$\Delta Y_i = \frac{F(S_i)}{r} \left[ \frac{1}{(a + H_{i-1})^r} \qquad \frac{1}{(a + H_i)^r} \right]$$

where b has been replaced by r+1.

Having noted from Equation B (and Figure 2) that $H_{i-1} = S_1D_1 + S_2D_2 + \ldots + S_{i-1}D_{i-1}$ and that $H_i = S_1D_1 + S_2D_2 + \ldots + S_{i-1}D_{i-1} + S_iD_i$, we substitute these expressions in the above equation for $\Delta Y_i$, and obtain the following expression for the layer contribution, $\Delta Y_i$, in terms of real dimensions:

$$\Delta Y_i = \frac{F(S_i)}{r} \left[ \frac{1}{(a + S_1D_1 + S_2D_2 + \ldots + S_{i-1}D_{i-1})^r} - \frac{1}{(a + S_1D_1 + S_2D_2 + \ldots + S_{i-1}D_{i-1} + S_iD_i)^r} \right] \quad \text{(F)}$$

where $r > 0$, $S \geq 0$, $F(0) = 0$, and $\partial F(S)/\partial S > 0$.

The restrictions just stated are necessary for the following reasons:

$r > 0$ so that $\Delta Y_q$ (for which $D_i = \infty$) will be finite.

$S \geq 0$ because it represents a strength and therefore cannot be negative.

$F(0) = 0$ so that $\Delta Y_i$ will approach zero as $S_i$ approaches zero, so that the contribution of a layer practically without strength will contribute practically nothing to the index.

If, for some value of $S_i$, $F(S_i)$ were allowed to decrease with an increase in $S_i$, then it would be possible that, for this value of $S_i$, $\Delta Y_i$ would also decrease with an increase in $S_i$, a violation of Condition 2. To prevent this possibility it is necessary that $F(s)$ increase with an increase in $S$. This will occur if $\partial F(s)/\partial S > 0$ for all values of $S$.

An additional consequence of the foregoing restrictions is the fact that $\Delta Y_i$ is never negative.

To show that an increase in either the thickness or the strength of the $i^{th}$ layer results in an increase in $\Delta Y_i$, as required by Condition 2, it is sufficient to show that $\partial \Delta Y_i / \partial D_i > 0$ and that $\partial \Delta Y_i / \partial S_i > 0$. By differentiating Equation F with respect to $D_i$ we find

$$\frac{\partial \Delta Y_i}{\partial D_i} = \frac{S_i F(S_i)}{(a + S_1 D_1 + \ldots + S_i D_i)^{r+1}} > 0 .$$

By differentiating Equation F with respect to $S_i$ we obtain

$$\begin{aligned}\frac{\partial \Delta Y_i}{\partial S_i} = F(S_i) &\left[ \frac{D_i}{(a + S_1 D_1 + \ldots + S_i D_i)^{r+1}} \right] \\ &+ \frac{1}{r} \frac{\partial F(S_i)}{\partial S_i} \left[ \frac{1}{(a + S_1 D_1 + \ldots + S_{i-1} D_{i-1})^r} \right. \\ &\left. - \frac{1}{(a + S_1 D_1 + \ldots + S_i D_i)^r} \right] > 0 .\end{aligned}$$

The inequality signs in the last two equations, proving that Condition 2 is satisfied, may be verified by inspection. For example, in the first equation all terms are positive; therefore the expression is positive. In the second equation the first term is clearly positive, while the sign of the second term is the same as the sign of the quantity within the brackets. The quantity within the brackets is positive because the denominator of the first fraction is obviously less than that of the second fraction; thus the second term of the equation is positive and consequently the entire expression is positive.

To show that $\Delta Y_i$ will decrease if the strength of an overlying layer is increased, as required by Condition 4, it is sufficient to show that $\partial \Delta Y_i / \partial S_k < 0$, where $k < i$. By differentiating Equation F we obtain

$$\frac{\partial \Delta Y_i}{\partial S_k} = D_k F(Si) \left[ \frac{-1}{(a + S_1 D_1 + \ldots + S_{i-1} D_{i-1})^{r+1}} + \frac{1}{(a + S_1 D_1 + \ldots + S_i D_i)^{r+1}} \right] < 0 .$$

The expression is negative because the negative term within the brackets has the smaller denominator, and therefore the greater absolute value.

Similarily, to show that $\Delta Y_i$ will decrease if the thickness of an overlying layer is increased, as required by Condition 4, it is sufficient to show that $\partial \Delta Y_i / \partial D_k < 0$, where again $k < 0$. Again differentiating Equation f, we obtain

$$\frac{\partial \Delta Y_i}{\partial D_k} = S_k \; F(S_i) \left[ \frac{-1}{(a + S_1 D_1 + \ldots + S_{i-1} D_{i-1})^{r+1}} + \frac{1}{(a + S_1 D_1 + \ldots + S_i D_i)^{r+1}} \right] < 0 .$$

This expression is the same as the expression for $\partial \Delta Y_i / \partial S_k$, above, except for the multiplier on the brackets, and therefore is negative.

Thus it has been shown that the index satisfies all four of the imposed conditions.

Finally, by substituting Equation F in Equation C, and combining terms, we find Y, which is the general form of the Texas Design Index.

$$Y = TDI = \frac{1}{r} \left[ \frac{F(S_1)}{a^r} - \frac{F(S_1) - F(S_2)}{(a + S_1 D_1)^r} - \frac{F(S_2) - F(S_3)}{(a + S_1 D_1 + S_2 D_2)^r} \right. \quad (G)$$

$$- \cdots - \frac{F(S_{q-1}) - F(S_q)}{(a + S_1D_1 + S_2D_2 + \ldots + S_{q-1}D_{q-1})^r} \qquad \text{(G)}$$

where $a>0$, $r>0$, $S\geq 0$, $F(0) = 0$, and $\partial F(S)/\partial S>0$.

The special form of Equation G used elsewhere in this report was obtained by letting $r = 1$, $F(S) = 1000\ S$, and $a = 1000$. When these substitutions are made, Equation G takes the following form:

$$TDI = S_1 - \frac{1000\ (S_1 - S_2)}{1000 + S_1 D_1} - \frac{1000\ (S_2 - S_3)}{1000 + S_1D_1 + S_2\ D_2} \qquad \text{(H)}$$

$$- \cdots - \frac{1000\ (S_{q-1} - S_q}{1000 + S_1D_1 + S_2D_2 + \cdots + S_{q-1}D_{q-1}}$$

The strength, S, of a material was assumed to be related to the Texas Triaxial Class of the material in accordance with the following equation:

$$S = \frac{50}{3}\ (7 - T) \qquad \text{(I)}$$

where T is the triaxial class obtained by plotting the rupture envelope of the material on the modified "Chart for Classification of Subgrade and Flexible Base Material" shown in Figure 6.

Equation (I), as well as the modified classification chart, will be discussed more fully in the next section. The choice of the special form for F (S) mentioned above, as well as the assignment of the values of 1 and 1000 to the constants a and r in Equation G, will be discussed in Section 10.

## 9. Assumed Relationship of Strength, S, and Triaxial Class, T

Figure 3 is a partial reproduction of the chart used by the Texas Highway Department for Classifying flexible pavement materials, excluding asphaltic concrete. To illustrate, briefly, the method of classification,* we have plotted the Mohr's rupture envelopes of three hypothetical matherials, A, B, and C. The "critical point" of each rupture line, from which the material is classified, is circled in the figure. Materials A and C are assigned class numbers 3.0 and 4.0 respectively, while material B has the class number 3.7.

From the above it is apparent that each class boundary can be labelled with a class number, 2.0, 3.0, etc., and the area labels, "Class 1," "Class 2,"

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*"Preparation of Soil and Flexible Base Materials for Testing" (Test Method Tex-101-E, Rev.: June 1964 Manual of Testing Procedures, Vol. 1, Texas Highway Department, Materials and Test Division.

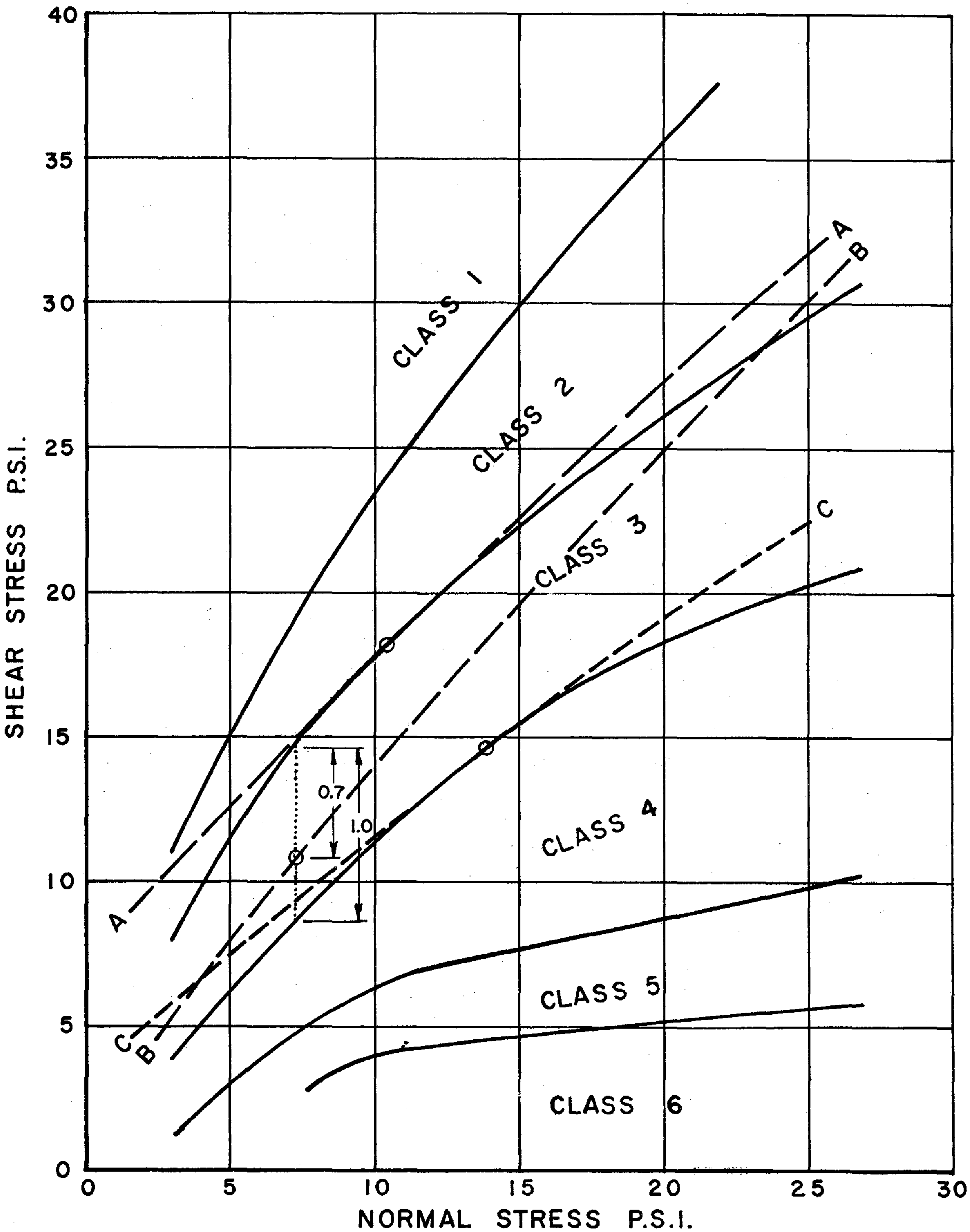



FIGURE 3: FLEXIBLE PAVEMENT MATERIALS CLASSIFICATION CHART USED BY TEXAS HIGHWAY DEPARTMENT.

etc., can be omitted without affecting the method of classifying the materials. These revisions are illustrated in Figure 4.

Since the theoretical lower and upper limits of the strength, S, are zero and infinity, respectively, it is desirable that the Triaxial Class, T, be assigned values to correspond to the lower and upper limits of S. Accordingly, the vertical -- or shear -- axis of the classification chart was assigned the class number 0, and the horizontal -- or normal stress -- axis was given the Class number 7, as indicated in Figure 5. Thus, when $T = 7$, $S = 0$ (since a rupture envelope coinciding with the horizontal axis would represent a material without shear strength); and when $T = 0$, $S = \infty$ (since a rupture envelope coinciding with the vertical axis represents a material of infinite shear strength). These additional revisions are shown in Figure 5.

The chart of Figure 5 has class lines labelled T = 0, 2, 3, 4, 5, 6, 7 -- a series from which the integer 1 is missing. A class line for $T = 1$ was supplied by the extrapolation procedure described in Appendix B, and is plotted in Figure 6.

The use of the revised chart shown in Figure 6 is recommended in lieu of the standard chart (Figure 1) for obtaining the Triaxial Class, T, to be used in Equation I. The revised chart yields the same results as the standard chart for materials of Class 2.0 and weaker. The presence of the Class 1 and Class 0 lines on the revised chart makes it possible to distinguish between materials stronger than class 2.0, a contingency not provided for in the standard chart.

As mentioned at the beginning of this article, the Texas Highway Department does not include asphaltic concrete in the Triaxial classification system: hence, the range of values of T for asphaltic concrete was not available. In an attempt to estimate an approximate value for asphaltic concrete, a standard Texas triaxial test was performed on a compacted specimen of the AASHO Road Test asphaltic concrete at a temperature of 110° F. (The temperature of 110° F corresponded to the average in-place temperature of this material at a depth of 2 inches on the 60 warmest days of the two-year traffic period at the Road Test.) Using the revised chart, this asphaltic concrete was found to have a class number of 1.0. Therefore, and in the absence of other data, a value of $T = 1.0$ was chosen to represent high-quality asphaltic concrete.*

Corresponding to $T = 1.0$, a value of $S = 100$ was chosen arbitrarily in order to set a scale for the variable, S. With this choice, then, we have the following correspondence between T and S:

| T | S |
|---|---|
| 0 | Infinite |
| 1 | 100 |
| 7 | 0 |

*Obviously further research is required to establish a range of values of S for asphaltic concrete.

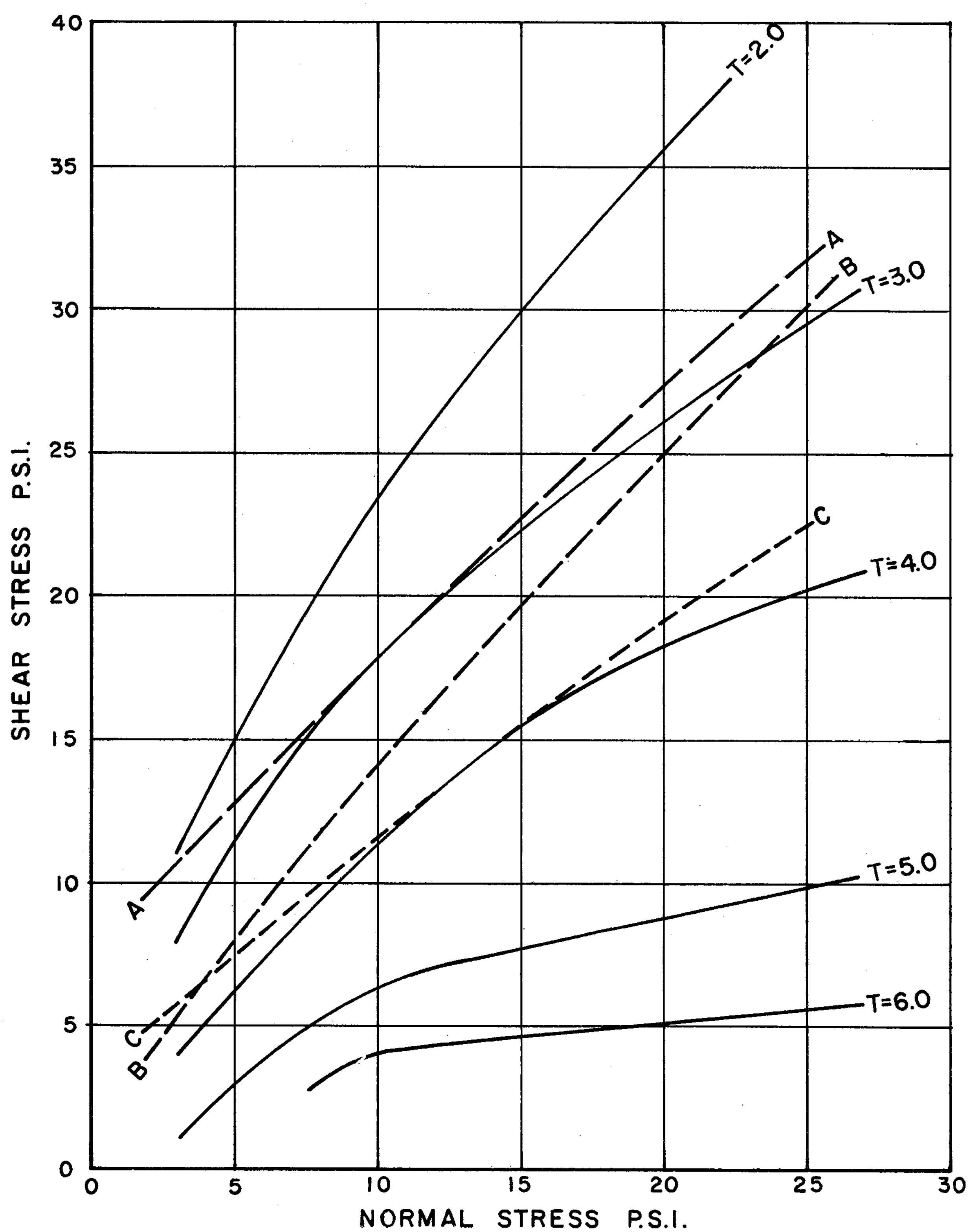



FIGURE 4: FLEXIBLE PAVEMENT MATERIALS CLASSIFICATION CHART WITH LINE LABELS REPLACING CLASS AREAS. THIS REVISION DOES NOT AFFECT THE CLASSIFICATION PROCEDURE.

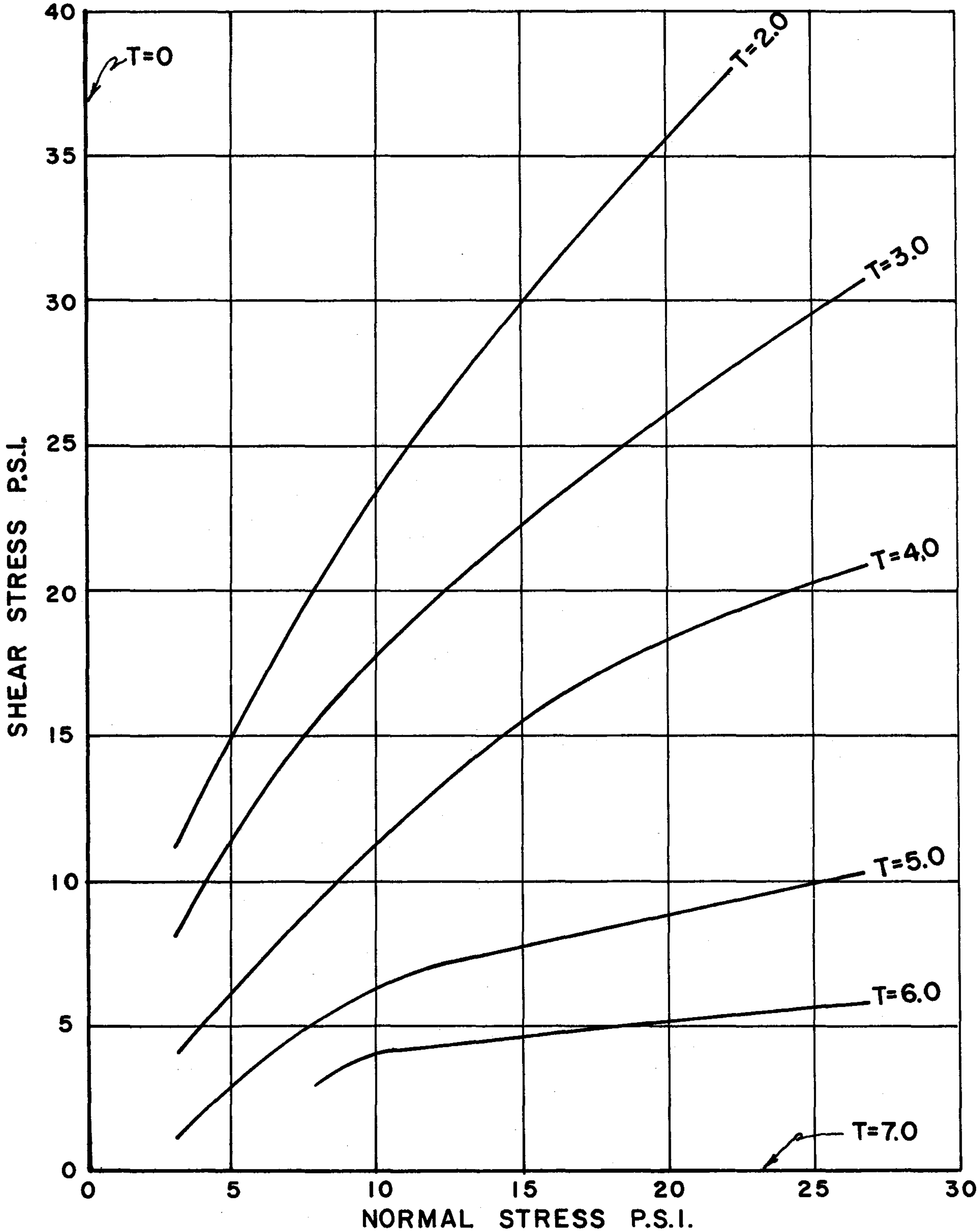



FIGURE 5: FLEXIBLE PAVEMENT MATERIALS CLASSIFICATION CHART WITH CLASS 0 AND CLASS 7 LINES DESIGNATED AS LIMITING VALUES OF STRENGTH.

Since values of T less than 1 rarely occur in untreated flexible pavement materials, it was decided to restrict the relationship of S and T to the interval, $1 \leq T \leq 7$, at least for the present. It was also decided to assume the simplest possible functional form of the relationship, and to test this function against pavement performance data. The relationship chosen was the linear equation,

$$S = A_0 + A_1 T \qquad (1 \leq T \leq 7)$$

where $A_0$ and $A_1$ are constants determined from the two conditions, (1) when $T = 1$, $S = 100$ and (2) when $T = 7$, $S = 0$. The resulting equation,

$$S = \frac{50}{3} (7 - T) \qquad (1 \leq T \leq 7) \qquad \text{(I)}$$

was given in Section 8 as Equation I.

## 10. Choice of F (S) and of Values for Constants in the TDI Equation

Equation H, the special form of the Texas Design Index used in this report, was obtained from the general equation (Equation G) by setting $F(S) = aS$, $r = 1$, and $a = 1000$. These choices were made after conducting a preliminary investigation of the properties of Equation G, and the effect on those properties of varying F (S), r and a. The choices were subjective; however, the equation selected -- Equation H -- was tested against the Road Test design index, using AASHO Road Test data, in the manner described in this section.

Since Road Test data were to be used, it was necessary to compute values of TDI for Road Test sections. This in turn required that triaxial test results be available for the Road Test materials. The necessary testing was performed at the Texas Transportation Institute, with the following results:

Table 2

| Material | Triaxial Class, T | Strength, S |
|---|---|---|
| HMAC | 1.0 | 100 |
| Base | 2.2 | 80 |
| Subbase | 3.5 | 58 |
| Subgrade | 5.6 | 23 |

As mentioned earlier, the asphaltic concrete was tested at $110^{\circ}$F. Base, subbase, and subgrade materials were tested at the "as constructed" moisture content and density. The corresponding values of S, computed from Equation I, are shown in the third column of the table. These values were used in Equation H to compute the TDI for Road Test sections.

Mathematical models, suitable for use in comparing the TDI with the Road Test index using data from any given traffic lane, were developed as follows:

The Road Test flexible pavement performance model reduces to the following special form when applied to sections located in a single loop and lane:

$$\log \rho = A_0 + A_1 \log (D + 1) \qquad (4)$$

where

$$D = 0.44 D_1 + 0.14 D_2 + 0.11 D_3$$

$\rho$ = the number of axle loads applied to a test section of design, D, while the serviceability index drops from 4.2 to 1.5, and

$A_0$, $A_1$ = constants associated with the lane, the values of which can be determined by analysis of $\rho$ and D data for the test sections in the lane.

A corresponding equation relating $\rho$ and TDI can be obtained from Equation 3 as follows:

$$\log \rho = B_0 + B_1 \text{ TDI} \qquad (5)$$

where $B_0$ and $B_1$ = constants which can be evaluated by analysis of $\rho$ and TDI data for the test sections in a given Road Test Traffic lane.

Equations 4 and 5 furnish a means for comparing the "goodness of fit" of the Road Test Design Index with that of the Texas Design Index, when the two equations are fitted to the same pavement performance data. The procedure followed was to compute D and TDI for each section in a Road Test traffic lane, and then perform two regression analyses, one using Equation 4 as the model and the other using Equation 5. The values of the root-mean-square-residual and correlation coefficient for the one model could then be compared directly with the corresponding values for the other.

Ten comparisons of the type just described were made, one for each of the ten lanes at the Road Test. The results indicated a somewhat better fit was achieved with the Road Test index, as can be seen from Table 3. However, until time is available for trying other variations on the TDI equation -- that is, other forms for F(S) and other values of a and r -- it was decided that Equation H was sufficiently promising to warrant its use in this progress report.

Table 3

Comparative Statistics
AASHO Road Test Design Index
and Texas Design Index
Using Road Test Performance Data

| Data From Loop | Lane | Root Mean Square Residual for Regression 1 | 2 | Correlation Coefficient for Regression 1 | 2 |
|---|---|---|---|---|---|
| 2 | 1 | 0.52 | 0.52 | 0.84 | 0.85 |
| 2 | 2 | .38 | .35 | .88 | .90 |
| 3 | 1 | .27 | .29 | .84 | .80 |
| 3 | 2 | .24 | .26 | .87 | .84 |
| 4 | 1 | .26 | .36 | .90 | .79 |
| 4 | 2 | .17 | .28 | .94 | .83 |
| 5 | 1 | .19 | .25 | .91 | .84 |
| 5 | 2 | .18 | .26 | .94 | .87 |
| 6 | 1 | .24 | .29 | .80 | .70 |
| 6 | 2 | .23 | .26 | .71 | .61 |
| Average | | .27 | .31 | .86 | .80 |

Model, Regression 1:

$$\log \rho = A_o + A_1 \log \text{(Road Test Design Index)}$$

Model, Regression 2:

$$\log \rho = B_o + B_1 \text{ (Texas Design Index)}$$

$\rho$ = No. axle applications required to reduce serv. index to 1.5.

$A_o, A_1, B_0, B_1$ = constants

## 11. Regional Effects

The model selected for relating pavement performance, Q, and TDI was given as Equation 3 (Section 6) and is repeated below:

$$Q = B \times TDI + Cr \tag{3}$$

where Q is to be estimated from deflection by means of Equation 2 (Section 5).

This section is concerned with the evaluation of the constant, B, and the variable, Cr, in Equation 3.

Indication of Regional Effect: Figure 7 is a plot of Q and TDI data from the 188 test sections selected for analysis. The circled points in the figure represent data from District 8, while the other points represent data from sections in the remaining 24 Districts. The very wide scattering of the points, as well as the tendency of the data from one District -- occupying a relatively small area of the state -- to plot in the upper portion of the scatter diagram, suggested the possible existence of a regional effect. It was therefore hypothesized that an effect associated with location did in fact exist, and steps were taken to test the hypothesis, as described below.

Estimate of the Slope, B: The first step toward testing the hypothesis required that an estimate by made of the slope, B, in Equation 3. It was reasoned that the regional effect -- if any -- would be less pronounced in a single District; therefore, an attempt was made to estimate B from data furnished by those Districts with a sufficient volume and range of data to support an estimate. Districts 8, 9 and 15 were selected for this purpose. Plots of Q and TDI data from those Districts are shown in Figures 8, 9, and 10.

Unfortunately, the "least-squares" regression technique normally used for estimating slopes was not strictly applicable to the problem because both variables (Q and TDI) were known to involve sizable errors, while use of the standard method requires that one variable be without appreciable error. As an aid to surmounting this difficulty, the authors used a technique that takes account of measurement errors in all variables entering into the analysis. The method is described in detail in Appendix C. (In what follows, the method of Appendix C will be referred to as the "multiple error" method to distinguish it from the standard "least-squares" method.)

In each of Figures 8, 9, and 10, regression lines for both the standard and the multiple error methods are shown. From a study of these figures, as well as Figure 7, it was felt that the slope of a regression line for "error free" data (had

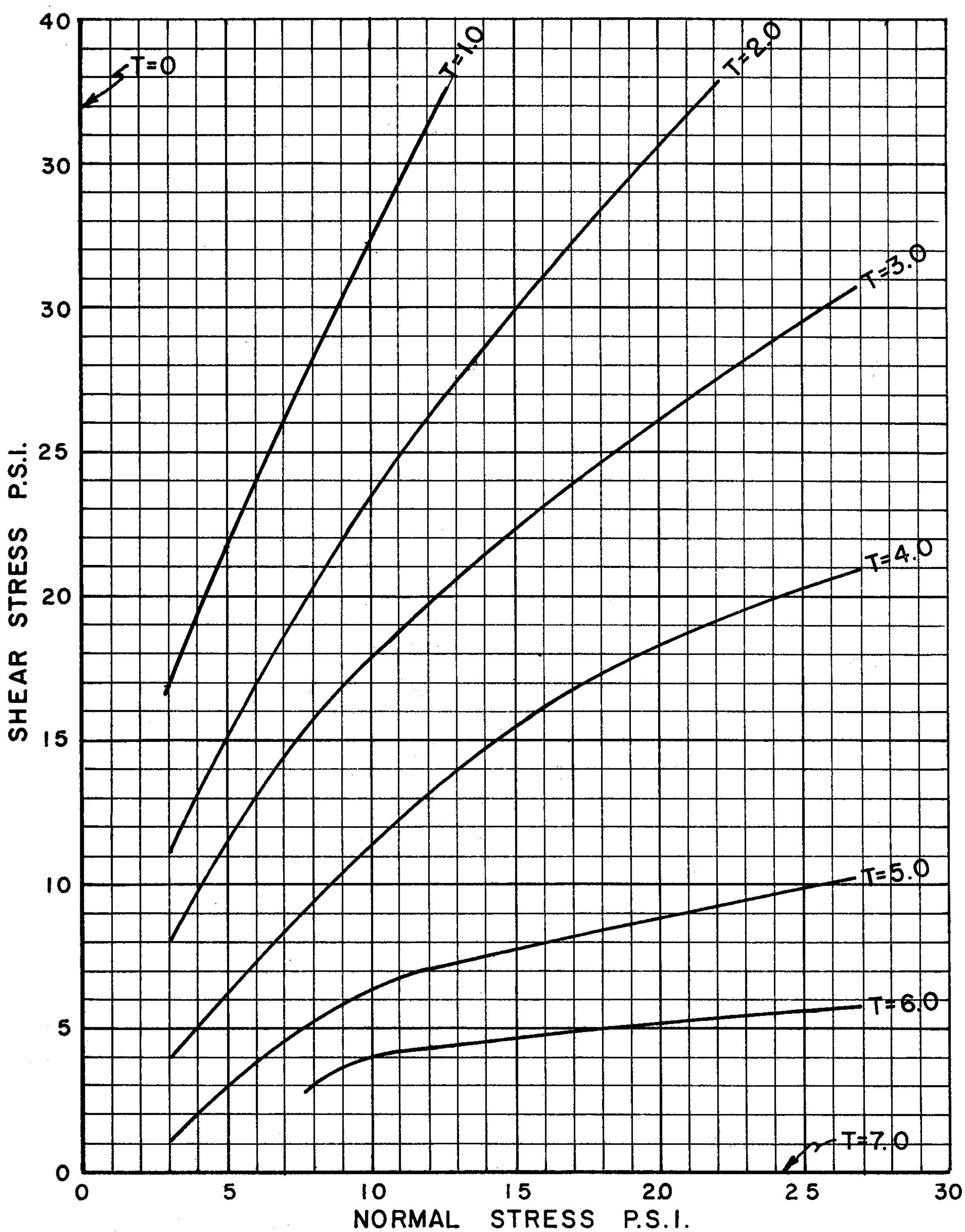



FIGURE 6: RECOMMENDED FLEXIBLE PAVEMENT MATERIALS CLASSIFICATION CHART (DIFFERS FROM FIGURE 5 IN THAT A CLASS 1 LINE HAS BEEN ADDED).

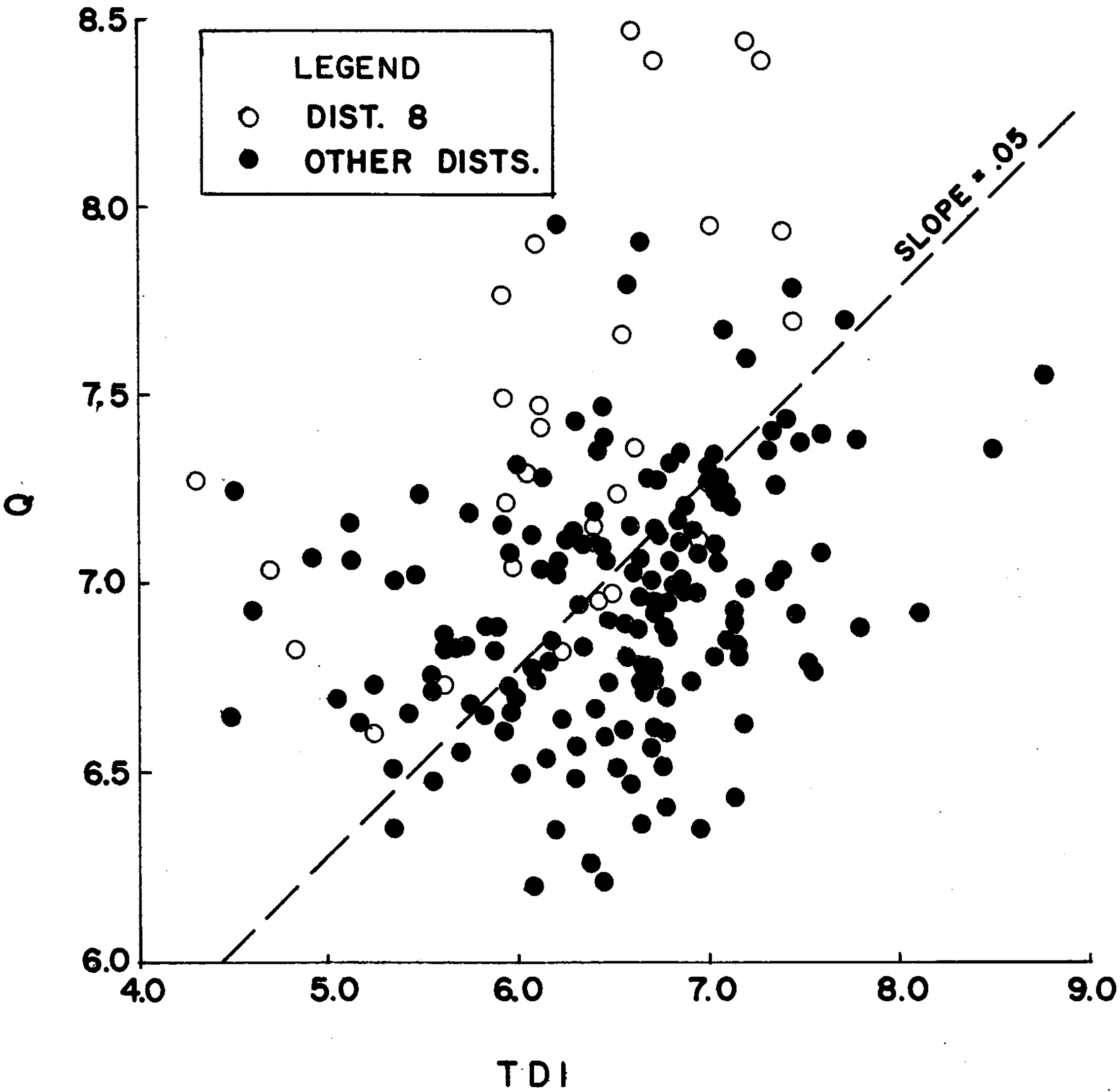



FIGURE 7: Q VS TDI (DATA NOT CORRECTED FOR REGIONAL EFFECT; COMPARE WITH FIGURE 15 WHICH IS A PLOT OF DATA CORRECTED FOR REGIONAL EFFECT).

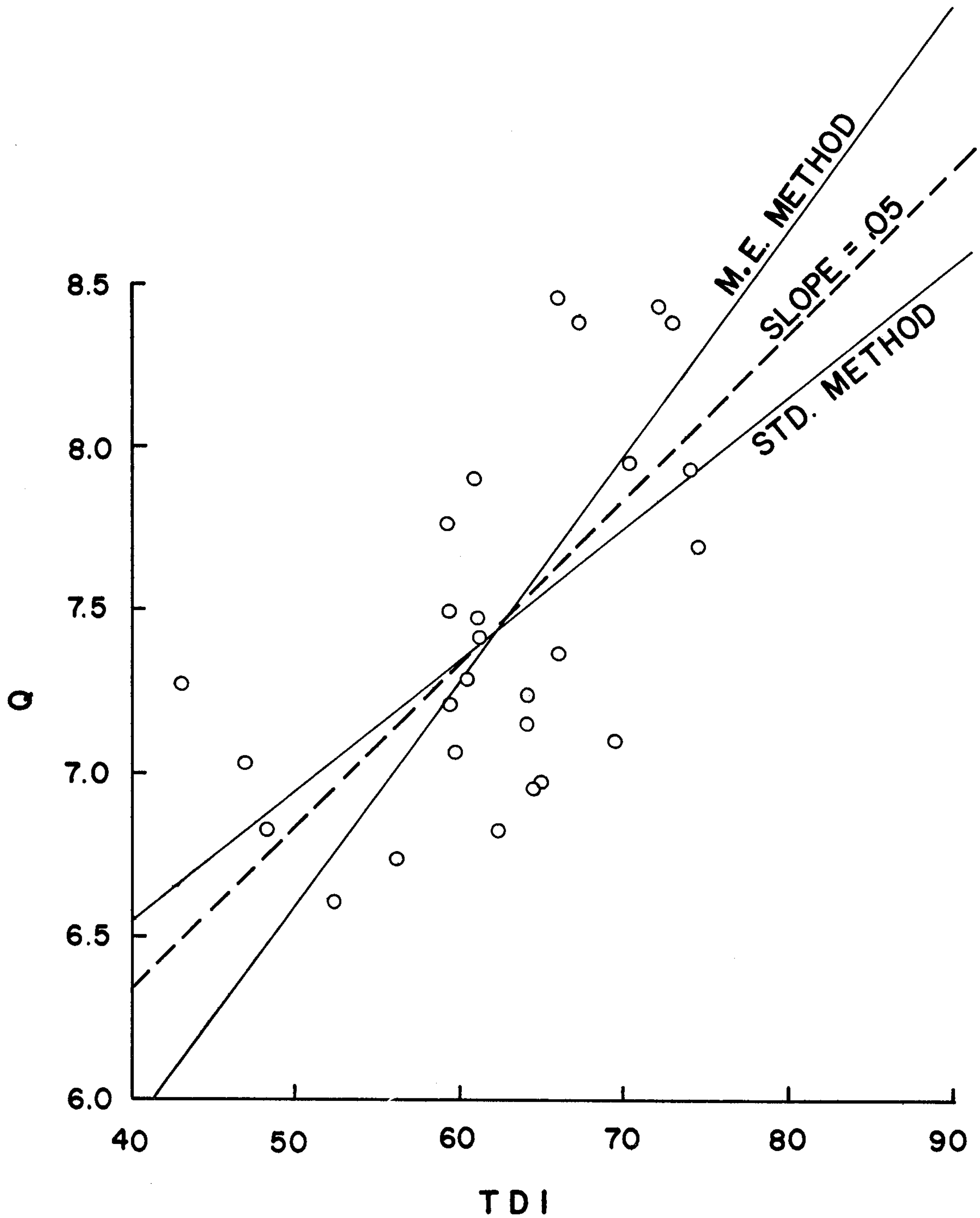



FIGURE 8: COMPARISON OF DISTRICT 8 STANDARD AND M. E. REGRESSION LINES (FROM THIS GRAPH AND FIGURES 9 AND 10. A SLOPE OF .05 WAS SELECTED FOR ALL REGIONS).

they been available) would lie somewhere between the limiting values indicated in the figures. The slope assigned to B was 0.05, as indicated by the dashed lines in Figures 7 through 10.

Estimates of Regional Constants from Individual Sections: By substituting .05 for B in Equation 3, and rearranging terms, that equation was rewritten as follows:

$$\hat{C}r = Q - .05 \times TDI \tag{6}$$

where $\hat{C}r$ = an estimate of Cr based on data from a single test section in the $r^{th}$ region.

Equation 6 provided a means for estimating a regional constant from the data provided by each of the 188 sections. The values so computed could then be examined for any correlation with location.

Contours of $\hat{C}r$: As a means for estimating, visually at least, the degree of correlation (if any) existing between $\hat{C}r$ and section location, each value computed from Equation 6 was written on a map of Texas at the location of the corresponding section. Contours of $\hat{C}r$ were then traced on the map, as indicated in Figure 11.

The fact that it was possible to draw the contours was taken as evidence that a location effect existed; for if the values of $\hat{C}r$ had been distributed at random (an indication of no regional effect), or if $\hat{C}r$ had been practically constant throughout (another indication of no regional effect), it would not have been possible to construct contours.

Designation of Regions: Within any area bounded by consecutive contour lines in Figure 11, the variation of $\hat{C}r$ is small compared to its variation across the state. Thus, if one neglects these relatively small variations, and assigns a constant value to Cr equal, say, to the mean value of $\hat{C}r$ within an area bounded by consecutive contours, that area may be regarded as a region -- for a region, according to previous definitions, is an area within which Cr has a fixed value.

However, an inspection of Figure 11 will reveal that within some of the areas bounded by consecutive contours, the volume of data is relatively meager. It was decided, therefore, to combine some of these areas into a single region in order to increase the number of sections, and hence the reliability of the average $\hat{C}r$ used as the regional constant, Cr.

Combining areas also reduced the number of regions, an aid to practical application of the regional concept in design. It should be noted, however, that combining areas in no way improved the reliability of the position of the regional boundaries in areas of the state where the volume of data was small.

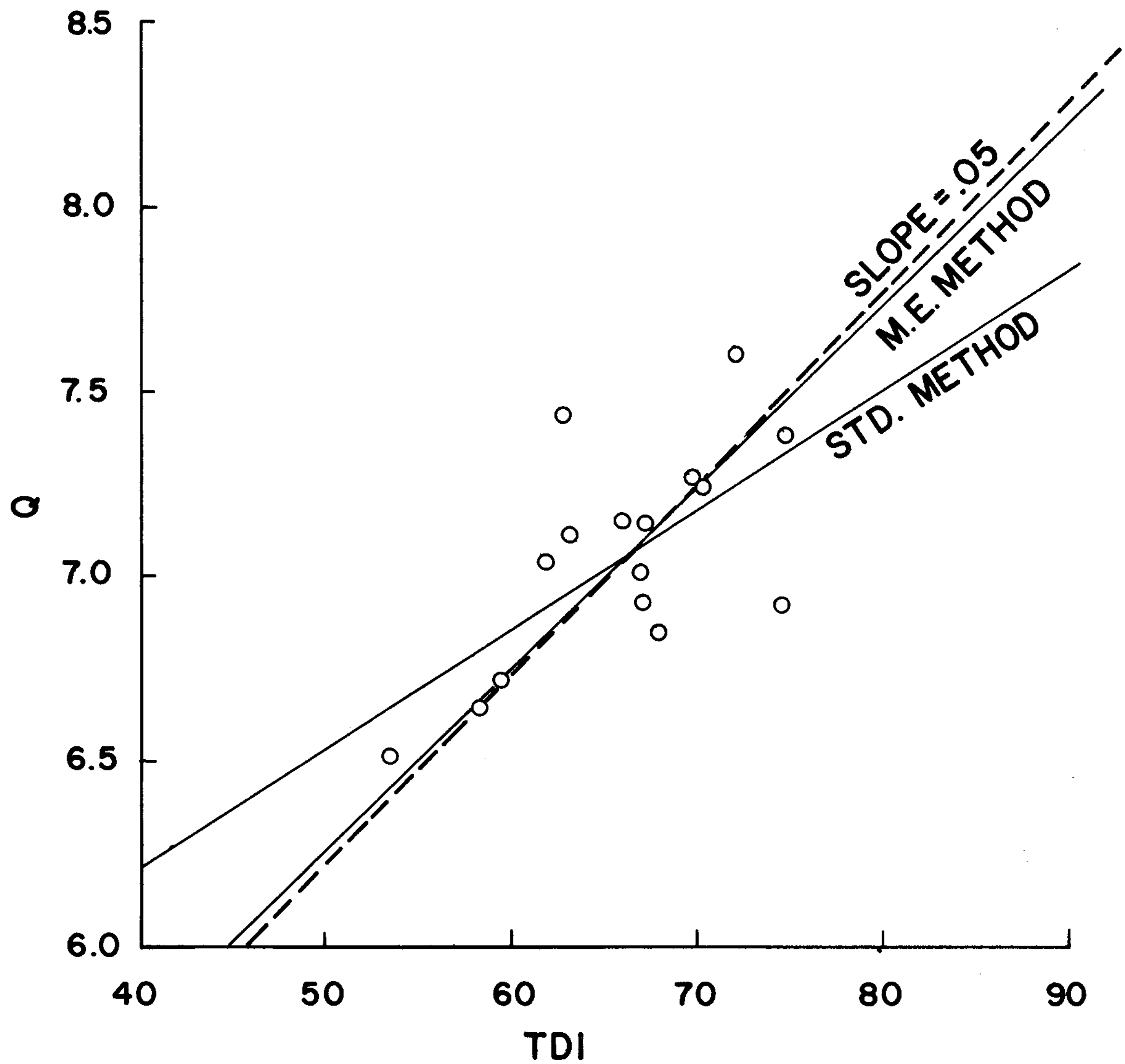



FIGURE 9: COMPARISON OF DISTRICT 9 STANDARD AND M. E. REGRESSION LINES.

The regional boundaries selected for use in connection with the analysis are shown in Figure 12. Region 2 is the area lying between the 3.6 and the 4.0 contours. Region 3, made up of two disconnected areas, is bounded by the 4.0 contours, and includes sections for which $\hat{Cr}$ is usually greater than 4.0. Region 1, also made up of two noncontiguous areas, is bounded by the 3.6 contour, and includes sections for which $\hat{Cr}$ is usually less than 3.6.

In transferring the 3.6 and 4.0 contours from Figure 11 to Figure 12, some irregularities were smoothed. Regional averages and other data used below and thereafter are based on the smoothed contours of Figure 12.

Regional Constants: The number of test sections in each region, the average value of $\hat{Cr}$ within the region, and the standard deviation of $\hat{Cr}$ about its mean within the region, are given in Table 4.

Table 4

| Region | No. Sections | Cr (Mean $\hat{Cr}$) | Standard Deviation |
|---|---|---|---|
| 1 | 62 | 3.36 | .26 |
| 2 | 80 | 3.80 | .24 |
| 3 | 46 | 4.36 | .37 |
| All | 188 | 3.80 | .47 |

Distribution of $\hat{Cr}$ Within Regions: Histograms showing the distribution of $\hat{Cr}$ in each of the three regions are shown in Figure 13. Also shown in the figure is a similar diagram for the state as a whole. Approximate distribution curves, fitted to the regional block diagrams in this figure, have been redrawn on a single graph in Figure 14.

Figure 14 tends to confirm the regional effect, for if Cr had been randomly distributed across the state, the three curves would have tended to peak at the same point. Instead their peaks are well separated, a circumstance that could hardly be attributed to chance considering the large number of sections in each region.

While it appears difficult to deny the regional effect, the reasons for the effect are not clear and warrant further investigation. No correlation between the three regions and geological, soil or rainfall maps could be found. Other factors that could cause the observed regional effect include differences in construction procedures, differences in testing procedures, differences in triaxial class estimating procedures, and interactions between these factors.

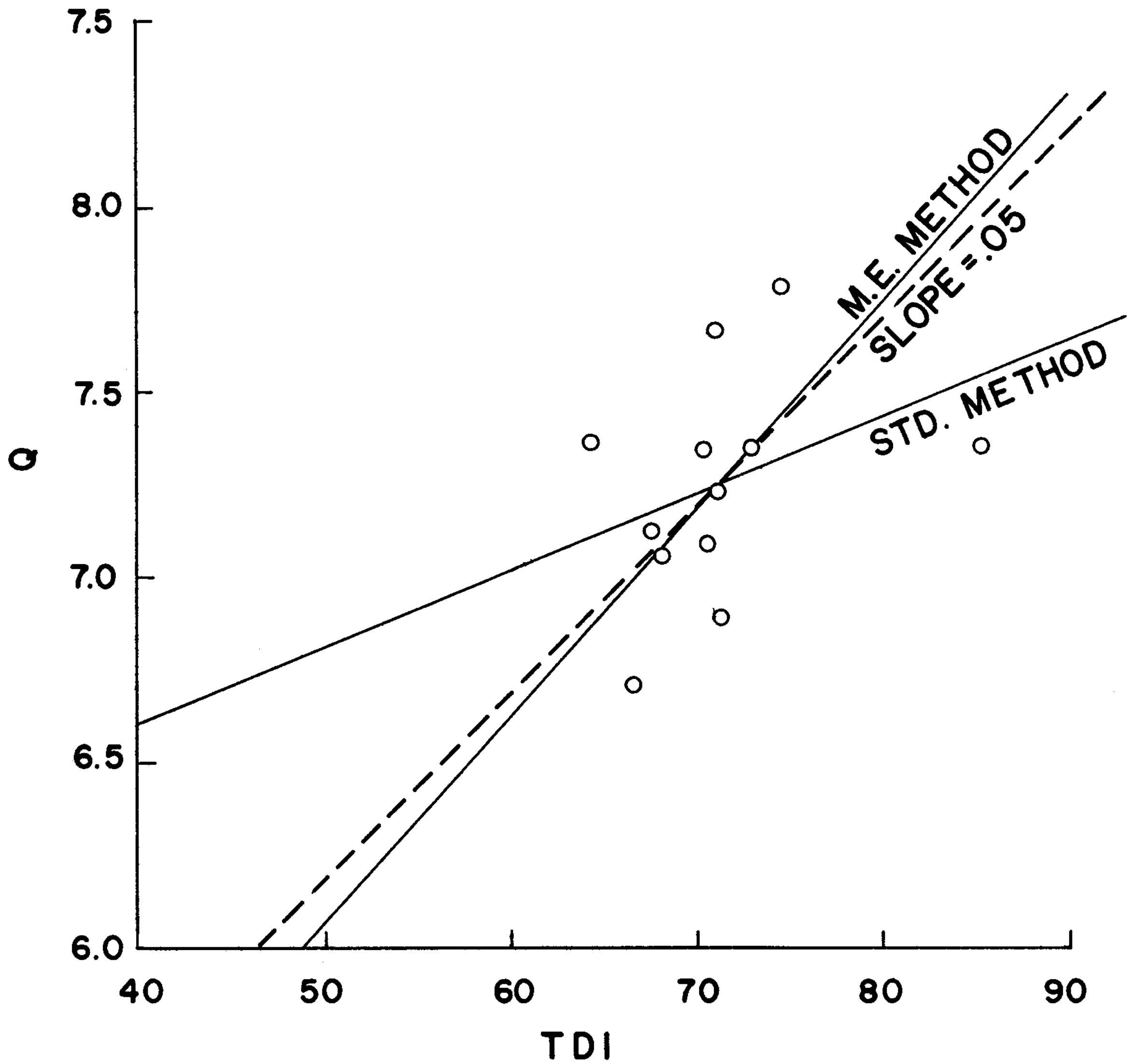
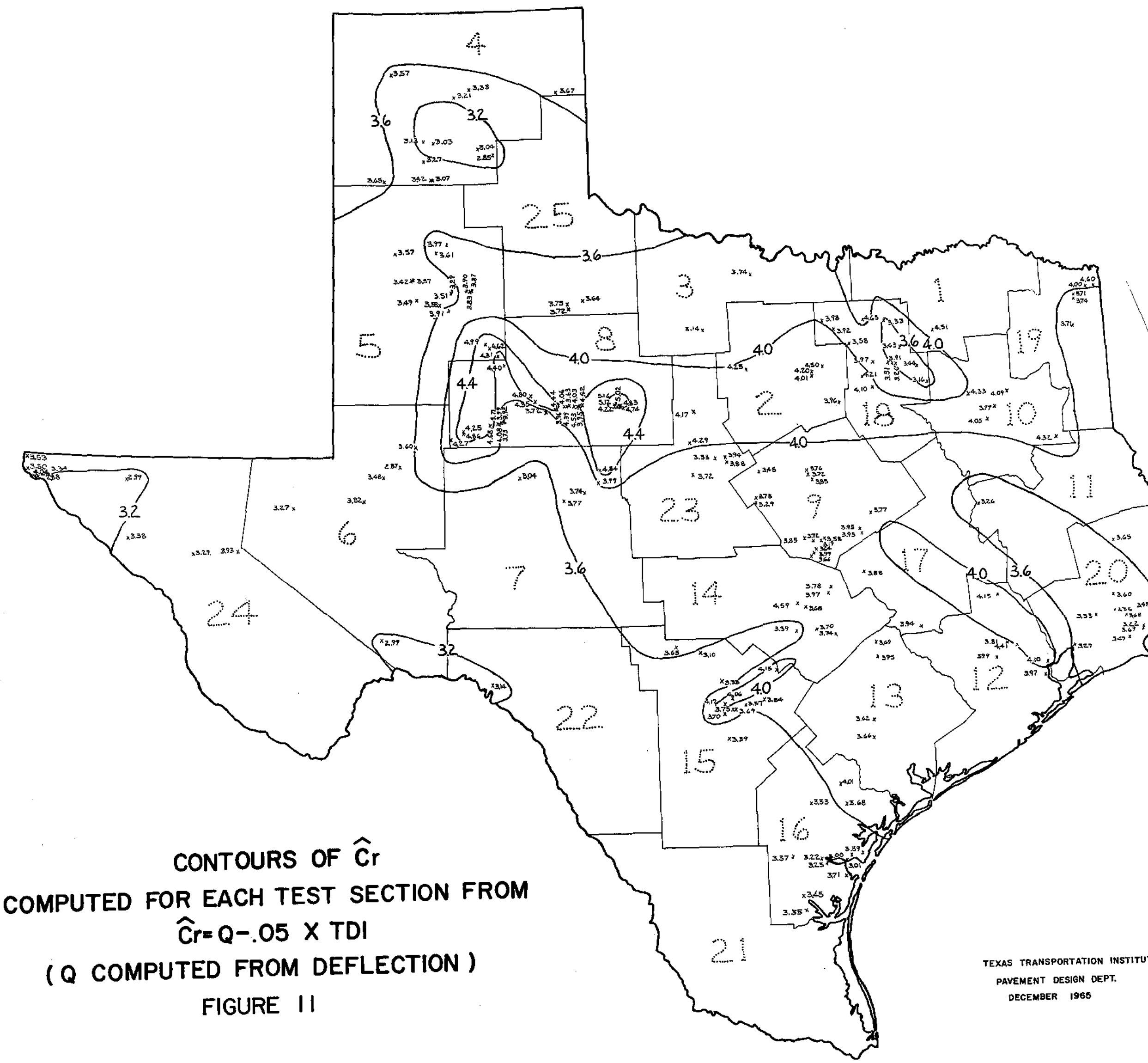


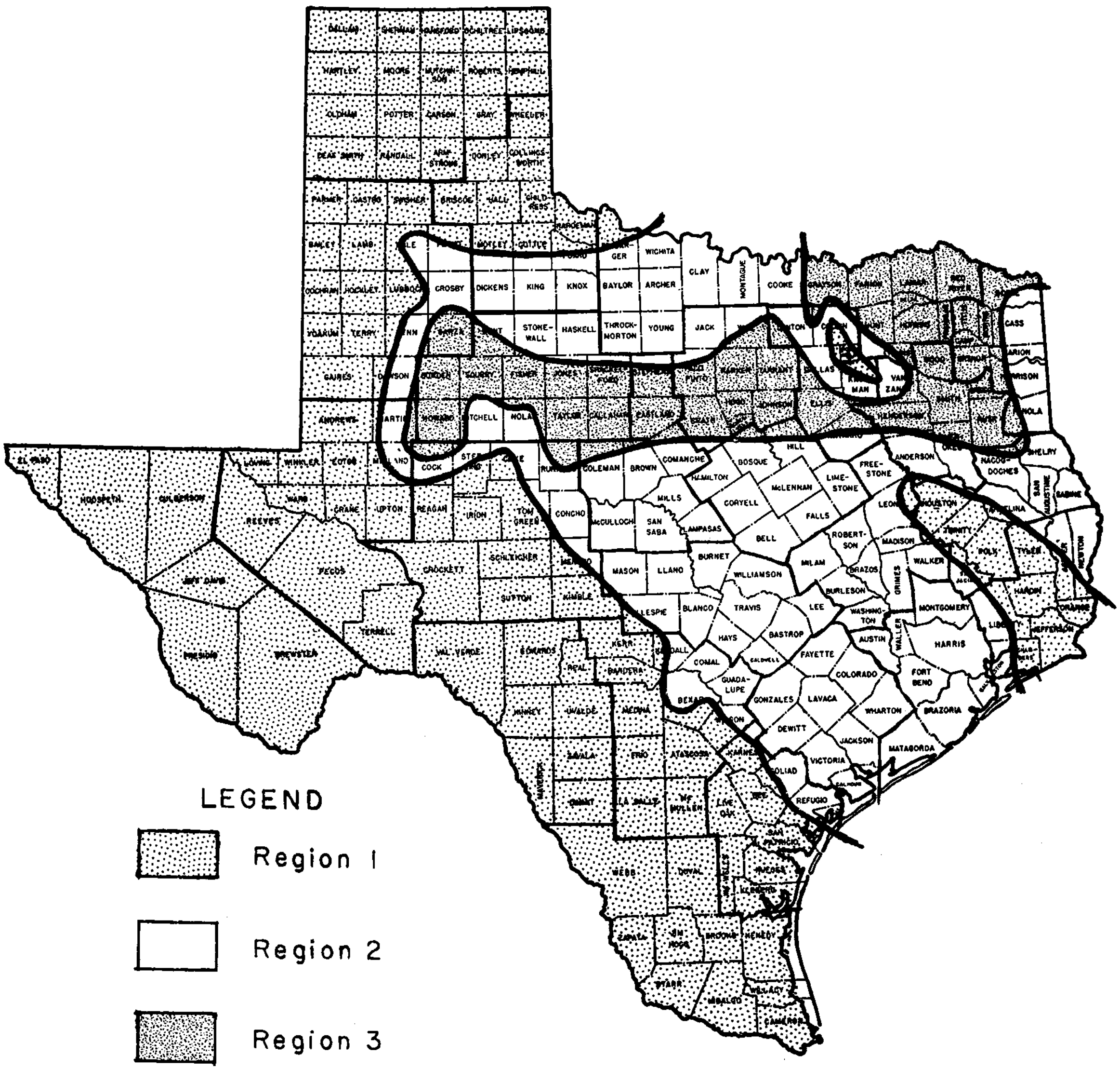

FIGURE 10: COMPARISON OF DISTRICT 15 STANDARD AND M. E. REGRESSION LINES.




CONTOURS OF $\hat{C}r$
COMPUTED FOR EACH TEST SECTION FROM
$\hat{C}r = Q - .05 \times TDI$
( Q COMPUTED FROM DEFLECTION )
FIGURE 11

TEXAS TRANSPORTATION INSTITUTE
PAVEMENT DESIGN DEPT.
DECEMBER 1965

Texas Transportation Institute
Pavement Design Dept.
December, 1965




REGIONS OF SIMILAR PAVEMENT PERFORMANCE
FIGURE 12

## 12. Analysis of Texas Data

In the preceding sections we have described many subjective decisions that were made in the steps that have brought us to this point in the report. Among these steps are the following:

1. Selection of a function to represent pavement performance (Equation 1, Section 4).

2. The substitution of a function of deflection for the pavement performance function (Equation 2, Section 5).

3. Formulation of an assumed relationship between performance, design, and region (Equation 3, Section 6).

4. Creation of an index (the TDI) to represent pavement design (Equation G, Section 8).

5. The modification of the Texas Highway Department's classification chart (Figure 6, Section 9).

6. The introduction of an arbitrary relationship between Triaxial Class, T, and strength, S (Equation I, Sections 8 and 9).

7. The choice of a form for F(S), and of values for the constants, a and r, in the general TDI equation (Section 10).

8. The estimate of the slope, B, in the performance Equation (Section 11).

9. The division of the state into regions of similar pavement performance (Section 11).

In this section we shall attempt to show -- this time with objectivity -- that the design equation resulting from the steps listed above is supported by the data.

We begin by writing Equation 3,(the design equation) in the following form as an hypothesis to be tested by the data from the 188 sections selected for analysis:

$$Q - Cr = B \times TDI \tag{7}$$

where Q is to be estimated from deflection data by means of Equation 2;

Cr = 3.36 for sections in Region 1, 3.80 for sections in Region 2, and 4.36 for Sections in Region 3;

Regions 1, 2 and 3 are the areas designated on the map reproduced in Figure 13;

TDI is to be computed from layer strength and thickness data by means of Equations H and I, and

B = a constant = .05.

To test the hypothesis of Equation 7, a regression analysis was performed using the following model:

$$Y = A_0 + A_1X \tag{8}$$

where Y = Q - Cr, and

X = TDI.

Since it was known that measurement errors are associated with both X and y, use was made of the "perpendicular error" regression technique described in Appendix C.

The results of the analysis are given below:

Table 5

$A_0$ = 0.00085

$A_1$ = 0.05001

rmsr = 0.2831

R = 0.723

From the data in Table 5 it was concluded that $A_0$ was not significantly different from zero and $A_1$ was not significantly different from .05. Though the root-mean-square-residual (or the standard deviation of the errors in Y estimated from X) was high, and the correlation coefficient, R, was low, it was felt that the analysis was sufficiently promising to warrant a progress report.

Substituting the values of zero and .05 for $A_0$ and $A_1$ in the regression model (Equation 8), we obtain

$$Y = .05X$$

Hence,

$$Q - Cr = .05 \times TDI, \tag{7}$$

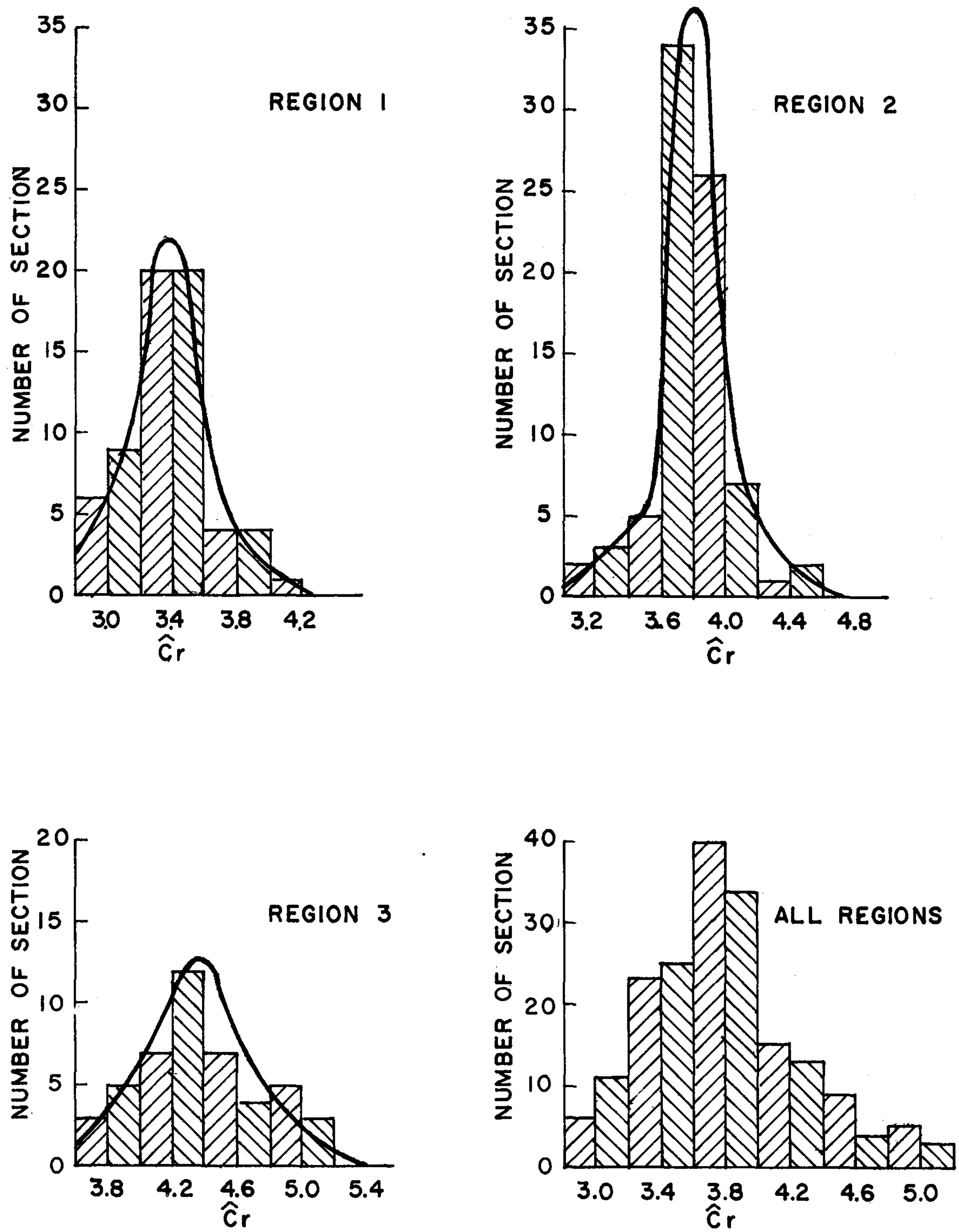



FIGURE 13: HISTOGRAMS SHOWING DISTRIBUTION OF $\hat{C}r$.

or

$$Q = .05 \times TDI + Cr. \qquad (3)$$

Thus the hypothesis has been substantiated.

Figure 15 is a plot of the data used in the analysis, and includes the regression line defined by Equation 7. The dashed lines are plotted above and below the regression line at a distance of one standard deviation. Approximately two-thirds of the data points are included in the band formed by the dashed lines.

While considerable scattering of the data is still present in Figure 15, the improvement in the scatter resulting from recognition of the regional effect can be seen by comparing this figure with Figure 7.

Figures 16, 17 and 18 are plots of the data from the individual regions. A separate regression could have been performed within each region. However, since the volume of data available within a single region would have been less than half that available throughout the state, it was felt that the combined analysis would yield more dependable results. The lines shown in the regional plots (Figures 16, 17 and 18) were predicted by Equation 7.

In Figure 19, the three regional lines predicted by Equation 7 have been plotted on a single graph for comparison. The hatched areas are two standard deviations in width (measured vertically). This figure, like the standard deviation of $\hat{Cr}$ data presented in Table 4, indicates some overlap between regions.

## 13. Summary of Use of Equations

In solving a particular design problem various equations developed in the report would be used in the following order:

1. Given (a) the expected traffic to be provided for in terms of the total equivalent number of 18-kip single axle loads, W, and (b) the initial and terminal serviceability index, $P_0$ and P, find Q from

$$Q = \log W - \log\log (P_0/P)$$

2. From the map of Figure 12 find the Region number. From the Region number and the table below find Cr.

Table 6

| Region | Cr |
|---|---|
| 1 | 3.36 |
| 2 | 3.80 |
| 3 | 4.36 |

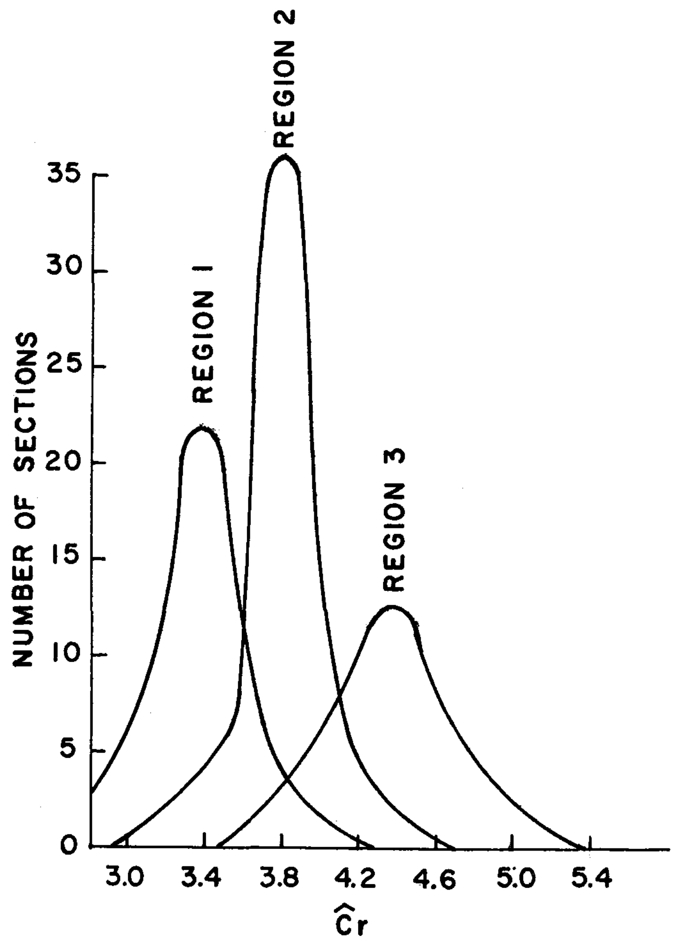



FIGURE 14: THE FREQUENCY CURVES OF FIGURE 13 REPLOTTED ON A SINGLE GRAPH. SEPARATION OF THE PEAKS IS EVIDENCE OF REGIONAL EFFECT.

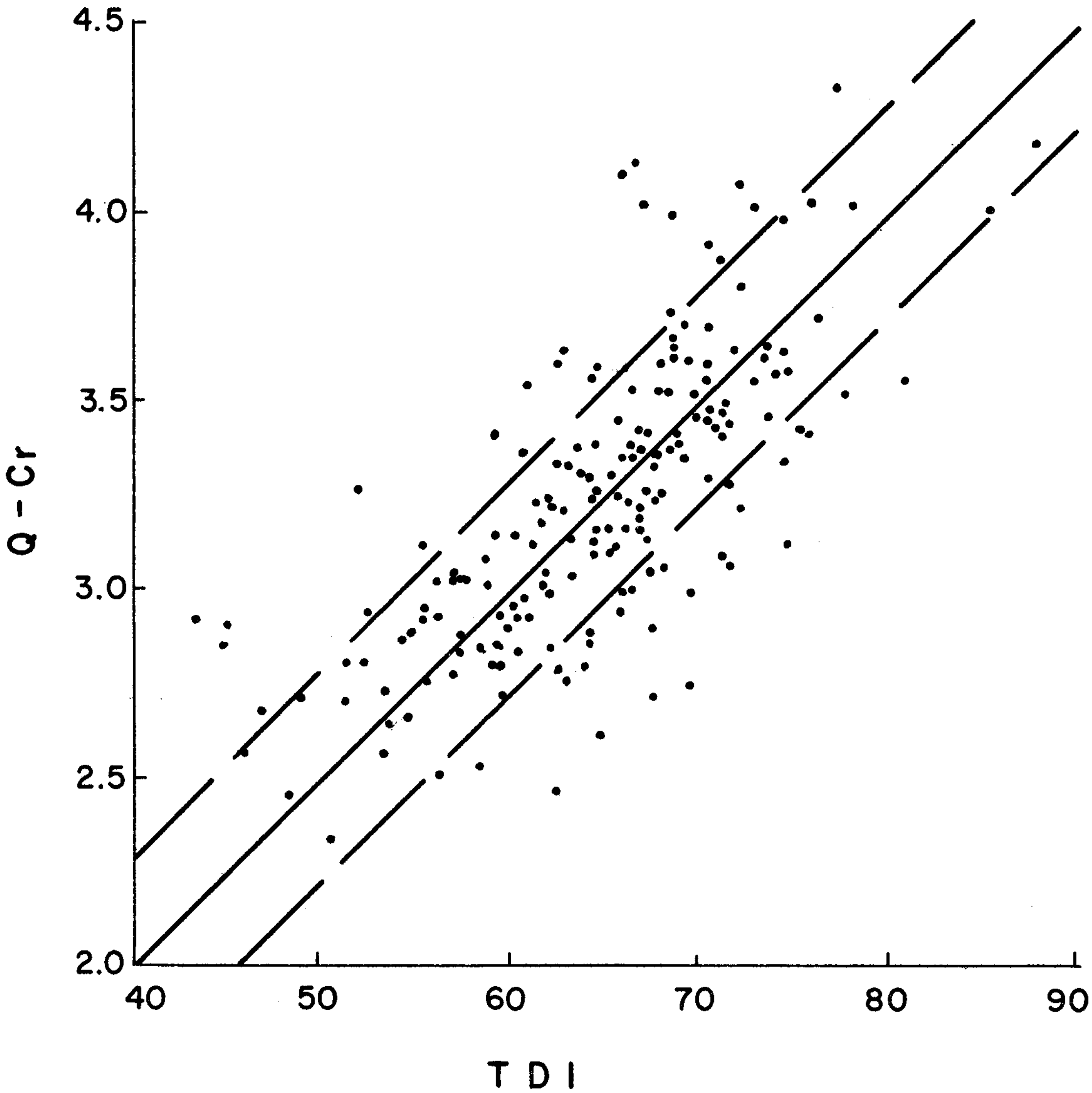



FIGURE 15: PLOT OF DATA AFTER CORRECTION FOR REGIONAL EFFECT. (COMPARE WITH FIGURE 7.)

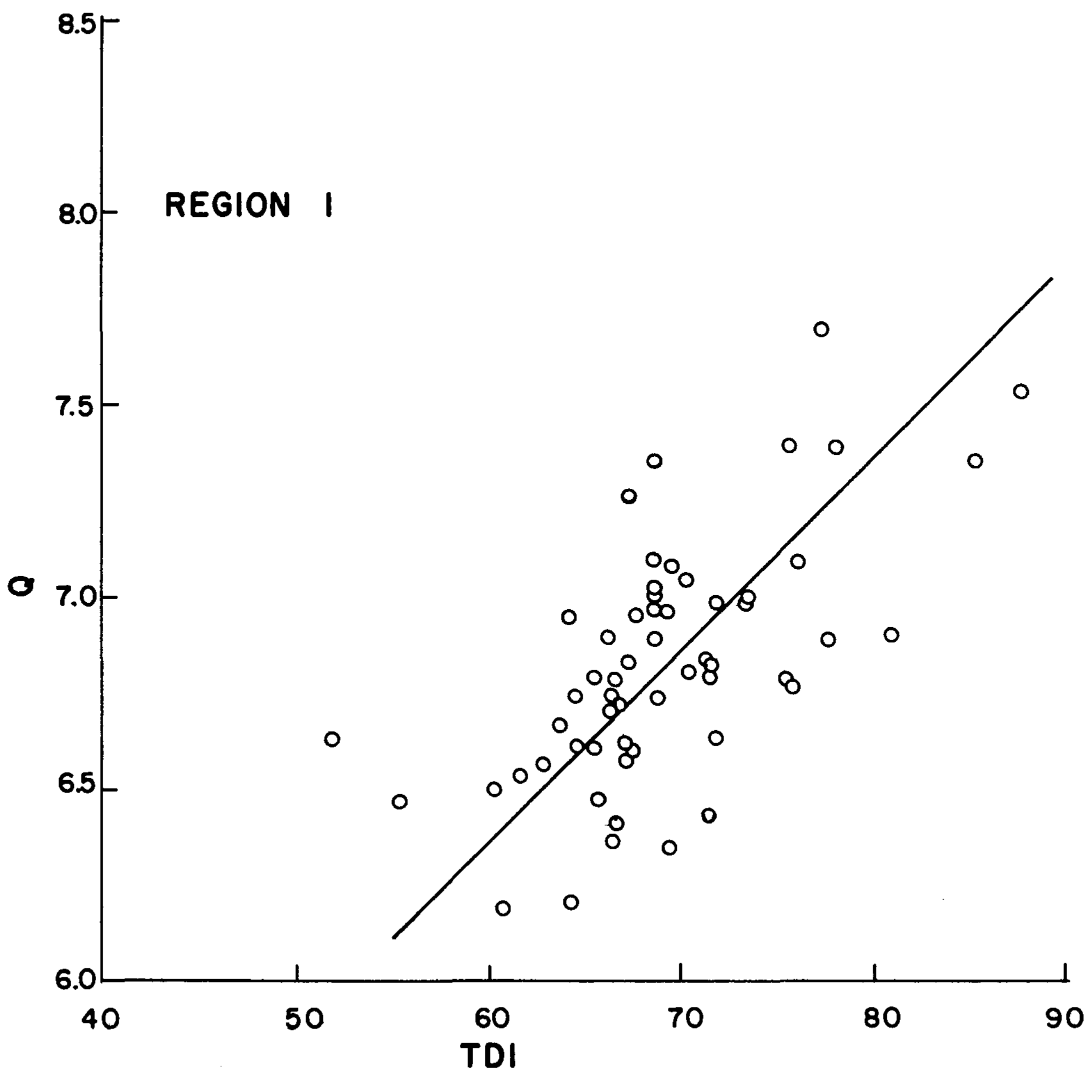



FIGURE 16: DATA FROM REGION 1, AND REGRESSION LINE PREDICTED BY EQUATION 7.

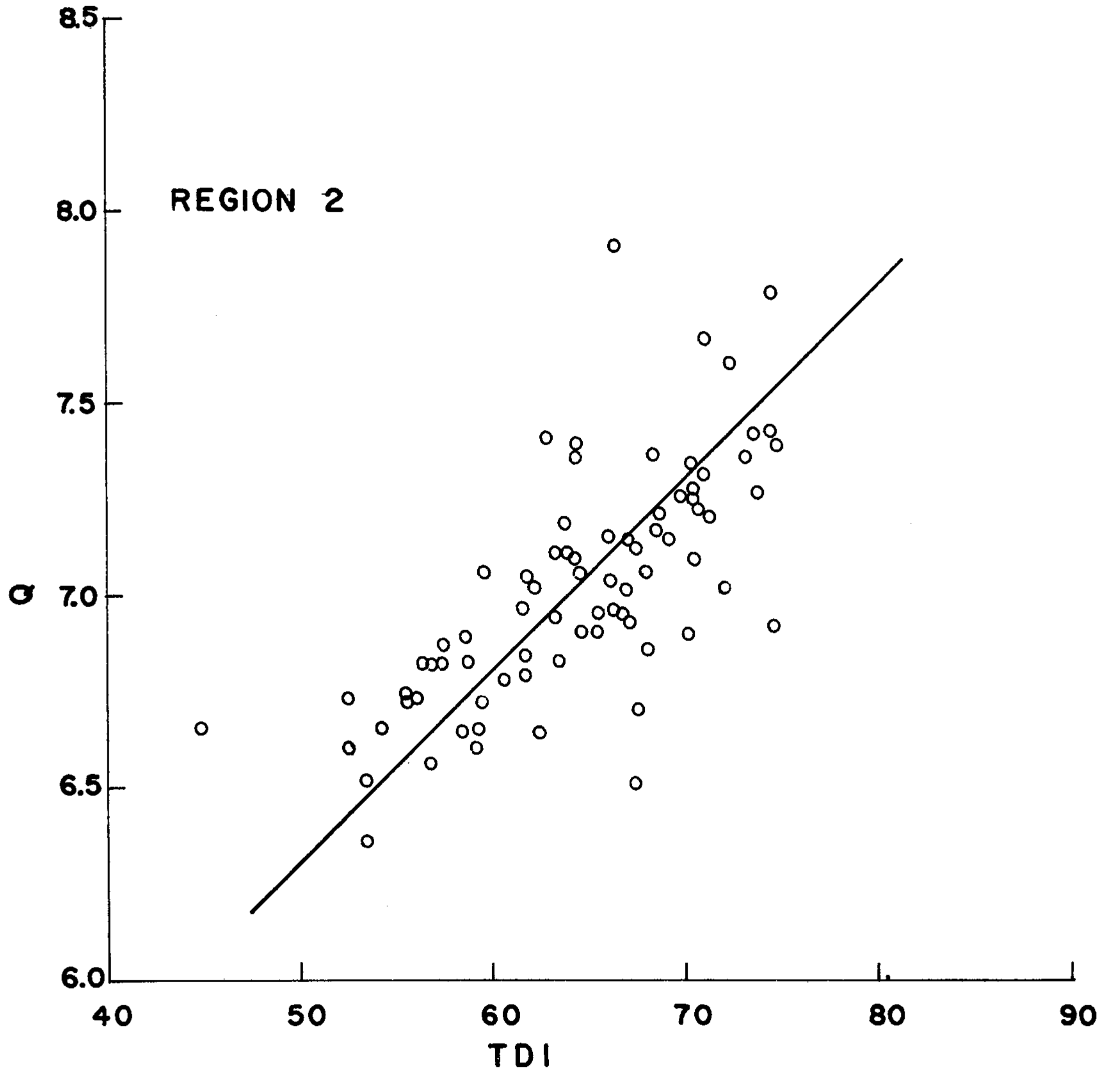



FIGURE 17: DATA FROM REGION 2 AND REGRESSION LINE PREDICTED BY EQUATION 7.

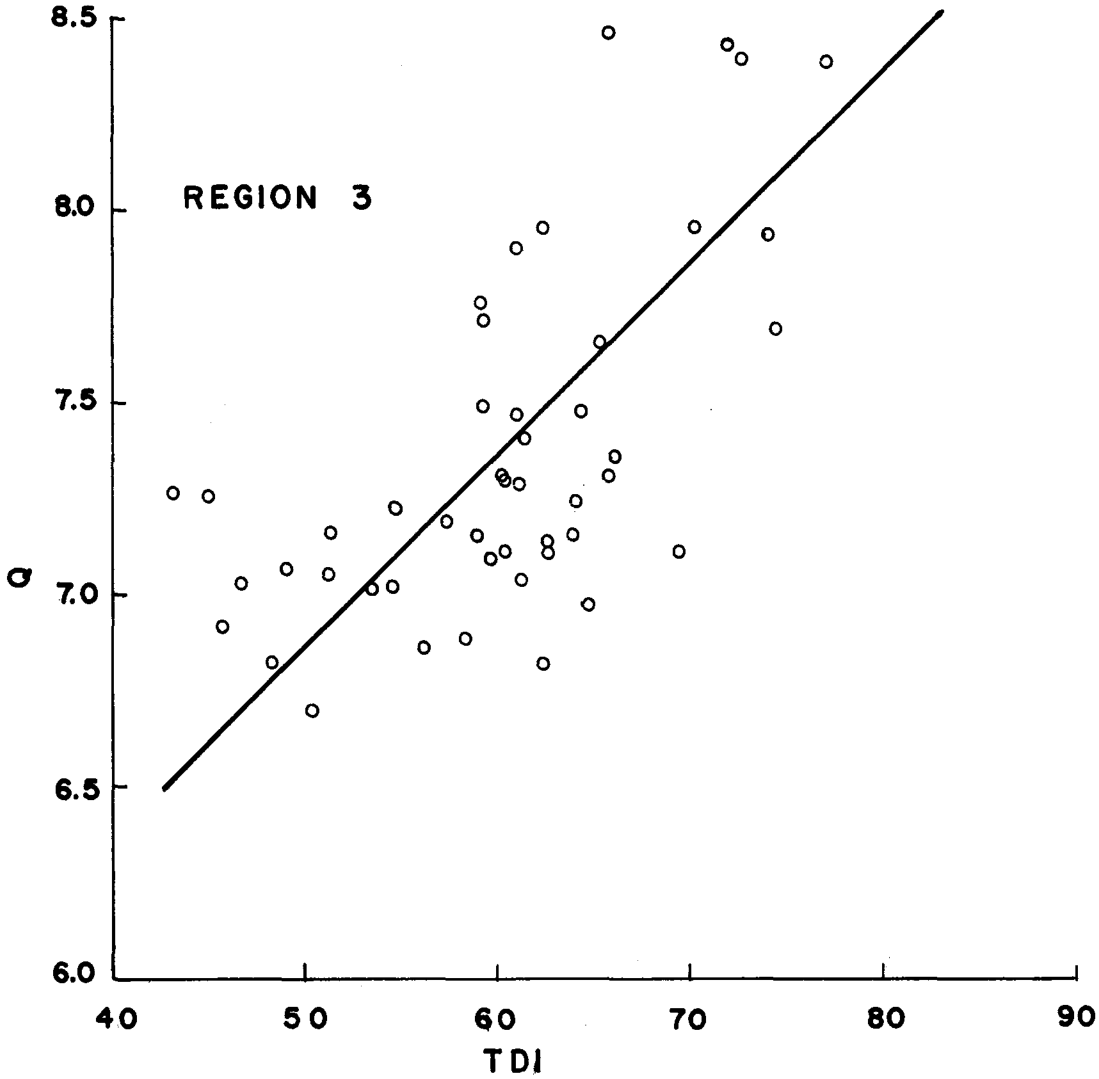



FIGURE 18: DATA FROM REGION 3 AND REGRESSION LINE PREDICTED BY EQUATION 7.

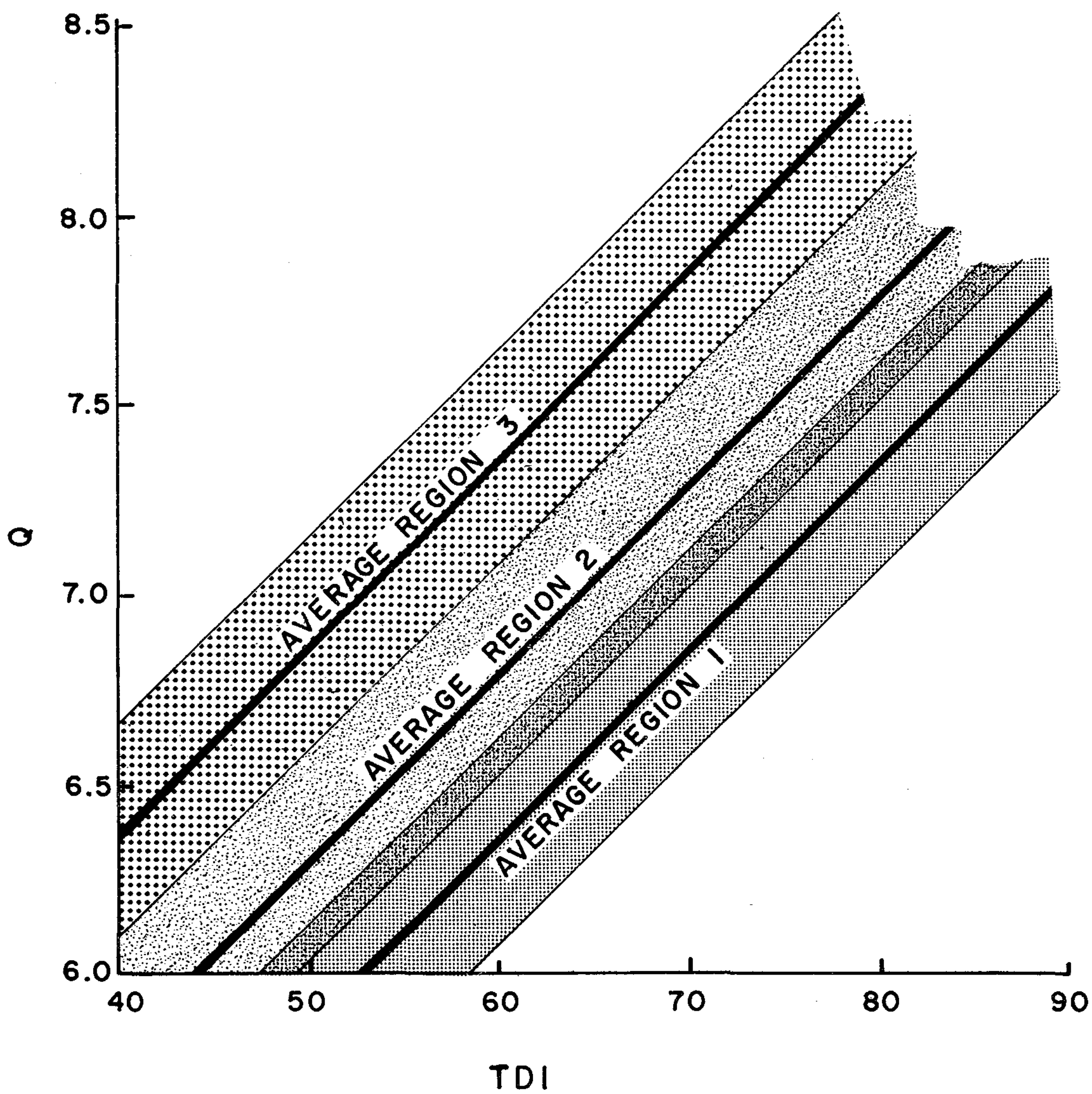



FIGURE 19: THREE REGIONAL LINES PREDICTED BY EQUATION 7 ON A SINGLE GRAPH. (HATCHED AREAS FOR EACH REGION ARE 2 STANDARD DEVIATIONS IN WIDTH. SOME OVERLAP OF REGIONS IS INDICATED.)

3. Given Q from Step 1 and Cr from Step 2, find the required TDI from

$$\text{Required TDI} = 20\,(Q - Cr)$$

4. Given the Triaxial Class for each material used in a pavement consisting of q layers (including the foundation layer), find the corresponding values of S from

$$S = \frac{50}{3}\,(7 - T)$$

where T is Triaxial Class. (Use T = 1 for asphaltic concretes.)

5. Assume values for the thicknesses, $D_1, D_2, \ldots, D_{q-1}$ of the layers above the foundation, and calculate a trial value of TDI from

$$TDI = S_1 - \frac{1000\,(S_1 - S_2)}{1000 + S_1 D_1} - \frac{1000\,(S_2 - S_3)}{1000 + S_1 D_1 + S_2 D_2} - \cdots$$

$$\cdots - \frac{1000\,(S_{q-1} - S_q)}{1000 + S_1 D_1 + S_2 D_2 + \ldots + S_{q-1} D_{q-1}}$$

6. If the computed TDI (Step 5) is not equal to the required TDI (Step 3) change thicknesses or strengths and recompute TDI from the formula given in Step 5. The following rules may assist in making changes:

(a) To increase (decrease) TDI, increase (decrease) either a thickness D or a strength, S.

(b) Making a change in a layer in the upper portion of the structure (say the surfacing or base) generally has a greater effect on TDI than changing a layer deeper in the structure (say the subbase).

Nomograms have been prepared for use in lieu of the above equations, and streamlined procedures have been developed to make the design process easier. These have been supplied to the Texas Highway Department.

## 14. Conclusions and Recommendations

1. It was concluded from the study of regional effects described in Section 11, and from the analysis and the plots presented in Section 12, that the basic design equation developed herein (Equation 3) is sufficiently reliable to warrant its investigation by the Texas Highway Department.

2. The reasons for the shapes taken by the contours of Figure 11, and of the three regions shown in Figure 12, are not clear. It therefore appears desirable to conduct additional investigations directed toward finding the cause or causes of the regional effects so clearly indicated by the data.

3. This report should be considered preliminary, and the design procedures suggested herein should be regarded as tentative, pending the termination of this project and the production of a final report.

## Appendix A

### Serviceability Loss as a Function of Deflection

The equation which expresses pavement performance as a function of deflection (Equation 2, Page 5), was derived in the manner described in this section.

First, the hypothesis was made that if a pavement is subjected to a series of deflections all of the same magnitude, then the accompanying loss in serviceability of the pavement depends only on the magnitude of the deflection and the number of times it occurs.

The mathematical form chosen to represent the hypothesis was a modified form of L. J. Painter's* equation for serviceability loss. Painter's equation follows:

$$\log(P_o/P) = bW \qquad (1A)$$

where P = the serviceability index of a pavement after W applications of an axle load of a given type (single or tandem) and weight have been applied,

$P_o$ = the serviceability index just prior to the first load application, and

b = the deterioration rate parameter.

The deterioration rate parameter, b, Painter took to be a function of the design of the pavement, the kind (single or tandem) and magnitude of the axle load, and the climatic variables. Alternatively, we assume that the deterioration rate parameter is a function of deflection only, and is of the following form:

$$b = 10^{A_o} \cdot U^{A_1} \qquad (2A)$$

where U is the magnitude of a deflection repeated W times as the serviceability index decreases from $P_o$ to P.

To test the hypothesis, and to evaluate the constants $A_o$ and $A_1$, advantage was taken of certain data available from the AASHO Road Test.

At the Road Test the serviceability index and the accumulated axle load applications were reported bi-weekly for each surviving test section.

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*Painter, L. J., "Analysis of AASHO Road Test Asphalt Pavement Data by the Asphalt Institute," Highway Research Record, Number 71, Highway Research Board, Washington, D. C., 1965.

The surface deflection of each section, induced by one of the test vehicles travelling at reduced speed, was also measured by Benkelman Beam at intervals of about two weeks. Thus, for each section, a series of nearly simultaneous values of serviceability index, deflection and accumulated axle applications were available for testing the hypothesis.

For this preliminary report the data chosen were taken from all sections in the single-axle lanes of Loops 3, 4, 5, and 6 that survived the two years of traffic testing. Twenty-one such sections were available. The design of these sections, and the magnitude of the axle loads acting on them, are given in Table 1A.

For the analysis of the Road Test data the model (Equations 1A and 2A) was written in the following form:

$$P_{i+1} = P_i \times 10^{-b \Delta w_i} \qquad (3A)$$

where $b = 10^{A0}\ \bar{U}_i^{A1}$,

$P_i$ = the serviceability index of a section at the time of the $i^{th}$ deflection determination,

$\Delta W_i$ = the number of axle applications (deflections) occurring in the time interval between the $i^{th}$ and the next deflection determination, and

$\bar{U}_i = (U_i + U_{i+1})/2$, where $U_i$ is the $i^{th}$ deflection measured. The unit of U is a thousandth of an inch.

We have assumed that the average deflection, $\bar{U}_i$, can be used in Equation 3A to represent a constant deflection acting during the interval between the measurement of $U_i$ and $U_{i+1}$, since this interval is relatively short (about two weeks as a rule).

It can be seen that Equation 3A is a recurrence formula from which P can be computed after each of a number of time intervals, if a starting value of P is given, if U and W are known for each interval, and if $A_0$ and $A_1$ are assigned **numerical** values. The variables involved in one such computation are plotted in Figure 1A.

The step-by-step procedure followed in estimating the most probable values of $A_0$ and $A_1$ is given below. All the numbered steps -- except the first -- were performed in a computer.

(1) A starting value of P for each section was selected from the data to correspond, approximately, to the start of the first (1959) spring thaw.

TABLE 1A

Design and Load Data
for the AASHO Road Test Sections
Furnishing Data Used in the
Deflection Analysis

| Figure | Loop | Lane | Single Axle Load (Kips) | $D_1$ | $D_2$ | $D_3$ | Section Number |
|---|---|---|---|---|---|---|---|
| 3A | 3 | 1 | 12 | 4 | 6 | 4 | 123 |
| 4A | 3 | 1 | 12 | 4 | 6 | 8 | 139 |
| 5A | 3 | 1 | 12 | 3 | 6 | 8 | 155 |
| 6A | 4 | 1 | 18 | 4 | 6 | 8 | 577 |
| 7A | 4 | 1 | 18 | 5 | 6 | 12 | 581 |
| 8A | 4 | 1 | 18 | 5 | 6 | 8 | 591 |
| 9A | 4 | 1 | 18 | 3 | 6 | 12 | 601 |
| 10A | 4 | 1 | 18 | 4 | 6 | 12 | 625 |
| 11A | 5 | 1 | 22.4 | 5 | 9 | 12 | 427 |
| 12A | 5 | 1 | 22.4 | 3 | 9 | 12 | 441 |
| 13A | 5 | 1 | 22.4 | 5 | 6 | 12 | 445 |
| 14A | 5 | 1 | 22.4 | 5 | 9 | 8 | 447 |
| 15A | 5 | 1 | 22.4 | 4 | 9 | 12 | 477 |
| 16A | 6 | 1 | 30 | 6 | 6 | 12 | 257 |
| 17A | 6 | 1 | 30 | 5 | 9 | 16 | 265 |
| 18A | 6 | 1 | 30 | 6 | 9 | 8 | 271 |
| 19A | 6 | 1 | 30 | 6 | 6 | 16 | 301 |
| 20A | 6 | 1 | 30 | 4 | 9 | 16 | 309 |
| 21A | 6 | 1 | 30 | 6 | 9 | 12 | 311 |
| 22A | 6 | 1 | 30 | 5 | 6 | 16 | 327 |
| 23A | 6 | 1 | 30 | 6 | 9 | 16 | 333 |

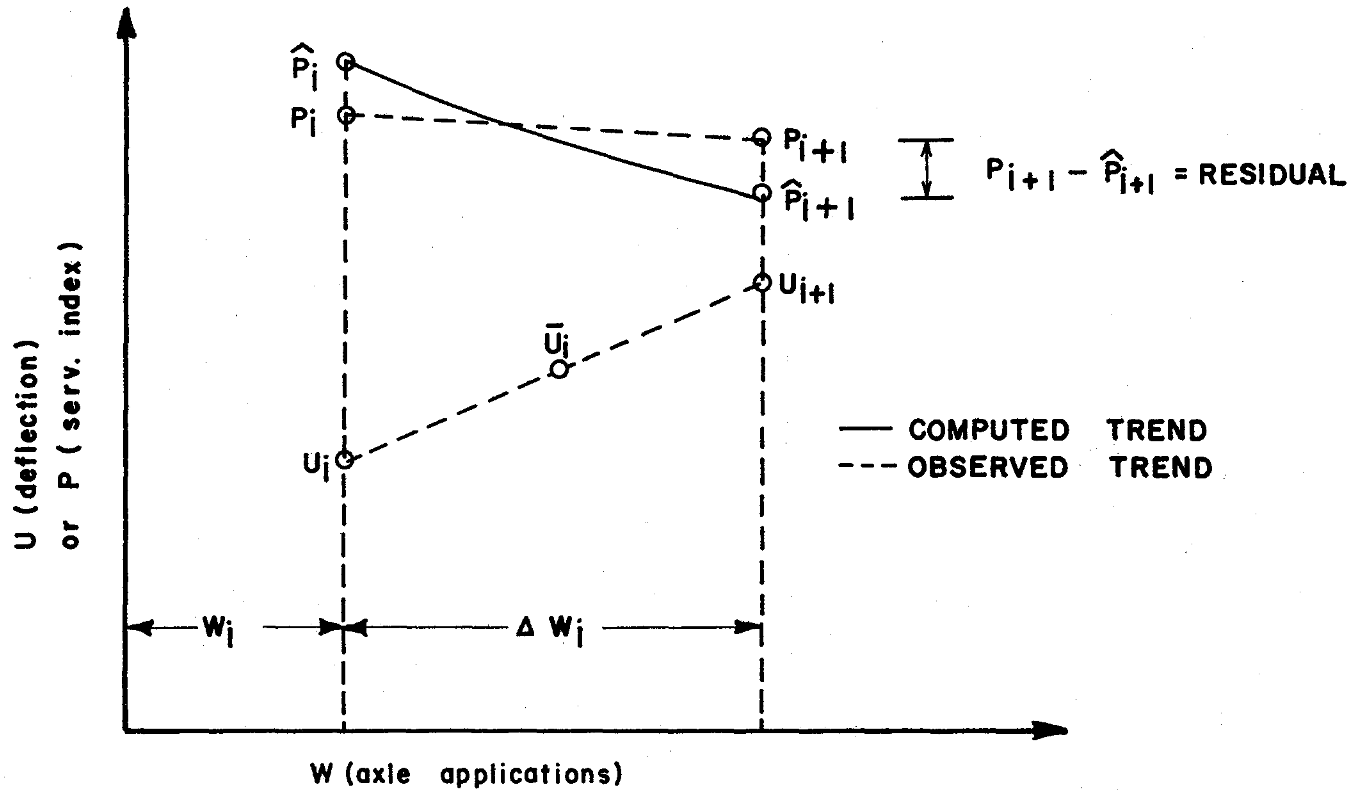



FIGURE 1A: VARIABLES APPEARING IN EQUATION 3 A.

(2) From a pre-selected set of values, a value was assigned to $A_o$.

(3) From a pre-selected range of values, a value was assigned to $A_1$.

(4) Using the values assigned above to $A_o$ and $A_1$, P was computed from Equation 3 A for the first section, for each day on which the deflection, U, of the section was measured, excepting the starting day, for which P had already been assigned (Step 1, above). The difference between the computed value and the corresponding observed value of P was computed.

(Since neither W nor P was reported on exactly the same day that the deflection was measured the "observed" values of P and W were found by interpolation).

(5) Step 4 was repeated for all sections, and the differences -- or residuals -- were summed over all sections.

(6) Steps 3 through 5 were repeated -- each time with a new value assigned to $A_1$ in accordance with a converging process -- until a value of $A_1$ was found for which the sum of the residuals differed from zero by less than 0.0001.

(7) The average absolute residual was computed and printed, together with the corresponding values of $A_o$ and $A_1$.

(8) Steps 2 through 7 were repeated until all the pre-selected values of $A_o$ had been used.

Table 2 A lists pairs of values of $A_o$ and $A_1$ determined as described above, together with the corresponding values of the average absolute residual. Figure 2 A is a plot of Average Absolute Residual versus $A_o$. Opposite each plotted point is given the value of $A_1$ corresponding to the plotted values of $A_o$ and Average Absolute Residual.

From Figure 2 A it was estimated that the least average absolute residual (about 0.3) was obtained when $A_o = -9$ and $A_1 = 3/2$. Substituting these values in Equation 3 A, we obtain

$$P_{i+1} = P_i \times 10^{-[10^{-9}(U_i)^{1.5} \Delta W_i]} \qquad (4A)$$

In Figures 3 A through 23 A predictions made from Equation 4 A are compared graphically with the observed data for the 21 test sections selected for this analysis.

Table 2A

Results of Parameter Variation Study

| Iteration No. | $A_0$ | $A_1$ | Sum of Residuals | Average Absolute Residual |
|---|---|---|---|---|
| 1 | -7 | 0.262 | 0 | 0.3418 |
| 2 | -8 | 0.882 | 0 | 0.3141 |
| 3 | -9 | 1.494 | 0 | 0.3029 |
| 4 | -10 | 2.099 | 0 | 0.3080 |
| 5 | -11 | 2.699 | 0 | 0.3246 |
| 6 | -12 | 3.293 | 0 | 0.3471 |
| 7 | -13 | 3.883 | 0 | 0.3740 |
| 8 | -14 | 4.470 | 0 | 0.4024 |
| 9 | -15 | 5.053 | 0 | 0.4310 |

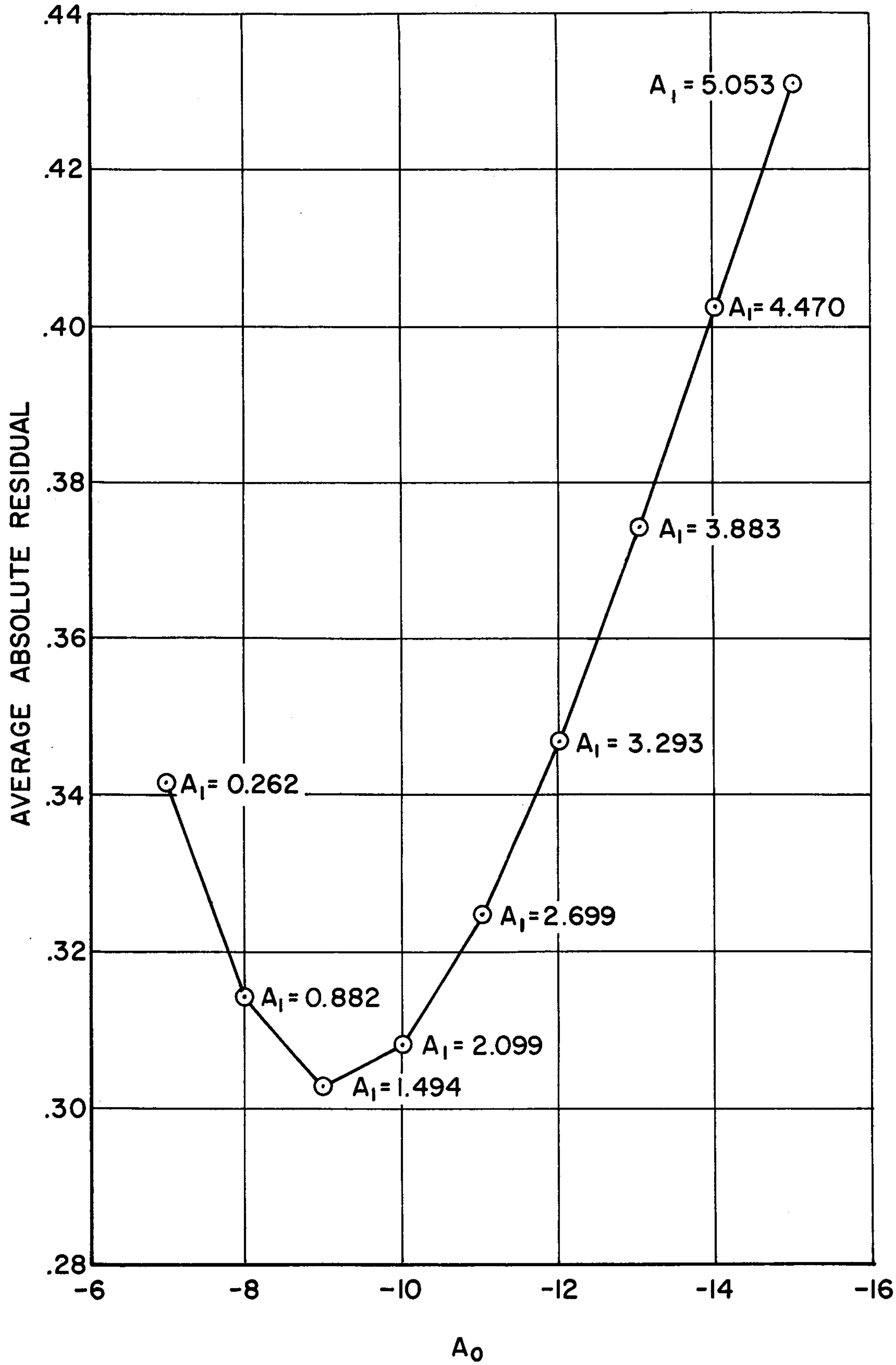
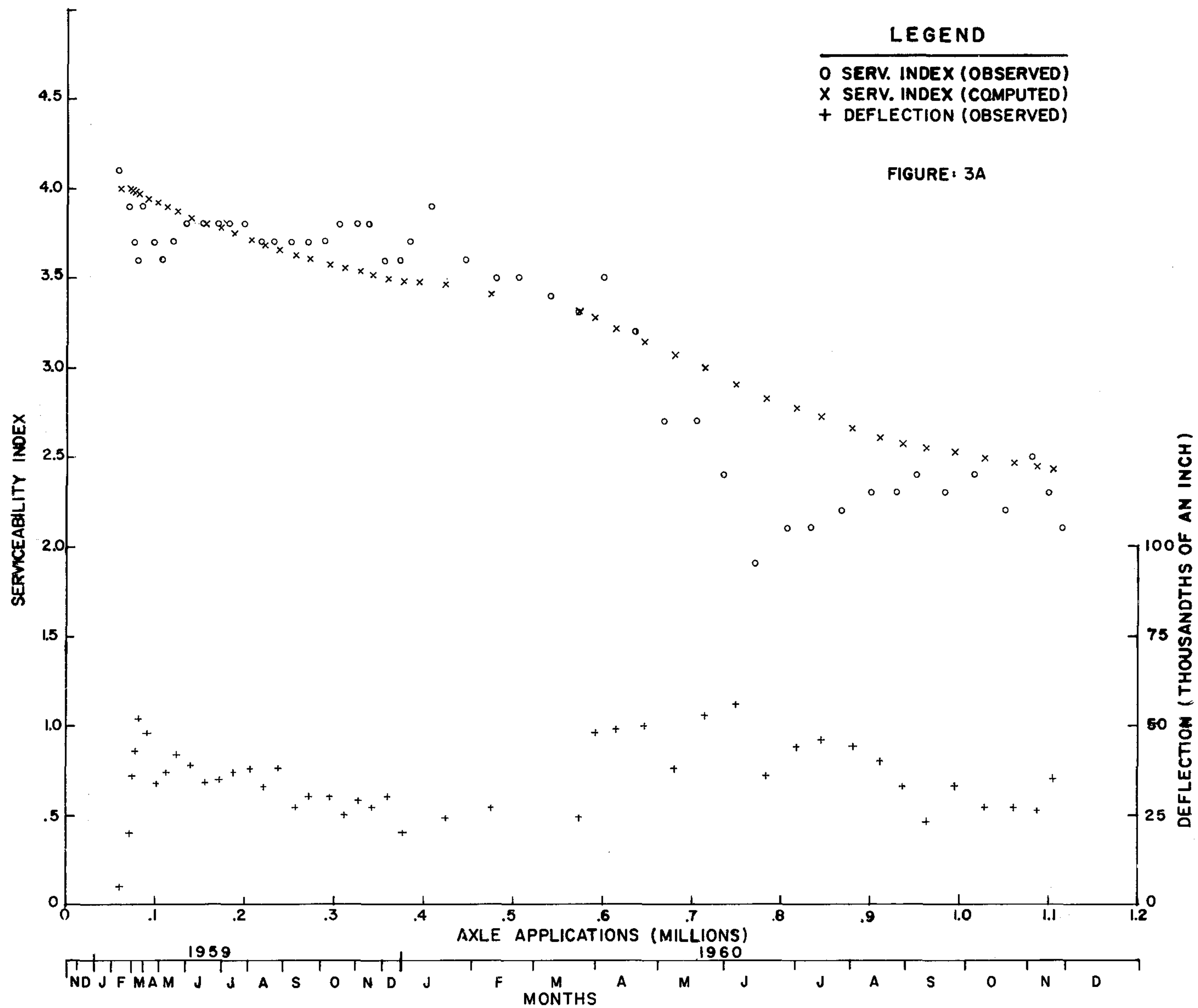
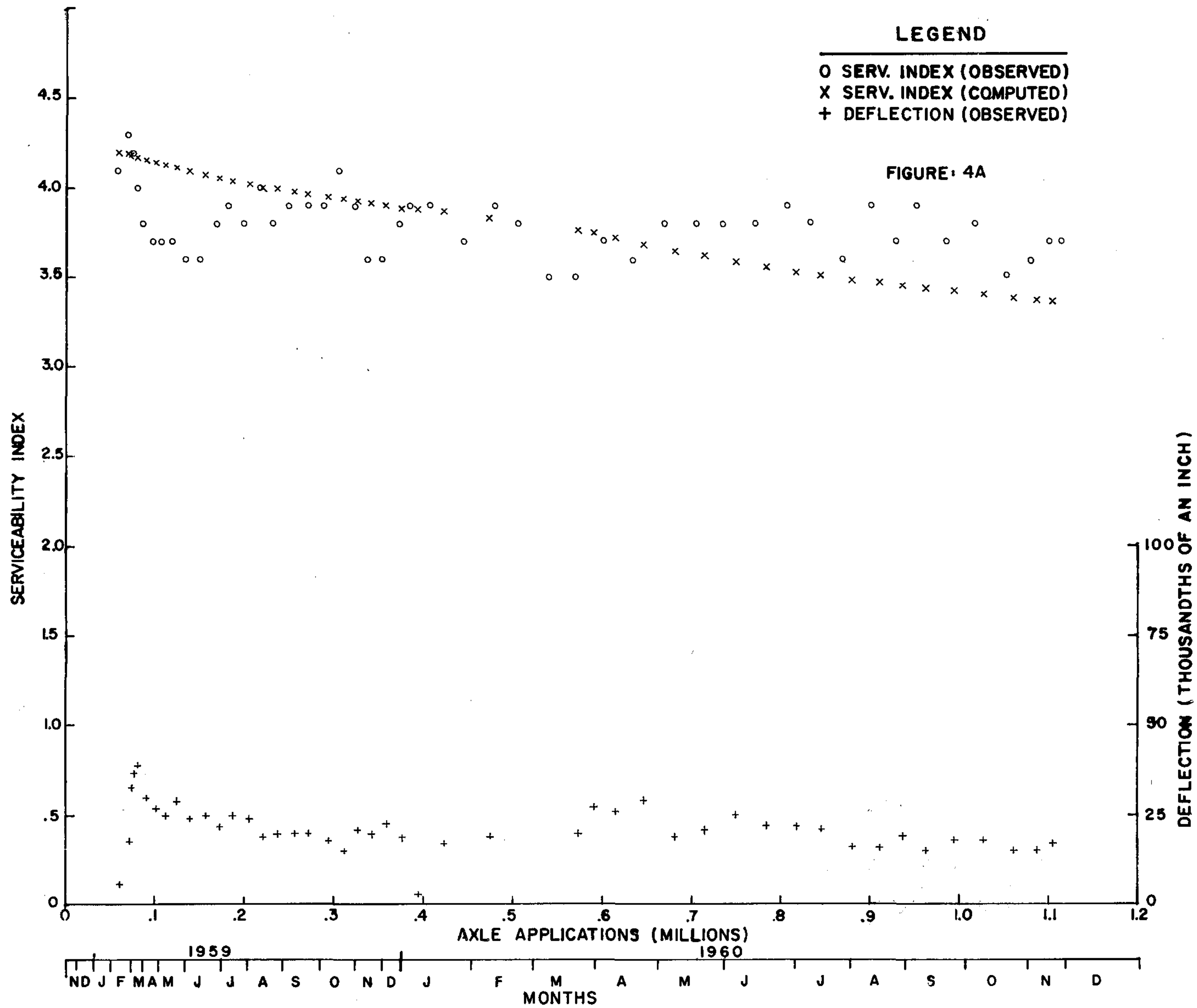
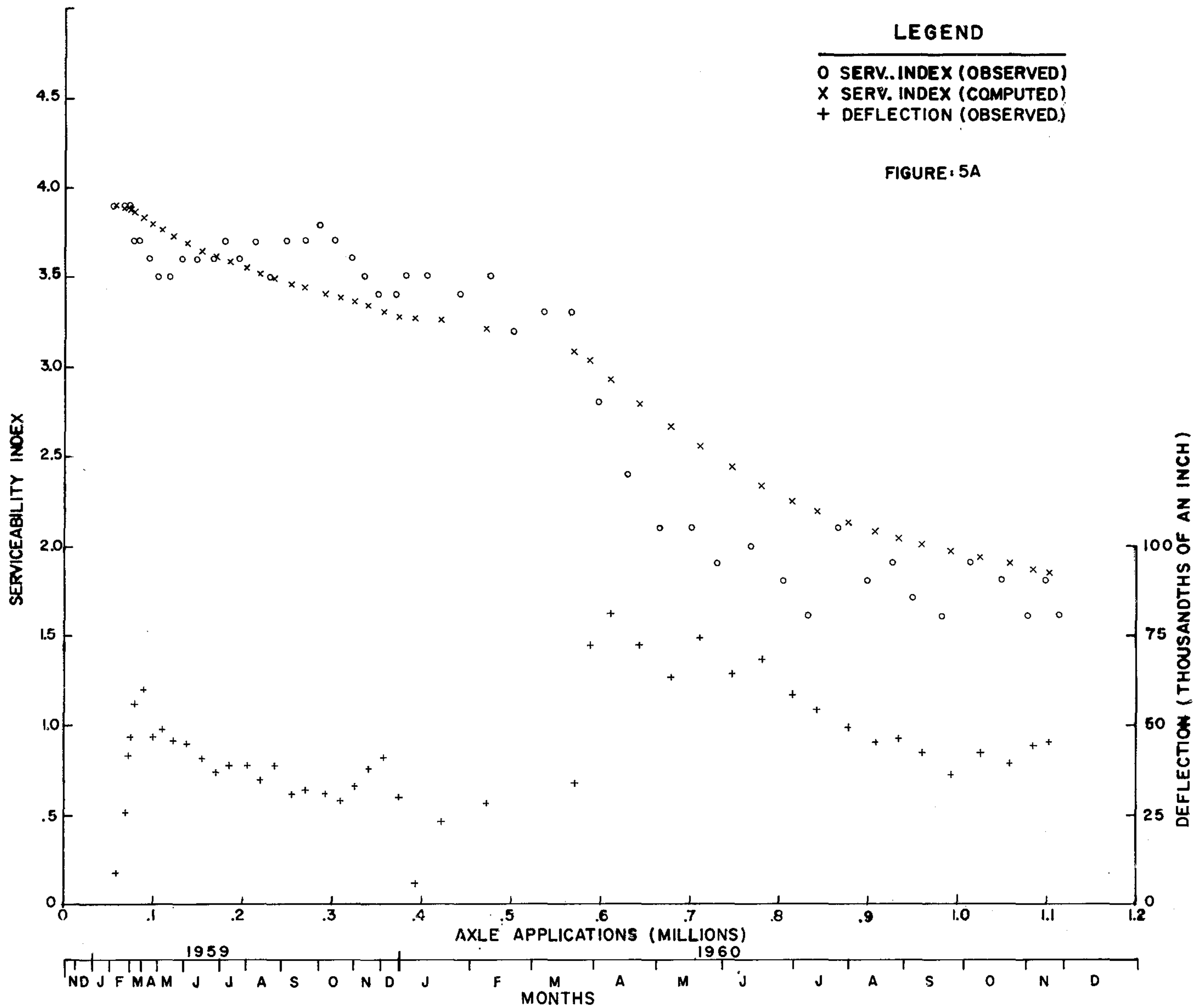
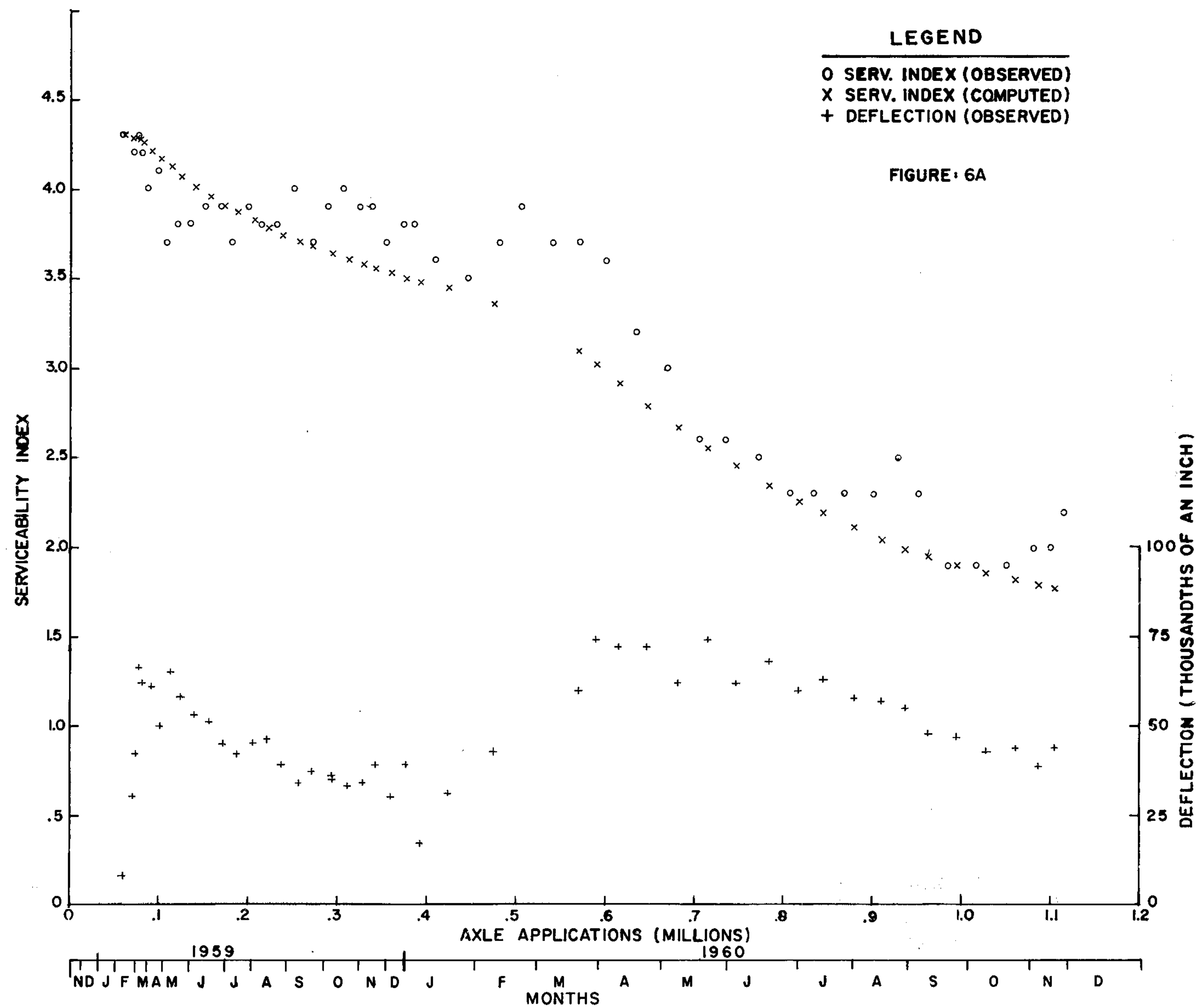


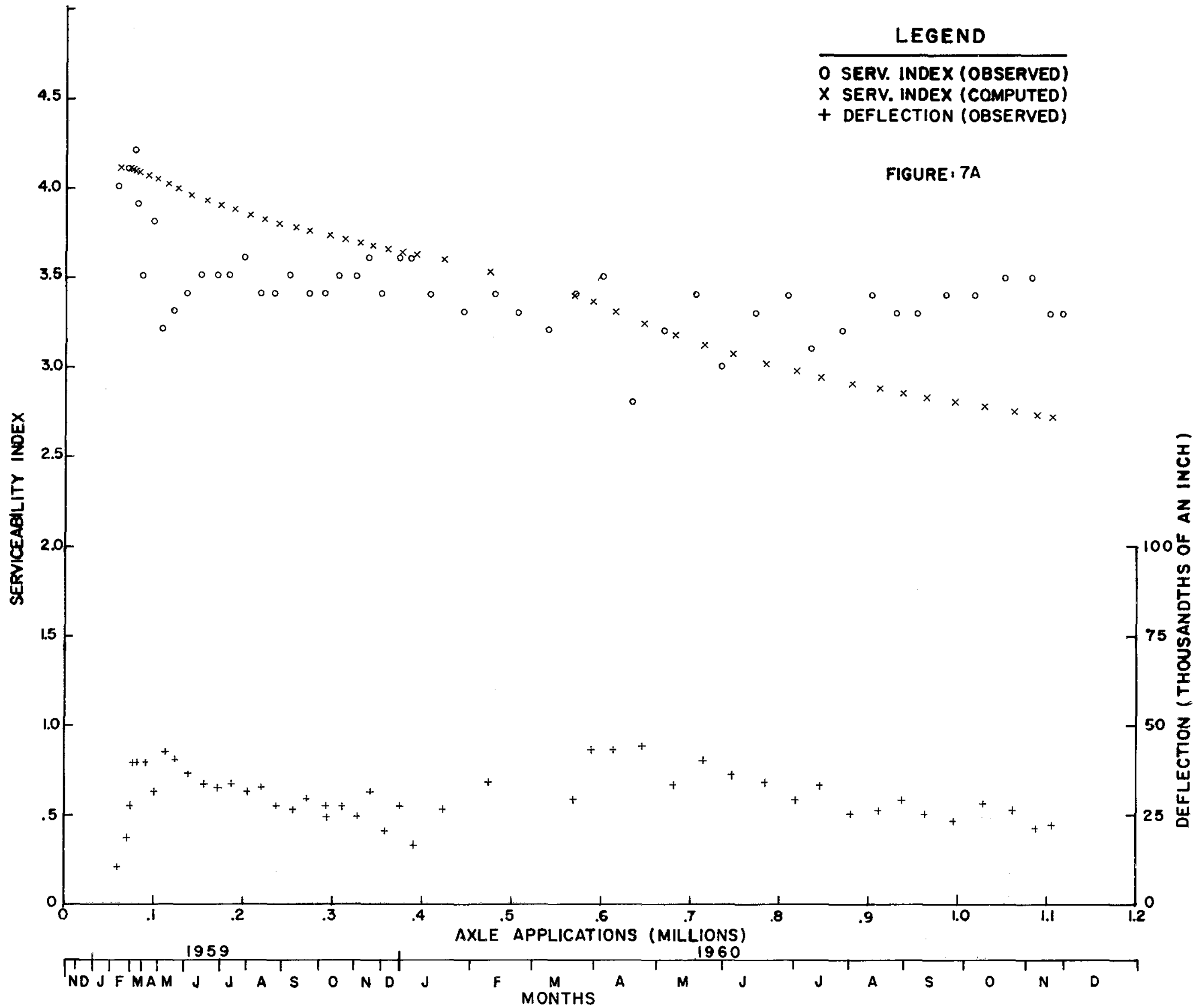

FIGURE 2 A: PLOT FROM WHICH VALUES OF − 9 AND 3/2 WERE SELECTED FOR THE CONSTANTS $A_0$ AND $A_1$, RESPECTIVELY.


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 3A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
25
50
75
100
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
N D J F M A M J J A S O N D J F M A M J J A S O N D

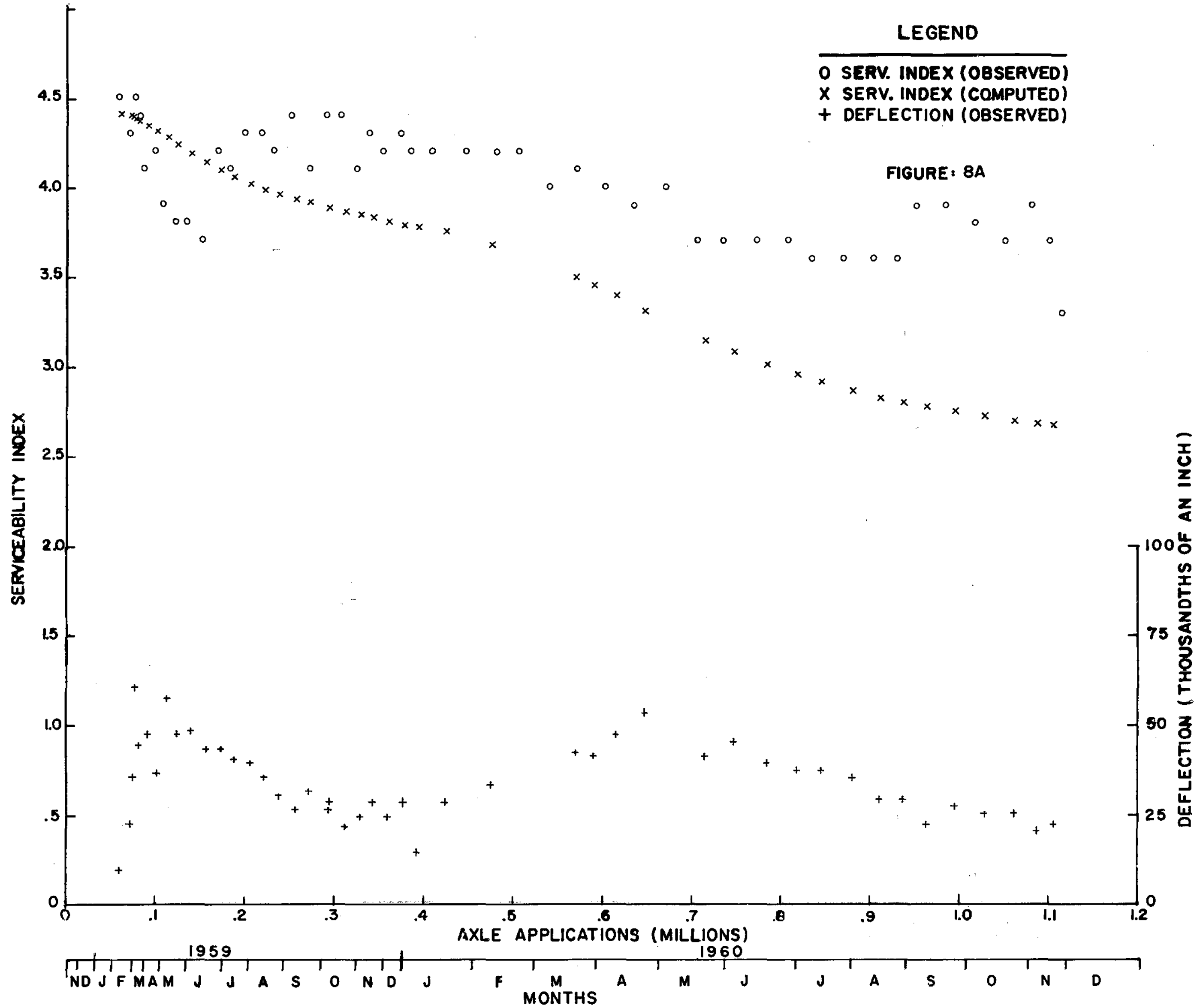

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 4A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
100
75
50
25
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
AXLE APPLICATIONS (MILLIONS)
1959
1960
ND J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 5A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 6A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS



LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 7A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
ND J F MAM M J J A S O N D J F M A M J J A S O N D
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 8A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS
0
.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
25
50
75
100
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
1.1
1.2
N D J F M A M J J A S O N D J F M A M J J A S O N D

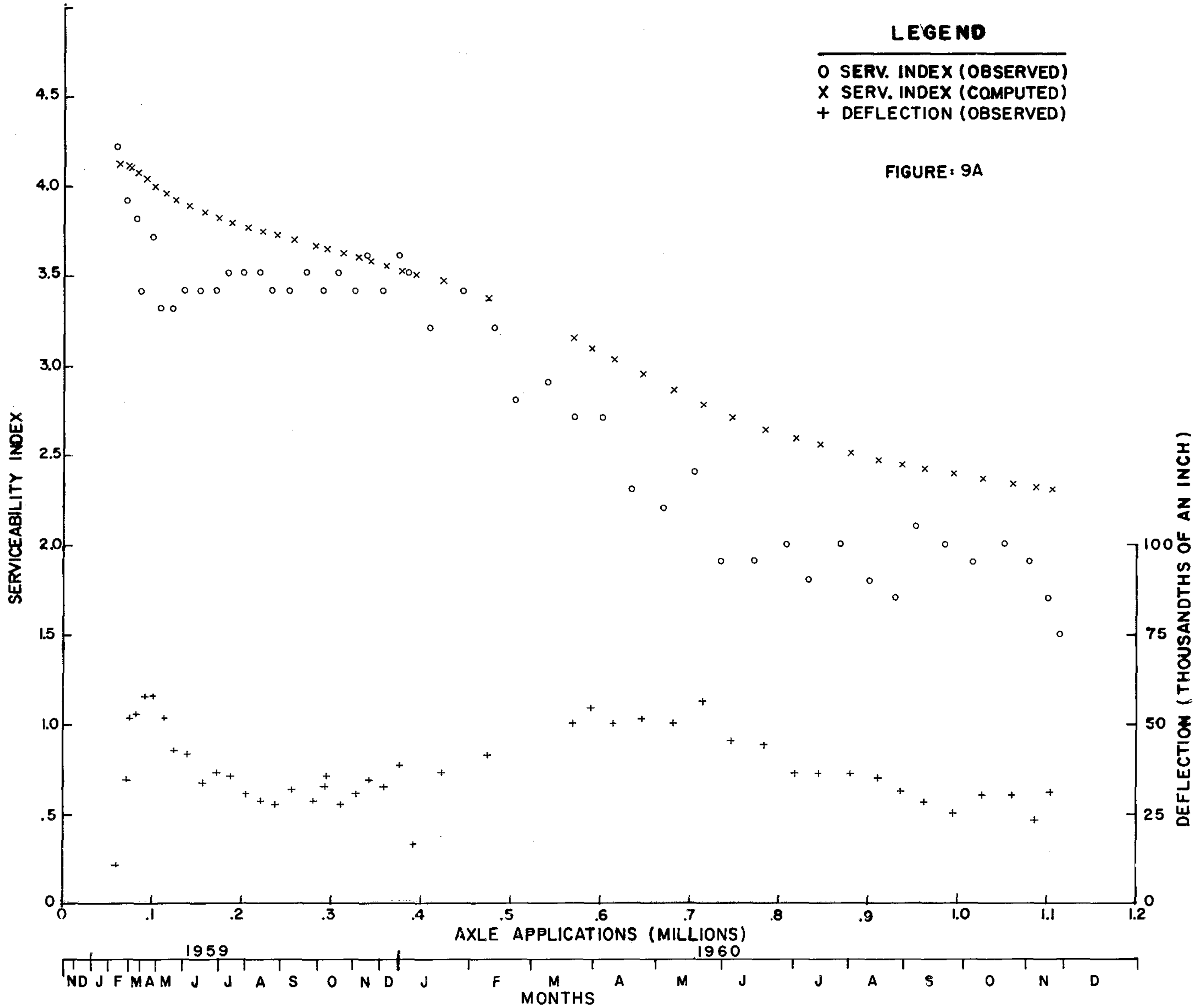
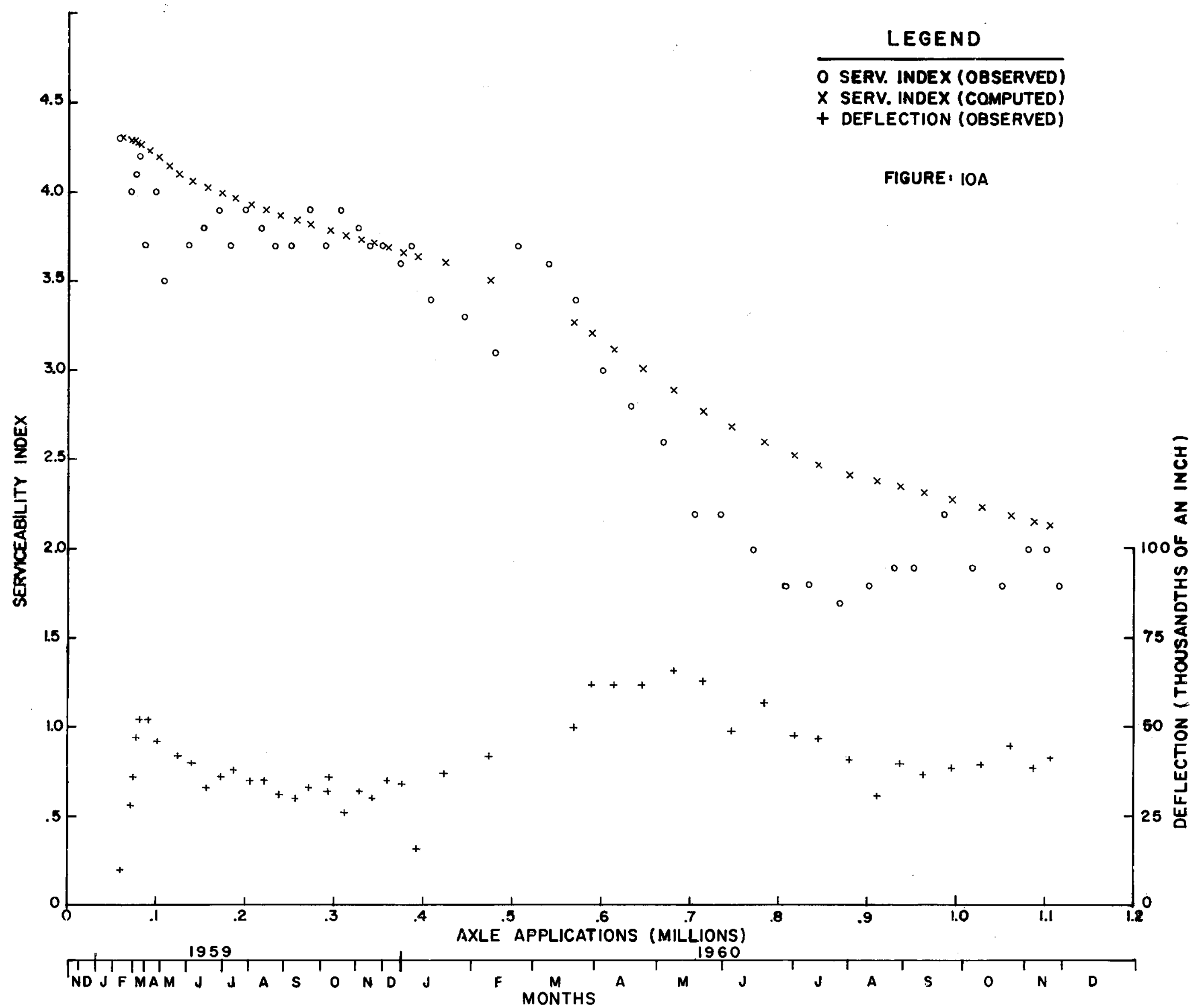
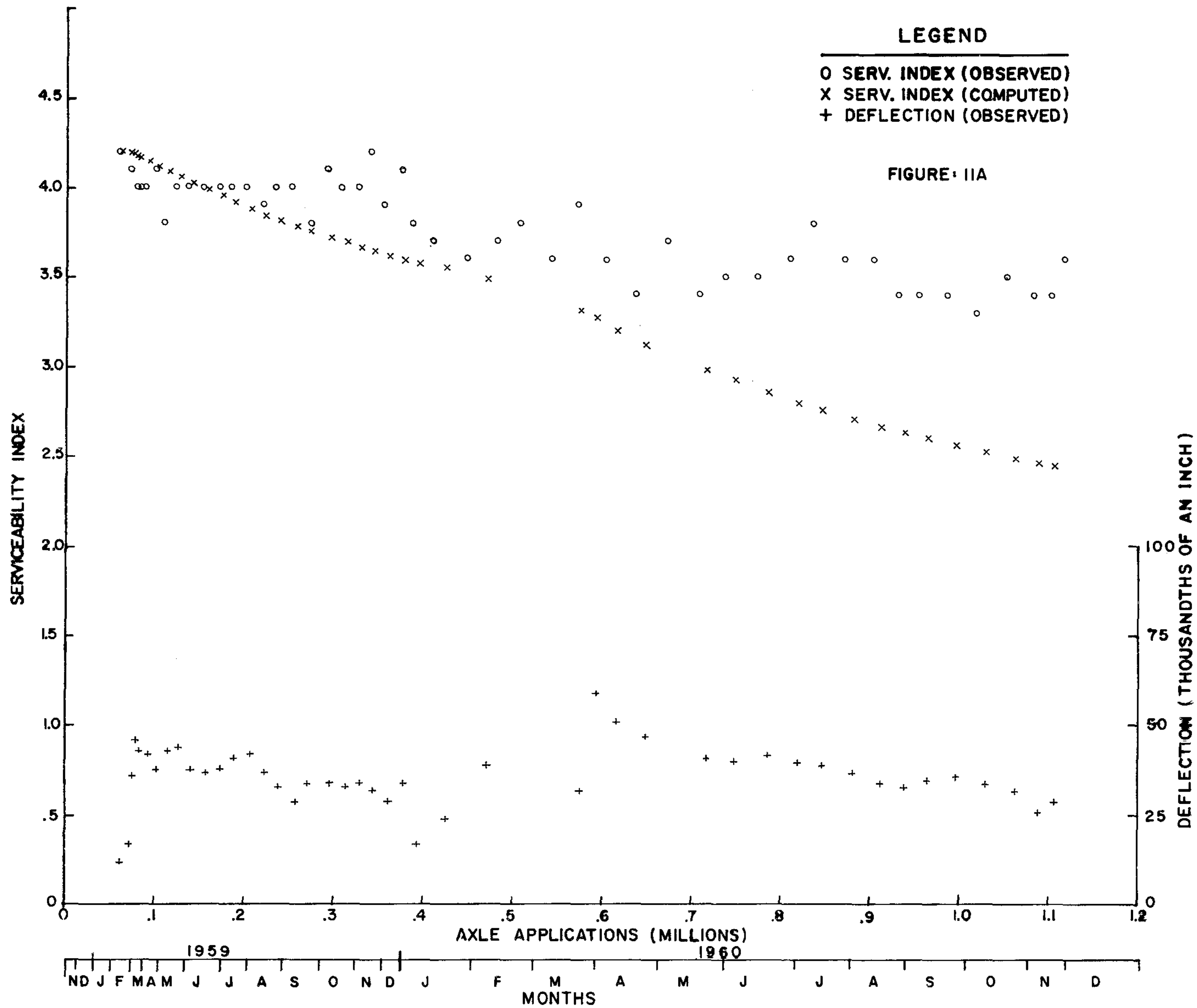
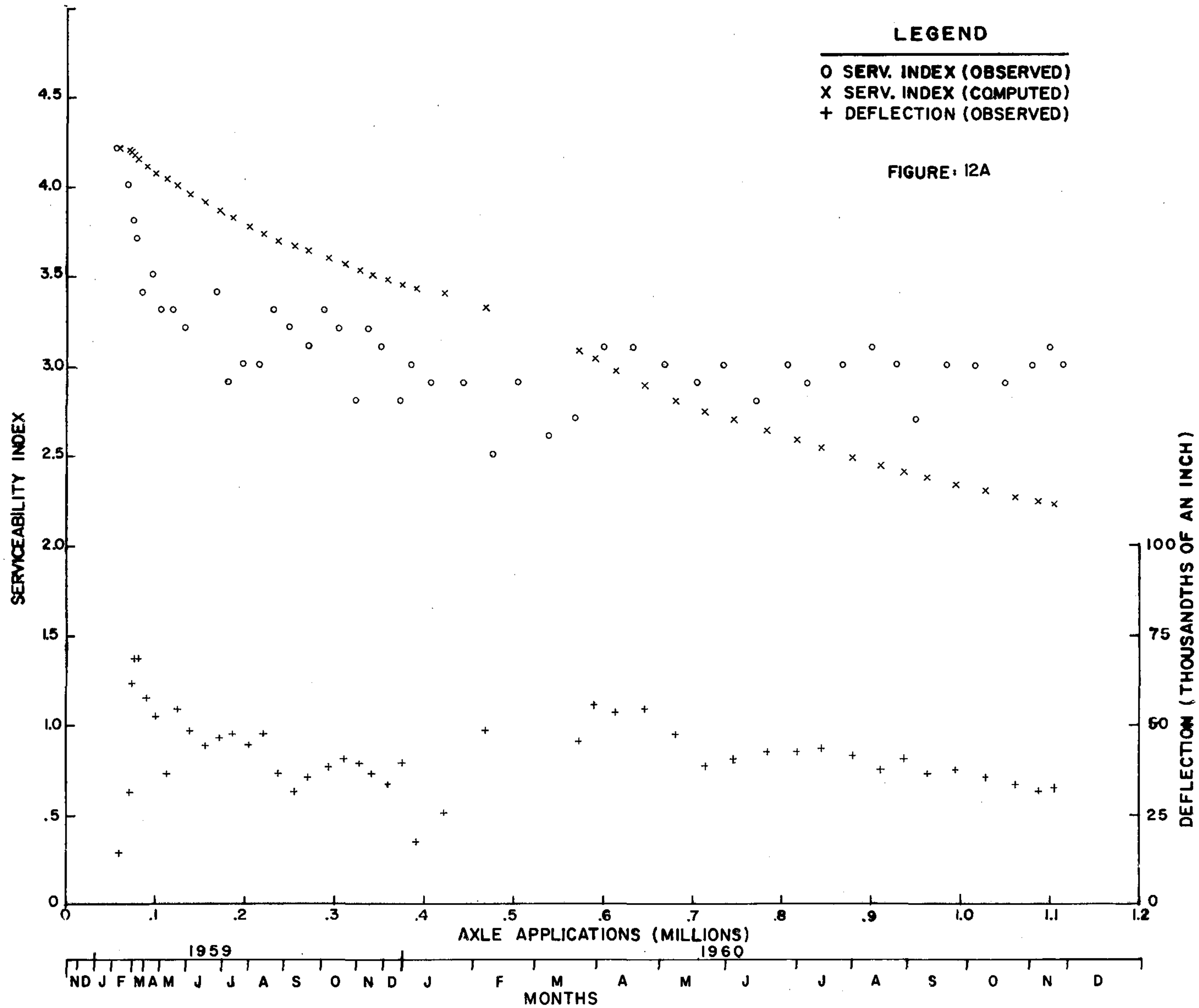

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
ND J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS

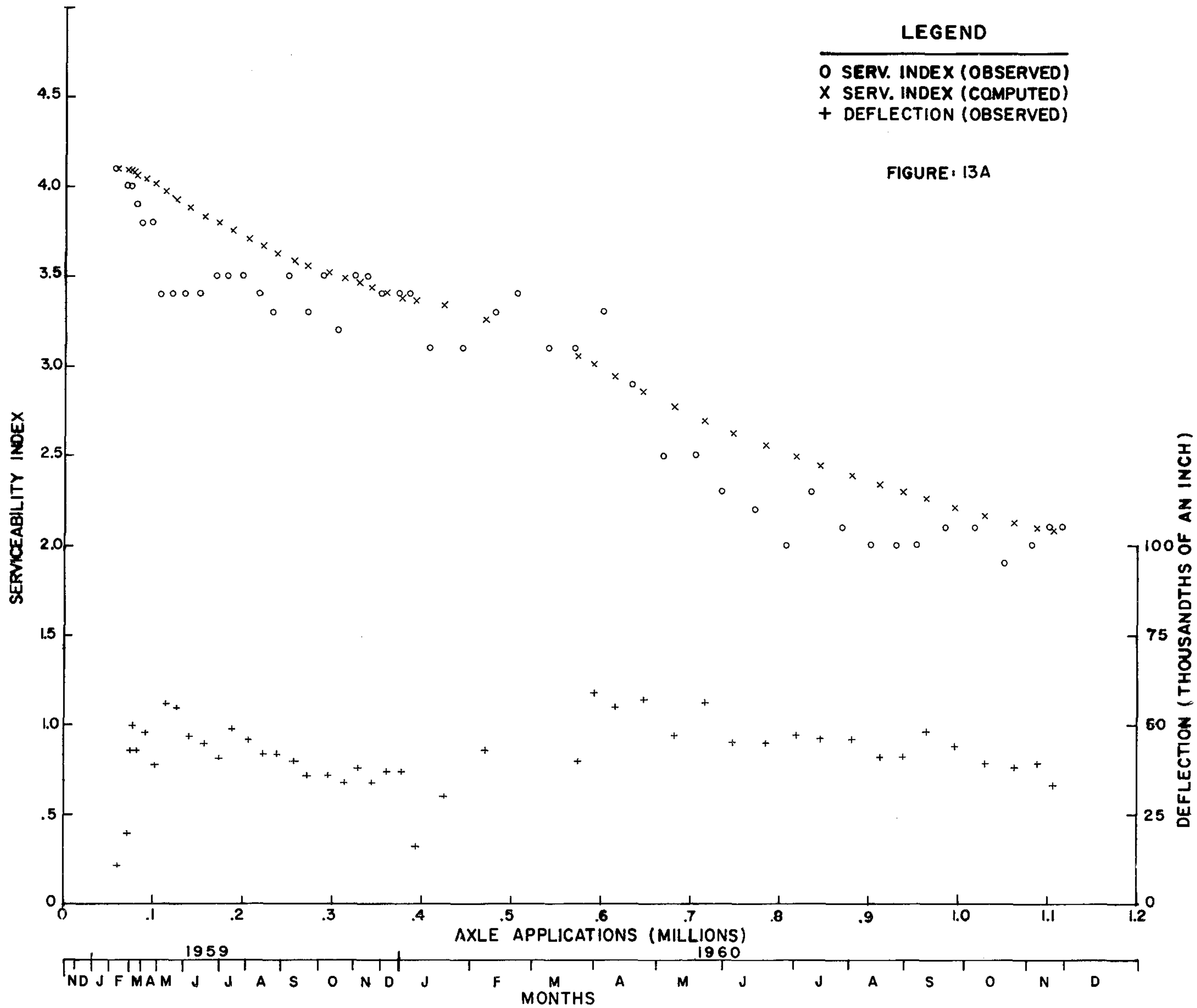
FIGURE: 9A

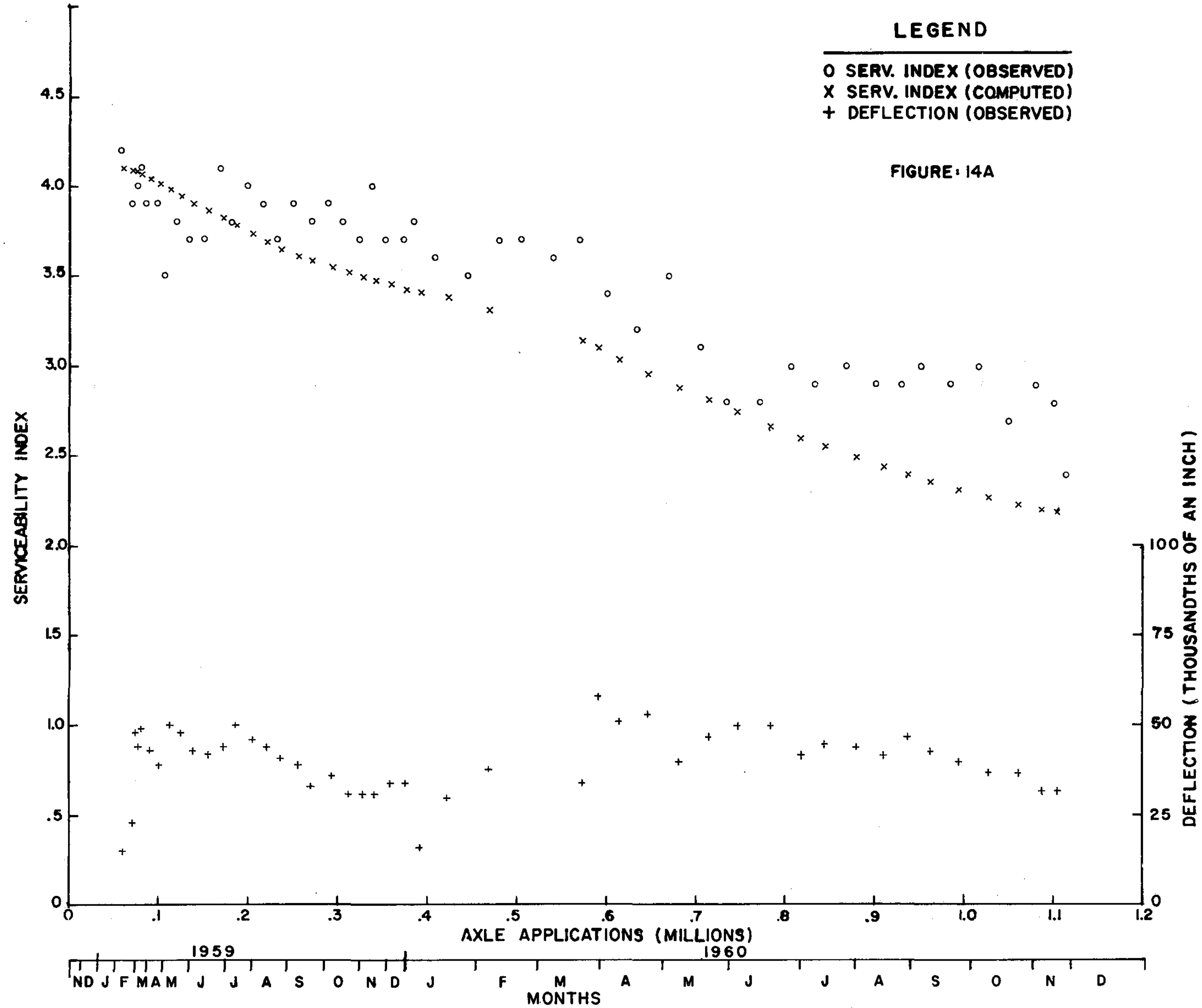

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 10A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
ND J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS

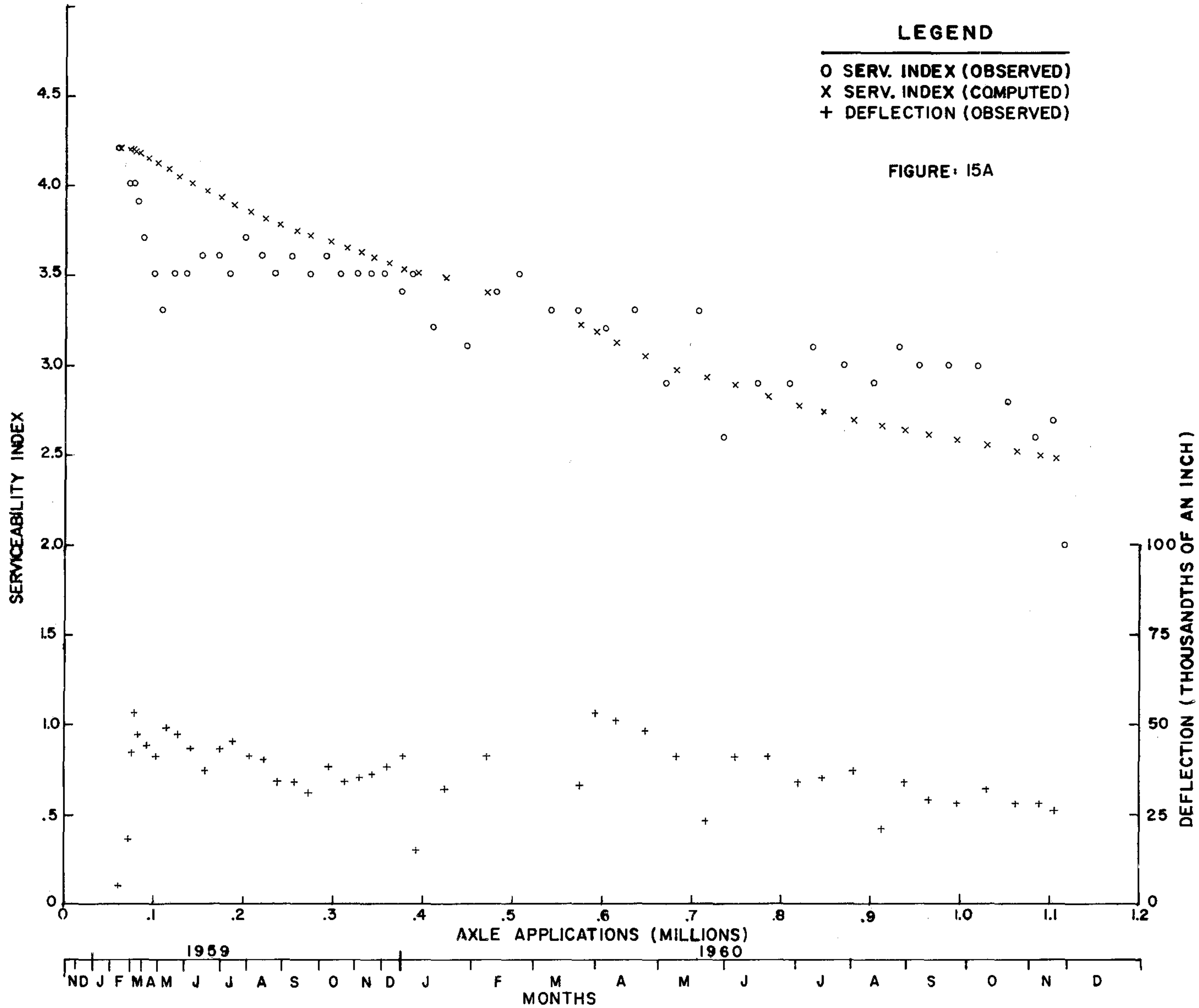

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 11A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS
N D J F M A M J J A S O N D J F M A M J J A S O N D

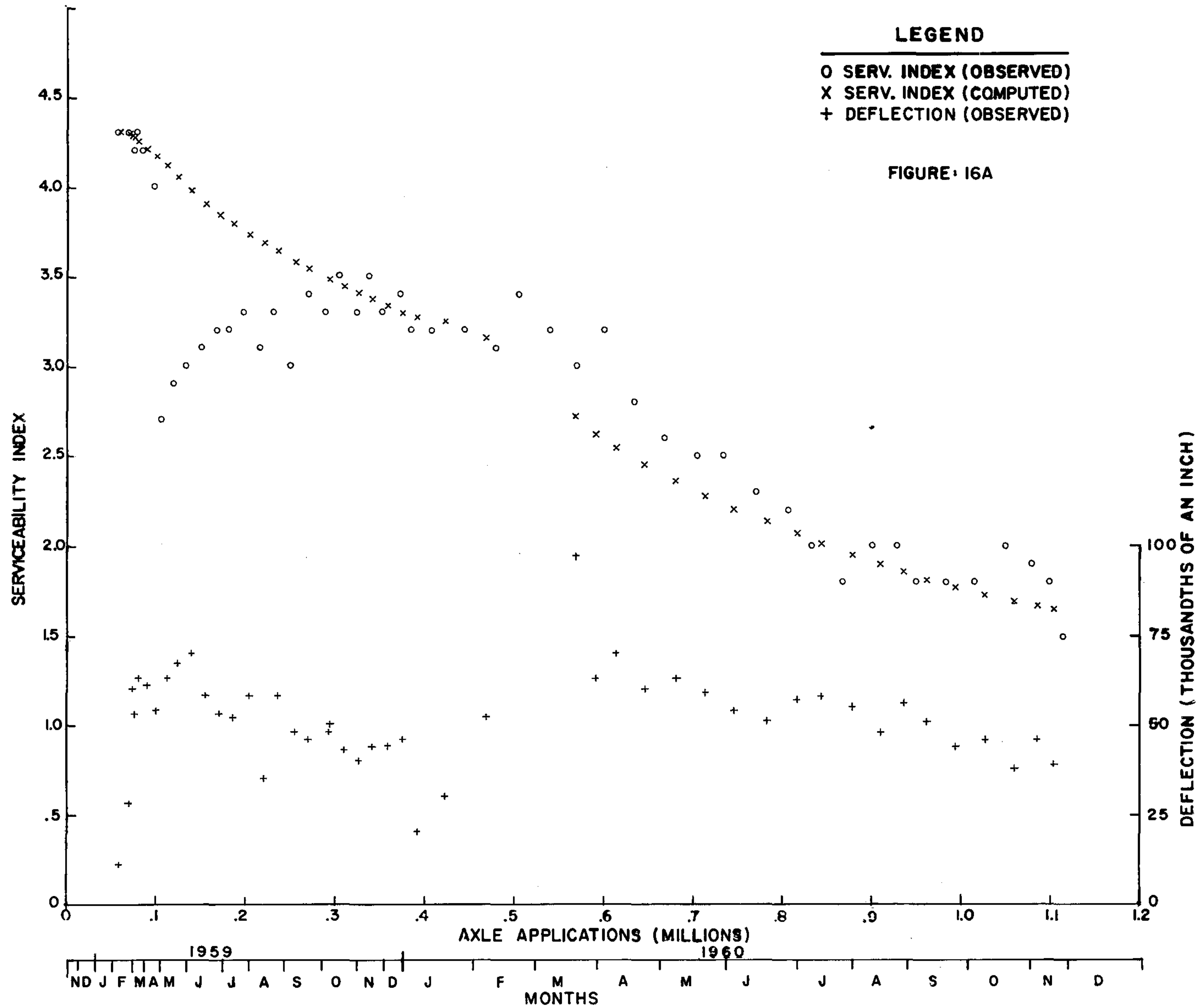


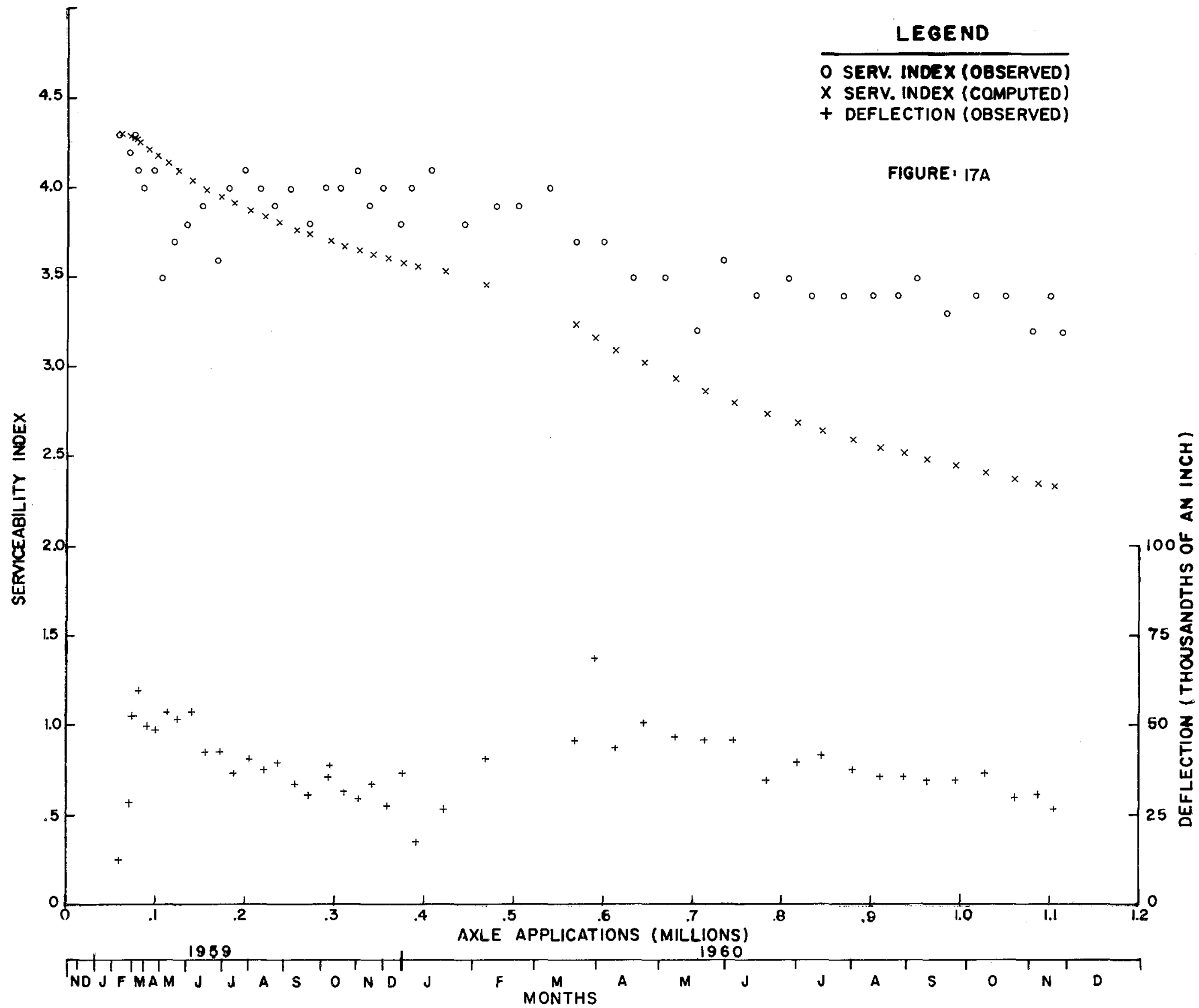
LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 12A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 13A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS
N D J F M A M J J A S O N D J F M A M J J A S O N D

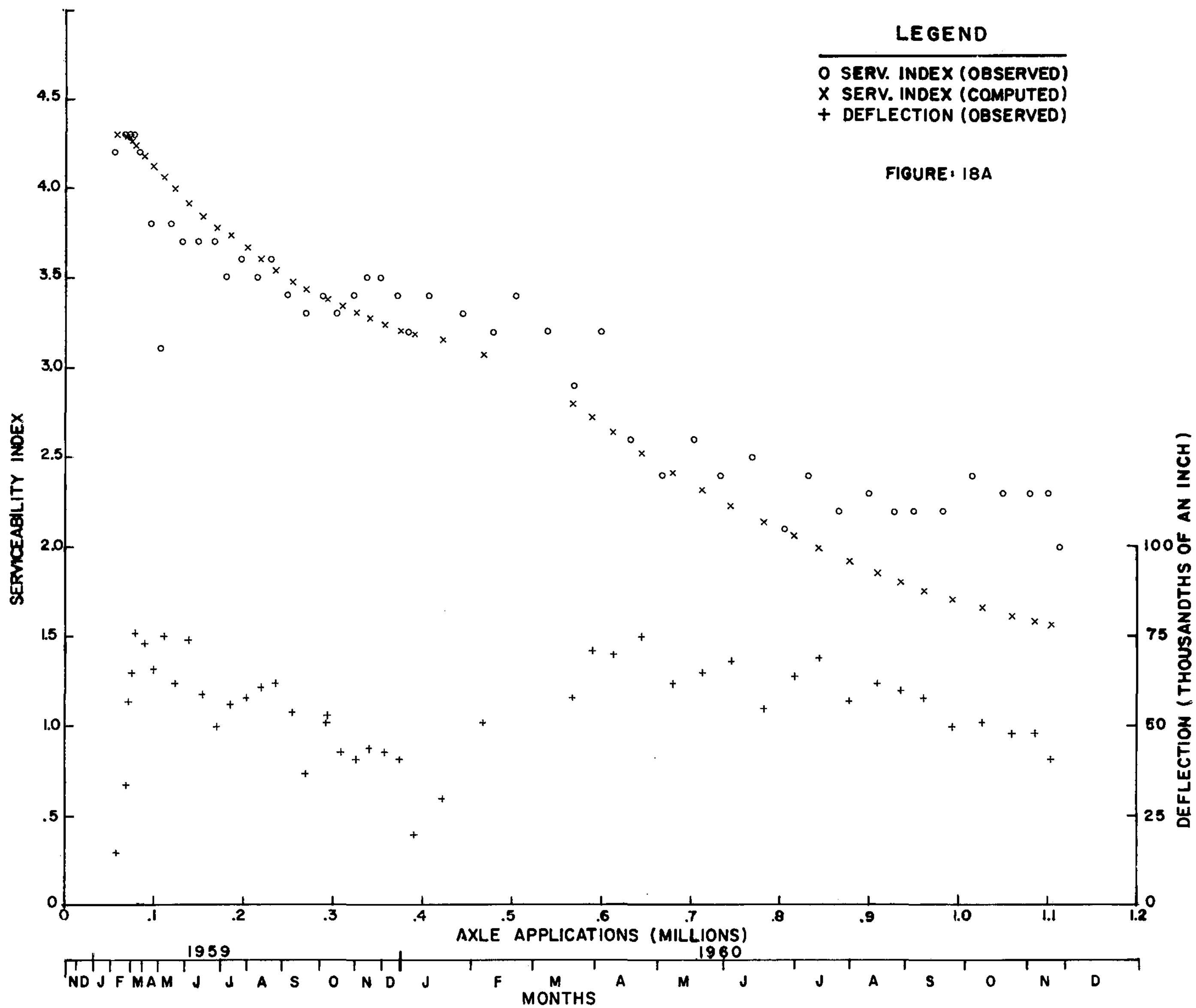

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 14A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS

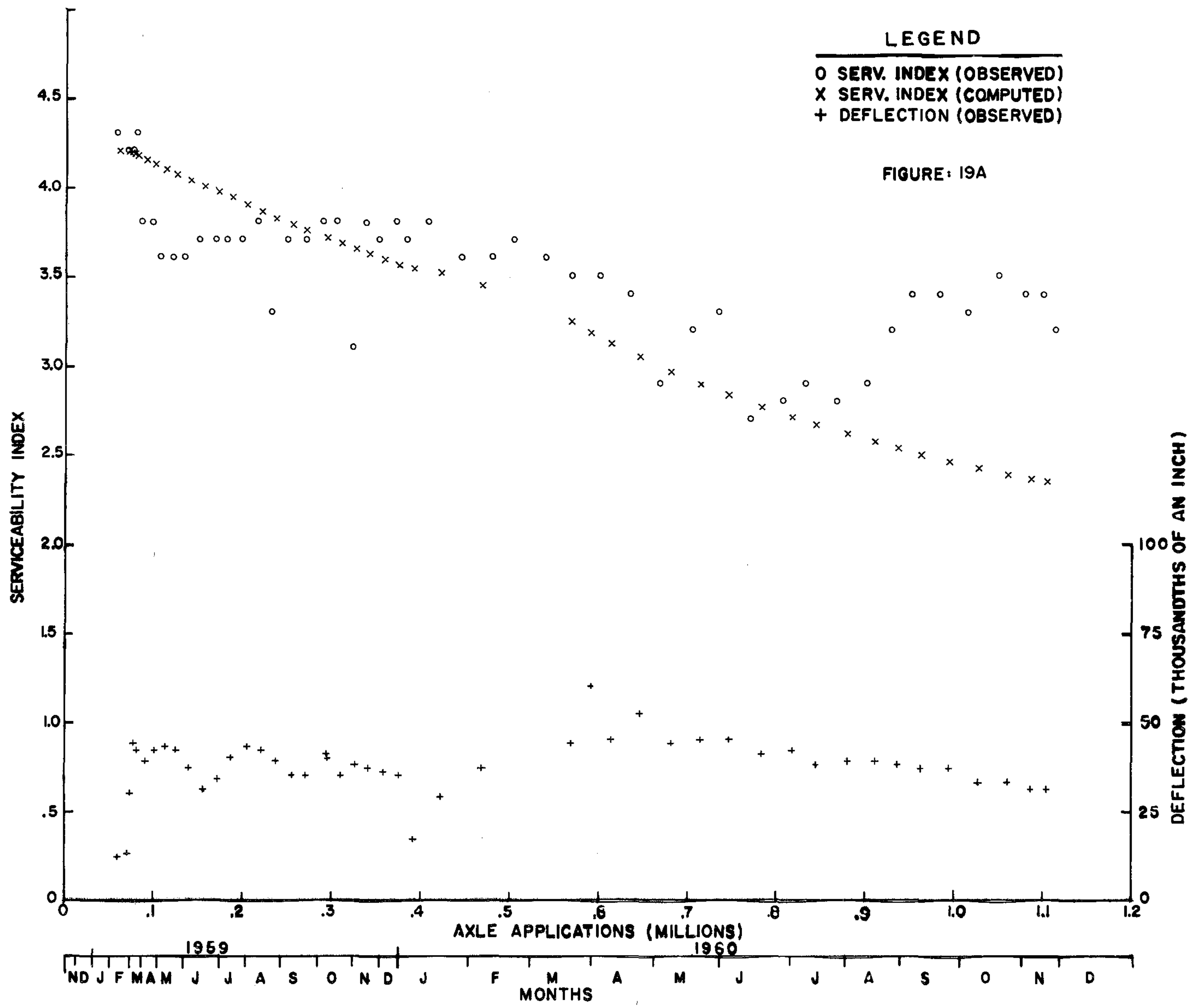

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 15A
SERVICEABILITY INDEX
DEFLECTION ( THOUSANDTHS OF AN INCH )
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 16A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS
N D J F M A M J J A S O N D J F M A M J J A S O N D

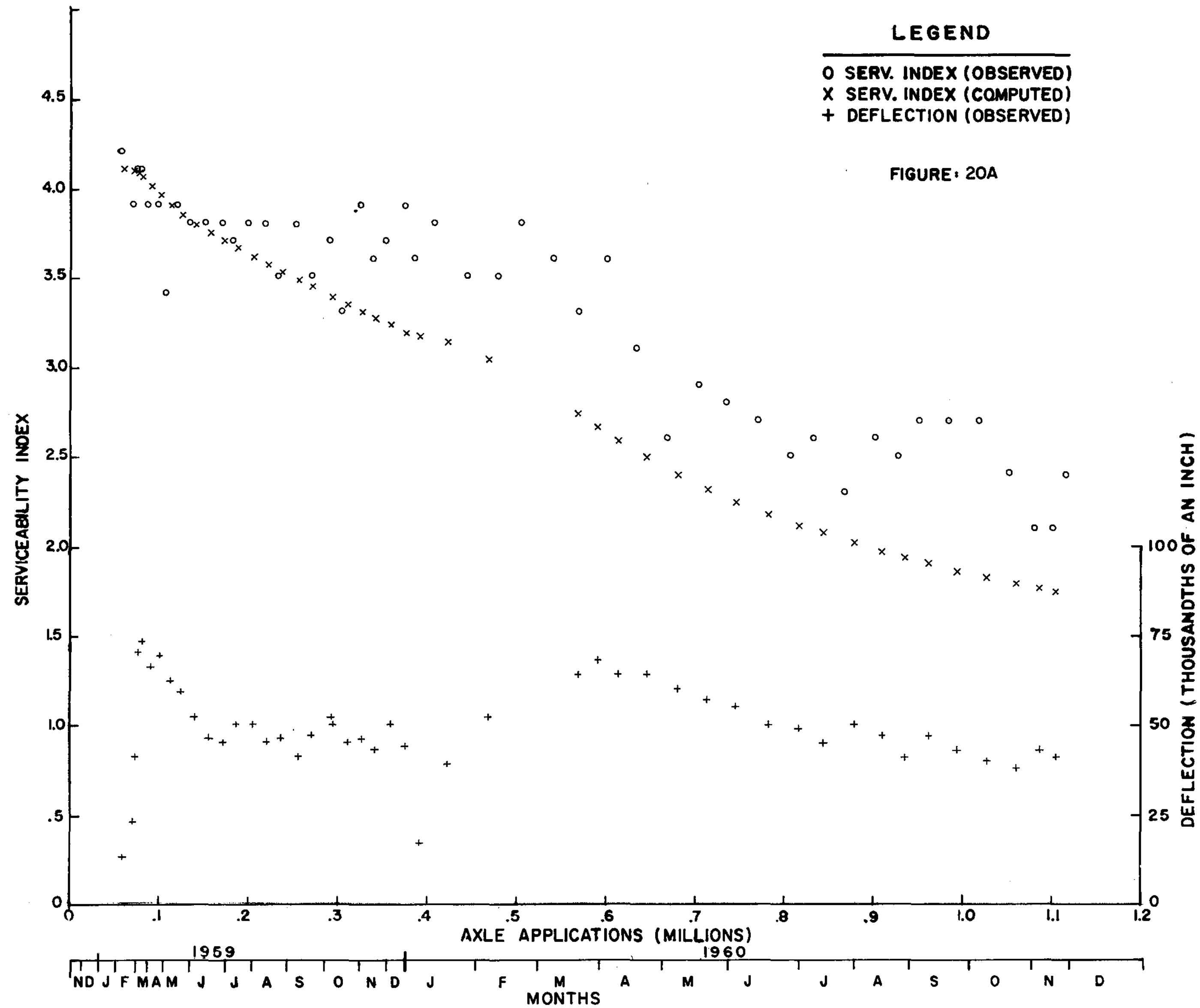


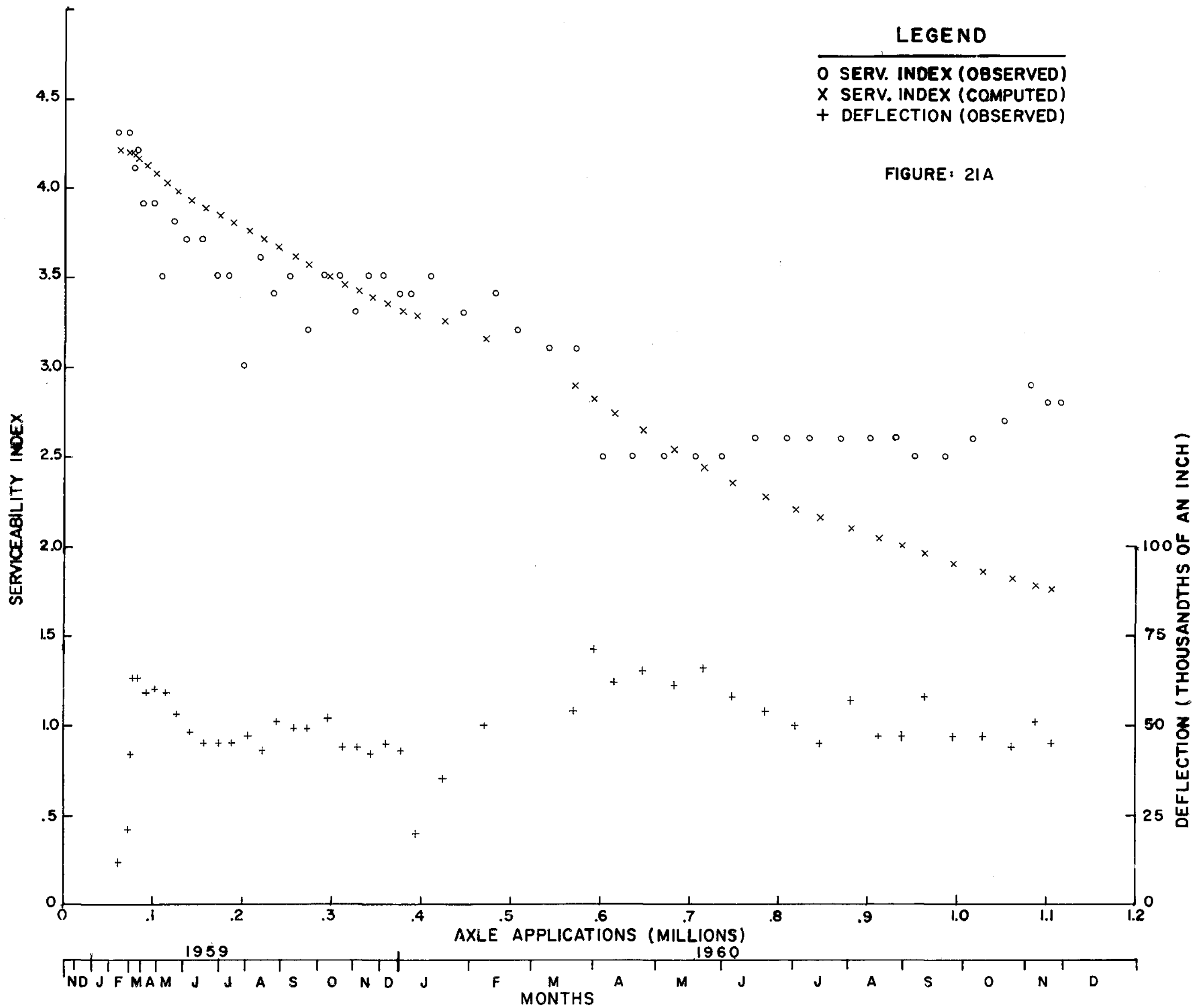
LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 17A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS

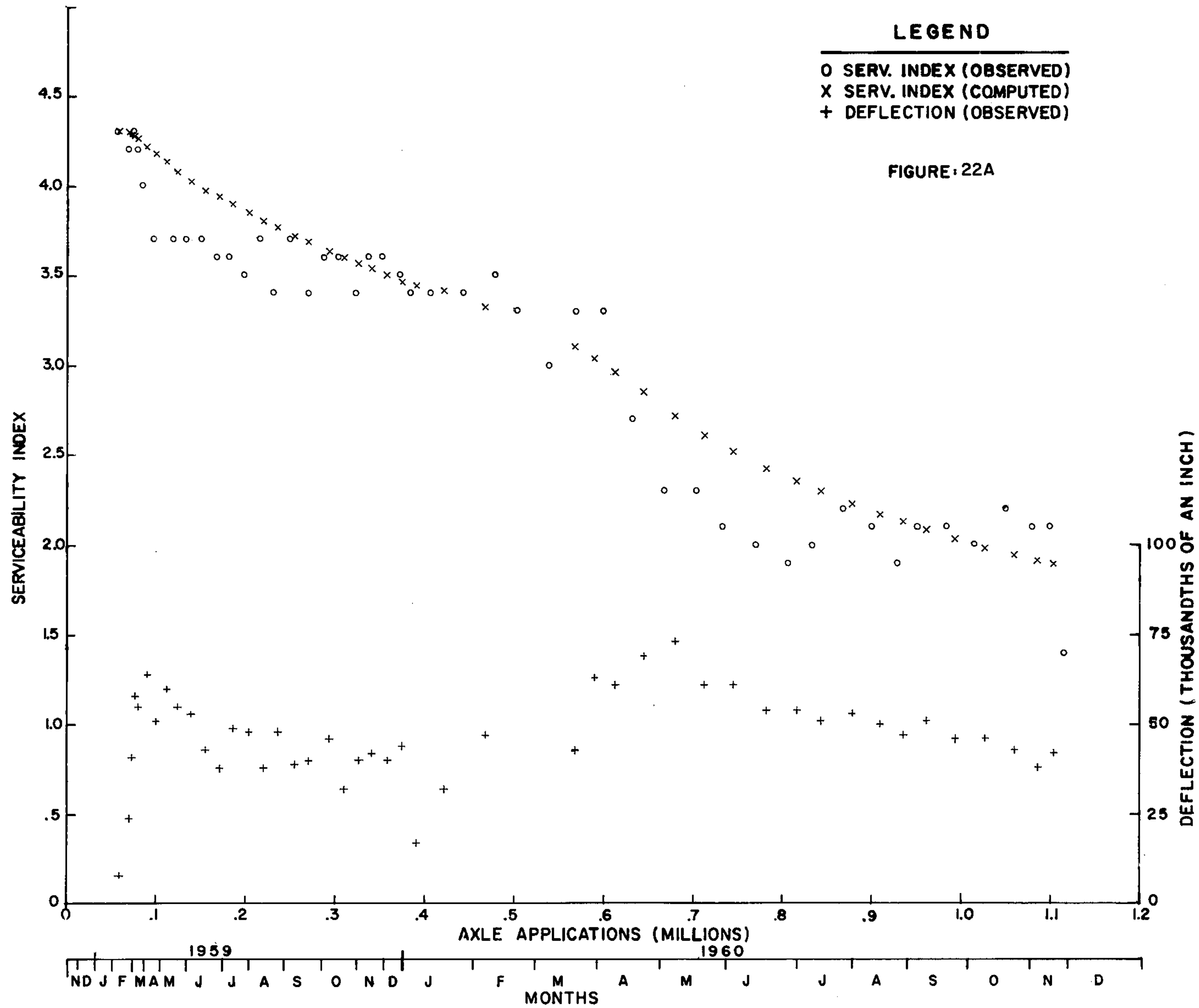

LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 18A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS

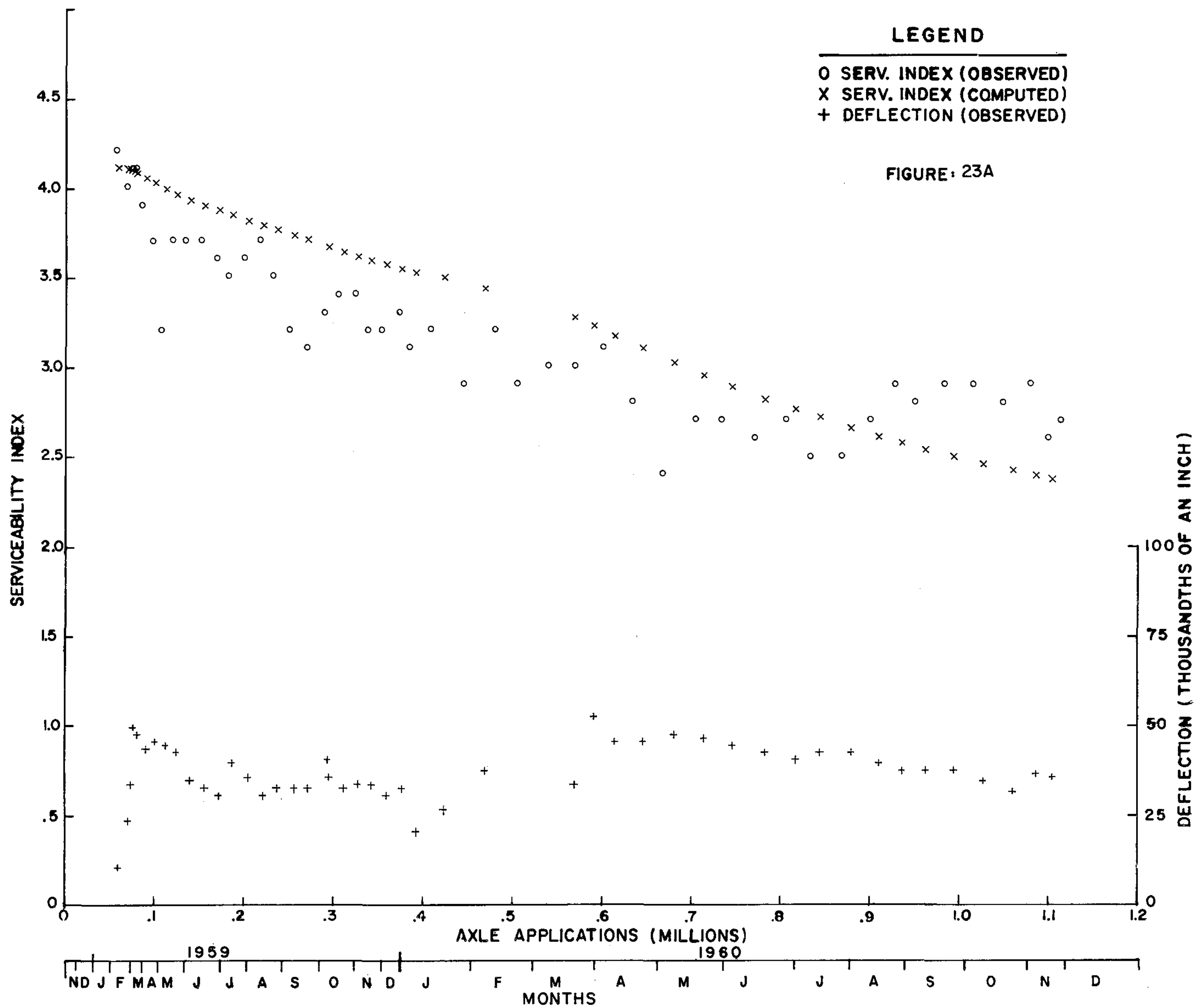


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 19A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 20A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
MONTHS
N D J F M A M J J A S O N D J F M A M J J A S O N D


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 21A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 22A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
.5
0
100
75
50
25
0
0 .1 .2 .3 .4 .5 .6 .7 .8 .9 1.0 1.1 1.2


LEGEND
O SERV. INDEX (OBSERVED)
X SERV. INDEX (COMPUTED)
+ DEFLECTION (OBSERVED)
FIGURE: 23A
SERVICEABILITY INDEX
DEFLECTION (THOUSANDTHS OF AN INCH)
AXLE APPLICATIONS (MILLIONS)
1959
1960
N D J F M A M J J A S O N D J F M A M J J A S O N D
MONTHS

By taking logarithms twice, and rearranging terms, Equation 4A can be written as follows:

$$\log(\Delta W_i) - \log\log(P_i/P_{i+1}) = 9 - 3/2 \log \bar{U}_i \qquad (5A)$$

For application to Texas test sections, we have used different symbols in Equation 5A. With the changed symbols the equation is

$$\log W - \log\log(P_o/P) = 9 - 3/2 \log U \qquad (6A)$$

where W = the equivalent number of 18-kip single axle loads applied to a section as its serviceability index drops from $P_0$ to P, and
U = the deflection produced by a 9000-pound dual wheel load (in thousandths of an inch), assumed constant throughout the life of the section.

Equation 6A was first given, in a slightly different form, as Equation 2 on page 5.

The assumption that U is constant throughout the life of a Texas section may be questionable. However, the severe freeze-thaw cycles that occurred at the Road Test obviously were largely responsible for the wide fluctuations in deflection trends observed there. Since such cycles are rare indeed in Texas, we feel that seasonal variations in deflections are relatively minor in this state.*

In defining U in Equation 6A as the deflection resulting from a 18-kip single axle load (9000-pound dual wheel load), we have assumed that the real traffic stream can be replaced by "an equivalent number" of 18-kip single axles, and that the deterioration stemming from the deflections produced by the real traffic is equivalent to the deterioration resulting from the deflections produced by the "equivalent number" of 18-kip axles.

Thus, in passing from Equation 5A (applicable to the AASHO Road Test) to Equation 6A (for use with Texas data) we have introduced two assumptions neither of which can be directly tested with data presently available. Nevertheless, we feel that the assumptions are reasonable, and that Equation 6A is a valuable asset to this project. Only through the use of Equation 6A was it possible to make any estimate (however imperfect) of the performance of an individual Texas Section. And only by studying the performance of individual sections is it possible to estimate regional effects.

As an indirect means for testing the validity of Equation 6A (or Equation 2), the equation was used in the following form for com-

---

*It is planned to check this assumption in future research.

puting P for the 188 Texas test sections considered in this report:

$$P_o = P \times 10^{bw} \tag{7A}$$

where $b = 10^{-9}U^{1.5}$.

Since values of P, W and U were known for each section, $P_o$ could be computed. The average of the 188 computed values was 4.2, a value already found to be the average serviceability index of a number of newly constructed highways in Texas, and reported as the average initial value of P at the AASHO Road Test.

## Appendix B

## Procedure Used in Revision of the Classification Chart

A series of coordinates (shear and normal stress) of the class lines, taken from Figure 5, are tabulated in Table 1B. These Data are plotted on a graph of shear stress versus triaxial class in Figure 1 B.

An inspection of Figure 1B shows that, in the interval $2 \leq T \leq 5$, a straight line can be fitted with acceptable accuracy to the four data points corresponding to a fixed value of normal stress. This circumstance was used in the extrapolation across the interval T = 2 to T = 1 indicated by the dashed lines in the figure. The extrapolation procedure, designed to preserve the continuity of the chart insofar as possible, is described below.

### Table 1B

Data from Figure 5 Used in Computing Coordinates of Class 1.0 Line of Figure 6

| | | | Constants | | | |
|---|---|---|---|---|---|---|
| Normal Stress, psi | Class Line | Shear Stress, psi | Straight Lines a | Straight Lines b | Curved Lines c | Curved Lines d |
| 3 | 2 | 11.3 | 17.90 | -3.300 | 16.94 | .5841 |
| | 3 | 8.0 | | | | |
| | 4 | 4.0 | | | | |
| | 5 | 1.4 | | | | |
| 5 | 2 | 14.9 | 23.00 | -4.000 | 21.71 | .5333 |
| | 3 | 11.4 | | | | |
| | 4 | 6.1 | | | | |
| | 5 | 3.2 | | | | |

Table 1B (Continued)

| Normal Stress, psi | Class Line | Shear Stress, psi | Constants: Straight Lines a | b | Curved Lines c | d |
|---|---|---|---|---|---|---|
| 10 | 2 | 23.1 | 34.13 | -5.567 | 32.17 | .4841 |
| | 3 | 17.8 | | | | |
| | 4 | 11.2 | | | | |
| | 5 | 6.4 | | | | |
| 15 | 2 | 29.8 | 44.50 | -7.300 | 41.95 | .4883 |
| | 3 | 22.2 | | | | |
| | 4 | 15.4 | | | | |
| | 5 | 7.9 | | | | |
| 20 | 2 | 35.3 | 52.33 | -8.667 | 49.34 | .4953 |
| | 3 | 26.0 | | | | |
| | 4 | 18.2 | | | | |
| | 5 | 9.0 | | | | |
| 25 | 2 | 41.0 | 60.83 | -10.167 | 57.36 | .5021 |
| | 3 | 29.4 | | | | |
| | 4 | 20.2 | | | | |
| | 5 | 10.0 | | | | |

The intercept, a, and the slope, b, of each straight line in Figure 1B are recorded in Table 1B. Thus, if $\mathcal{T}$ represents the shear stress in the interval $2 \leq T \leq 5$, the equation for $\mathcal{T}$ (with normal stress fixed) is

$$\mathcal{T} = a + bT \quad (2 \leq T \leq 5) \qquad (1B)$$

where a and b are given in Table 1B.

Coordinates for each dashed line in Figure 1B were then computed from the formula

$$\mathcal{T}' = \frac{c}{T^d} \qquad (0 \leq T \leq 2) \qquad (2B)$$

where $\mathcal{T}'$ is the shear stress in the interval $0 \leq T \leq 2$, and the constants

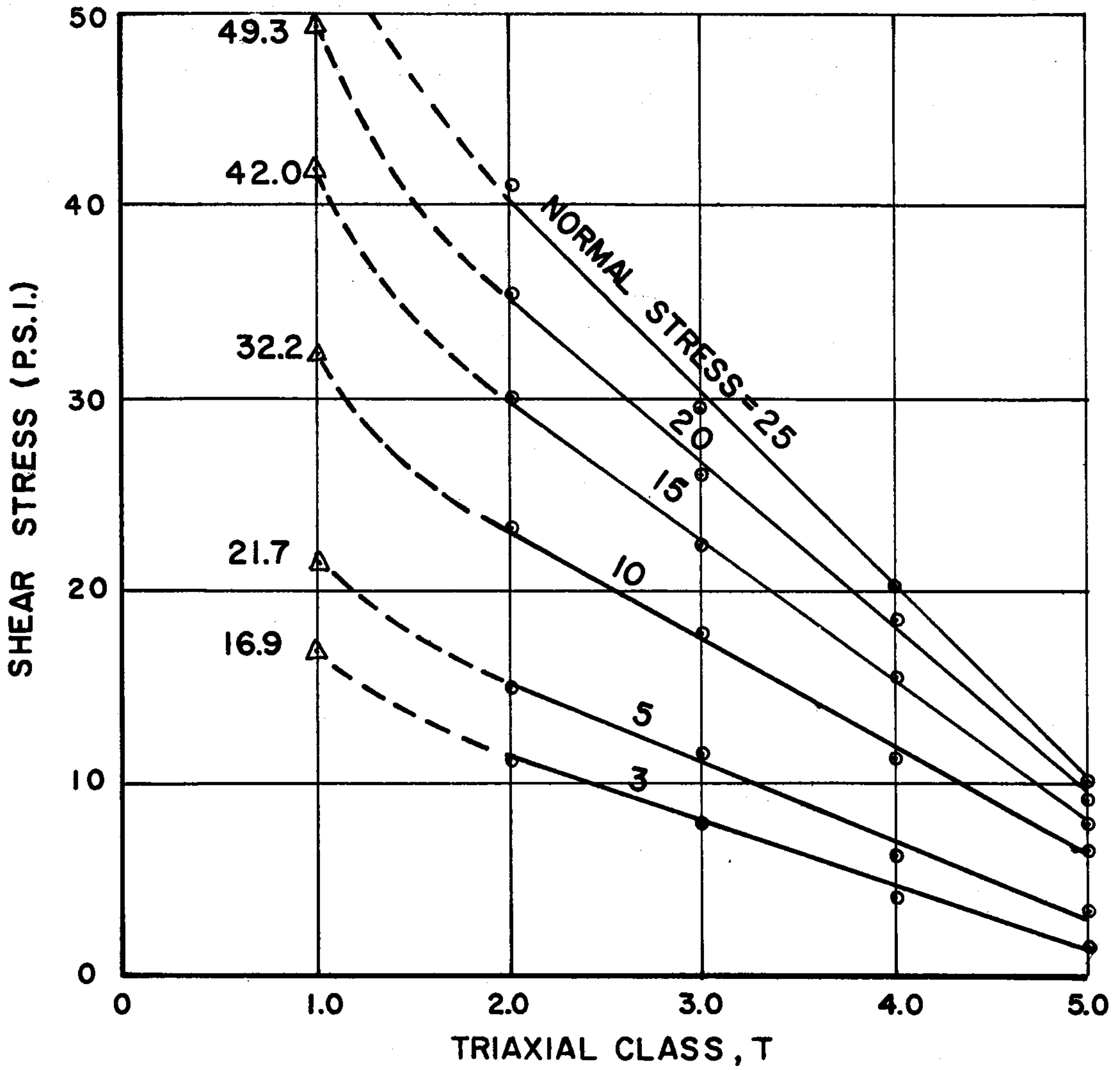



FIGURE 1B: THE DATA FOR PLOTTING THE CIRCLED POINTS WERE TAKEN FROM THE STANDARD CLASSIFICATION CHART (FIGURE 3). THE SOLID LINES WERE FITTED TO THESE POINTS. DASHED LINES WERE COMPUTED FROM EQUATION 2 B.

c and d are determined from the conditions that when T = 2,

$$\tau' = \tau$$

and
$$\frac{d\tau'}{dT} = \frac{d\tau}{dT} \qquad (T = 2)$$

It will be noted from the formula for $\tau'$ that the shear stress approaches infinity as T approaches zero, as required by the stipulation made in Section 9 that the Class 0 would be assigned to a material of infinite strength.

From the formula it is also evident that when T = 1, $\tau' = c$. Thus, the coordinates of the Class 1 line shown in Figure 6 are given in Table 1B in the columns headed "Normal Stress" and "c", the latter being the shear stress when T = 1. These coordinates are also given in Figure 1B.

## APPENDIX C

## MULTIPLE ERROR REGRESSION TECHNIQUE

### 1C INTRODUCTION

In the classical least-squares method of fitting a linear model to data collected in an experiment involving several variables, it is assumed that the values of all but one - the dependent or response variable - are known precisely. Frequently, however, there are errors of measurement in all the variables, and when this is the case the classical method yields a biased estimate of the regression coefficients. Since the objective of most experiments is to obtain unbiased estimates of these coefficients, it is apparent that measurement errors in the independent variables should not be ignored.

The regression technique described herein accounts for errors in all variables. It is essentially the same as a method described by J. Johnston,[9] but includes a new concept - that of the "quality" of a variable. Because of the introduction of this concept, and because it was desired to confirm Johnston's results by independent means, it was necessary to perform the mathematical operations described in the following sections.

The reader who does not desire to follow the derivations will find the gist of the method in Sections 5, 6, 7 and 8.

## 2C ASSUMPTIONS

Let it be supposed that an experiment involves a set of p variables, the true values of which are known to be linearily dependent. We name the variables $X_1, X_2, \ldots., X_j, \ldots., X_p$.

In the course of the experiment we measure the whole set of variables from time to time (or from place to place, depending on the nature of the experiment). At one of these times (or places) we obtain the ith set of measurements, $X_{i1}, X_{i2}, \ldots., X_{ij}, \ldots., X_{ip}$.

Corresponding to the ith set of measurements, there is a set of true values, $\hat{X}_{i1}, \hat{X}_{i2}, \ldots., \hat{X}_{ij}, \ldots., \hat{X}_{ip}$, and a set of measurement errors, $e_{i1}$, $e_{i2}, \ldots., e_{ij}, \ldots., e_{ip}$.

We assume that the measurement errors are random, independent, and normally distributed, with a mean value of zero. The measurement error, $e_{ij}$, is defined by

$$X_{ij} = \hat{X}_{ij} + e_{ij} \quad . \tag{1}$$

The true values of the variables, according to our assumption of linear dependence, satisfy the equation

$$A_o + \sum_{j=1}^{p} A_j \hat{X}_{ij} = 0 \quad , \tag{2}$$

where $A_o, A_1, \ldots., A_p$ are constants.

## 3C DERIVATION OF EXPRESSION TO BE MINIMIZED

Equations 1 and 2 lead directly to a relationship involving the measured values and measurement errors, as follows:

$$A_o + \sum_{j=1}^{p} A_j X_{ij} = \sum_{j=1}^{p} A_j e_{ij} \quad . \tag{3}$$

By squaring Equation 3 and then summing over the index i, we obtain,

$$\sum_{i=1}^{n} (A_o + \sum_{j=1}^{p} A_j X_{ij})^2 = \sum_{i=1}^{n} (\sum_{j=1}^{p} A_j e_{ij})^2 \quad , \tag{4}$$

where n is the total number of times the set of variables has been measured.

We may simplify Equation 4 somewhat by eliminating $A_o$, as indicated below.

Noting that the error term is independent of $A_o$, we differentiate Equation 4 with respect to $A_o$, and solve the resulting equation for $A_o$, obtaining

$$A_o = - \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{p} A_j X_{ij} \quad ,$$

or, more briefly,

$$A_o = - \sum_{j=1}^{p} A_j \bar{X}_j \quad , \tag{5}$$

where $\bar{X}_j$ is the mean of the n measured values of the variable $X_j$.

By substituting the right side of Equation 5 for $A_o$ in Equation 4, we obtain an equation in terms of the deviations of the measured variables from their means:

$$\sum_{i=1}^{n} (\sum_{j=1}^{p} A_j V_{ij})^2 = \sum_{i=1}^{n} (\sum_{j=1}^{p} A_j e_{ij})^2 \quad , \tag{6}$$

where $V_{ij} = X_{ij} - \bar{X}_j$ = the deviation of $X_{ij}$ from the mean of the n measured values of $X_j$.

The right side of Equation 6 can be expanded into the sum of a series of terms of the type

$$A_j\ A_k \sum_{i=1}^{n} e_{ij}\ e_{ik} \quad .$$

But, for large n, every term for which $j \neq k$ has an expected value of zero as a consequence of our previously stated assumption regarding the measurement errors, $e_{ij}$ . If we neglect terms for which $j \neq k$, there will remain in the series only terms of the type,

$$A_j^{\,2} \sum_{i=1}^{n} e_{ij}^2 \quad (j = 1, \ldots., p).$$

Thus, Equation 6 may be written in the following form:

$$\sum_{i=1}^{n} \left( \sum_{j=1}^{p} A_j\ V_{ij} \right)^2 = A_1^{\,2} \sum_{i=1}^{n} e_{i1}^2 + \ldots. + A_p^{\,2} \sum_{i=1}^{n} e_{ip}^2 \quad . \tag{7}$$

Without loss of generality, we separate each constant, $A_j$, into two arbitrary factors, $C_j$ and $M_j$, and define one of the factors as indicated below:

$$A_j \equiv C_j\ M_j \quad , \tag{8}$$

where $M_j$ is defined by

$$M_j \equiv \left[ \sum_{i=1}^{n} e_{ip}^2 \Big/ \sum_{i=1}^{n} e_{ij}^2 \right]^{1/2} \quad . \tag{9}$$

We also introduce a new variable, $Z_{ij}$, defined by

$$Z_{ij} \equiv M_j\ V_{ij} \quad . \tag{10}$$

From Equations 8 and 10 it can be seen that

$$A_j\ V_{ij} = C_j\ M_j\ V_{ij} = C_j\ Z_{ij} \quad . \tag{11}$$

From Equations 8 and 9 it is clear that

$$A_j^2 \sum_{i=1}^{n} e_{ij}^2 = C_j^2 M_j^2 \sum_{i=1}^{n} e_{ij}^2 = C_j^2 \sum_{i=1}^{n} e_{ip}^2 \quad . \tag{12}$$

In Equation 7 we now make the following substitutions --

| For | Substitute | Basis |
|---|---|---|
| $A_j V_{ij}$ | $C_j Z_{ij}$ | Eq. 11 |
| $A_j^2 \sum_{i=1}^{n} e_{ij}^2$ | $C_j^2 \sum_{i=1}^{n} e_{ip}^2$ | Eq. 12 |

with the following result:

$$\sum_{i=1}^{n} \left( \sum_{j=1}^{p} C_j Z_{ij} \right)^2 = \left( \sum_{j=1}^{p} C_j^2 \right) \left( \sum_{i=1}^{n} e_{ip}^2 \right) \quad . \tag{13}$$

We rewrite Equation 13 in the following form:

$$\sum_{i=1}^{n} \left[ \frac{C_1 Z_{i1} + \dots + C_p Z_{ip}}{(C_1^2 + \dots + C_p^2)^{1/2}} \right]^2 = \sum_{i=1}^{n} e_{ip}^2 \quad . \tag{14}$$

(It is of interest to note that if we let $p = 3$, and regard the quantities $Z_{i1}$, $Z_{i2}$ and $Z_{i3}$ as the rectangular coordinates of a point in three dimensional space, then the expression enclosed in brackets represents the perpendicular distance from the point $(Z_{i1}, Z_{i2}, Z_{i3})$ to the plane, $C_1 Z_1 + C_2 Z_2 + C_3 Z_3 = 0$).

In the interest of further simplification, we define the constant $B_j$ by

$$B_j \equiv \frac{C_j}{(C_1^2 + \dots + C_p^2)^{1/2}} \quad , \tag{15}$$

and write Equation 14 in terms of the new set of constants, as follows:

$$\sum_{i=1}^{n} (B_1 Z_{i1} + \dots + B_p Z_{ip})^2 = \sum_{i=1}^{n} e_{ip}^2 \quad . \tag{16}$$

From the definition (Equation 15) of $B_j$ it is clear that

$$B_1^2 + B_2^2 + \ldots. + B_p^2 - 1 = 0 \quad . \tag{17}$$

Values of the coefficients $B_1$, ...., $B_p$ can be estimated by minimizing the left side of Equation 16, subject to the constraint expressed by Equation 17. From them, estimates of the coefficients $A_o$, $A_1$, ...., $A_p$ can be computed as shown in the next section.

## 4C PROCEDURE FOR ESTIMATING THE COEFFICIENTS $B_j$ and $A_j$

To minimize Equation 16, subject to Equation 17, we employ "Lagrange's method of multipliers"[10], as indicated below.

$$\text{Let } \alpha = \sum_{i=1}^{n} (B_1 Z_{i1} + \ldots. + B_p Z_{ip})^2 \quad , \tag{18}$$

$$\beta = B_1^2 + \ldots. + B_p^2 - 1 = 0, \text{ and} \tag{19}$$

$-\lambda$ = the Lagrange multiplier.

According to the Lagrange technique, $\alpha$ will have an extreme value when the p + 1 parameters ( $\lambda$, $B_1$, $B_2$, ...., $B_p$) have values determined by the following p + 1 equations:

$$\begin{aligned} \frac{\partial\alpha}{\partial B_1} - \lambda \frac{\partial \beta}{\partial B_1} &= 0 \quad , \\ \frac{\partial\alpha}{\partial B_2} - \lambda \frac{\partial \beta}{\partial B_2} &= 0 \quad , \\ &\vdots \\ \frac{\partial \alpha}{\partial B_p} - \lambda \frac{\partial \beta}{\partial B_p} &= 0 \quad . \end{aligned} \tag{20}$$

By performing the indicated operations on Equations 18 and 19 we form a set of p linear equations corresponding to the p differential equations of Equations 20, and write the result in matrix form as follows:

$$\begin{bmatrix} W_{11} - \lambda & W_{12} & \cdots & W_{1p} \\ W_{21} & W_{22} - \lambda & \cdots & W_{2p} \\ \vdots & & & \\ W_{p1} & W_{p2} & \cdots & W_{pp} - \lambda \end{bmatrix} \text{X} \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad , \tag{21}$$

$$\text{where } W_{jk} = \sum_{i=1}^{n} Z_{ij} Z_{ik} = W_{kj}. \tag{22}$$

Equation 21 has a nontrivial solution if (and only if) the determinant of the p x p matrix in Equation 21 is zero. This determinant can be made zero by choosing an appropriate value of $\lambda$. But since there are p values of $\lambda$ that will make the determinant zero, it is necessary to choose the particular value that will result in minimizing $\sum_{i=1}^{n} e_{ip}^2$. We submit, without proof, that the smallest positive value of $\lambda$ is the root desired.*

We also assume that the reader is familiar with methods for finding the roots of the determinant of the symmetrical, p x p matrix in Equation 21.[11]

Let $\hat{\lambda}$ be the smallest positive value of $\lambda$ that will make the determinant zero. We substitute $\hat{\lambda}$ for $\lambda$ in Equations 21, divide each equation (except the last) by $B_p$, and form p - 1 linear equations which we express in matrix form below:

$$\begin{bmatrix} W_{11}-\hat{\lambda} & W_{12} & \cdots & W_{1,p-1} \\ W_{21} & W_{22}-\hat{\lambda} & \cdots & W_{2,p-1} \\ \vdots & & & \vdots \\ W_{p-1,1} & W_{p-1,2} & \cdots & W_{p-1,p-1}-\hat{\lambda} \end{bmatrix} \times \begin{bmatrix} B_1/B_p \\ B_2/B_p \\ \vdots \\ B_{p-1}/B_p \end{bmatrix} = \begin{bmatrix} -W_{1p} \\ -W_{2p} \\ \vdots \\ -W_{p-1,p} \end{bmatrix} \qquad (23)$$

These p-1 equations can be solved for the p - 1 ratios, $B_j/B_p$ (j = 1, ...., p - 1).

Now according to Equation 15,

$$\frac{B_j}{B_p} = \frac{C_j}{C_p} \quad ,$$

and, according to Equation 8,

$$\frac{C_j}{C_p} = \frac{A_j/M_j}{A_p/M_p} \quad ,$$

from which we conclude that

$$\frac{B_j}{B_p} = \frac{A_j/M_j}{A_p/M_p} \quad . \qquad (24)$$

---

* Johnston[10] presents a proof of this statement.

We note from Equation 9 that $M_p = 1$, and (without loss of generality) we also let $A_p = 1$. By substituting 1 for $M_p$ and $A_p$ in Equation 24, we obtain

$$\frac{B_j}{B_p} = \frac{A_j}{M_j} \quad .$$

Thus, we have the following for finding the estimate, $\hat{A}_j$, of $A_j$:

$$\hat{A}_j = \left(\frac{B_j}{B_p}\right) M_j \quad . \tag{25}$$

Equation 25 is the last step in the solution of the problem. Application of this regression technique presupposes some knowledge of measurement errors for each variable (see, for example, Equation 9). To make the technique somewhat easier to apply, we shall discuss in the next section the concept of the "quality" of a variable, and will present an alternate to Equation 9 for computing the $M_j$.

## 6C QUALITY OF A VARIABLE

We define the quality, $Q_j$ of the variable, $X_j$, as the ratio of the variance of $X_j$ to the variance of the errors made in measuring $X_j$. The equivalent mathematical definition is the dimensionless ratio given below:

$$Q_j \equiv \frac{\sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2}{\sum_{i=1}^{n} e_{ij}^2} \quad . \tag{26}$$

It may be seen from Equation 26 that if the experiment is so designed that $X_j$ varies widely about its mean, and if the errors made in measuring $X_j$ are small, then the quality of the variable is high. On the other hand, if $X_j$ varies only slightly from its mean and the measurement errors are large, then the quality is low.

From the foregoing it is clear that the quality of a variable depends not only on the precision with which it can be measured, but also upon the design of the experiment.

## 6C Use of the Quality Ratios

From Equations 26 and 9 it can be shown that the constant, $M_j$, can be defined in terms of the quality ratio, $Q_j/Q_p$, as follows:

$$M_j \equiv \left[ \frac{Q_j}{Q_p} \cdot \frac{\sum_{i=1}^{n} (X_{ip} - \bar{X}_p)^2}{\sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2} \right]^{1/2} . \qquad (27)$$

The unknown in Equation 27 is the quality ratio, $Q_j/Q_p$. To compute $M_j$ (a necessary step if the multiple error regression technique is to be used), the investigator must estimate this ratio. This may be difficult, but probably less so than estimating the ratio of the sums of the squared errors as required by Equation 9. Therefore Equation 27, rather than Equation 9, was used for computing $M_j$ in the analysis of Dynaflect data described in this report.

This section describes how the equations derived in the preceding sections may be used in estimating the constants in the regression model.

The model is given below:

$$A_o + A_1X_1 + A_2X_2 + \ldots. + A_jX_j + \ldots. + A_{p-1}X_{p-1} + X_p = 0$$

Steps to be followed in estimating the regression coefficients, $A_j$, are given below in sequence.

1. To each variable, $X_j$, assign a quality ratio, $Q_j/Q_p$, where $Q_j$ is defined as follows:

$$Q_j \equiv \frac{\sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2}{\sum_{i=1}^{n} e_{ij}^2} \quad .$$

2. Compute p values of $\sum_{i=1}^{n} V_{ij}^2$ $(j = 1, \ldots., p)$ from

$$\sum_{i=1}^{n} V_{ij}^2 = \sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2 \quad .$$

3. Compute p values of $M_j$ $(j = 1, \ldots., p)$ from

$$M_j \equiv \left[ \frac{Q_j}{Q_p} \cdot \frac{\sum_{i=1}^{n} (X_{ip} - \bar{X}_p)^2}{\sum_{i=1}^{n} (X_{ij} - \bar{X}_j)^2} \right]^{1/2} \quad .$$

4. Compute the $p^2$ elements of the symmetrical determinant,

$$\begin{vmatrix} W_{11} & W_{12} & \cdots & W_{1p} \\ W_{21} & W_{22} & \cdots & W_{2p} \\ \vdots & & & \\ W_{p1} & W_{p2} & \cdots & W_{pp} \end{vmatrix} \quad ,$$

from the equation

$$W_{jk} = M_j \, M_k \sum_{i=1}^{n} V_{ij} \, V_{ik} = W_{kj} \quad .$$

5. Find the least positive value, $\hat{\lambda}$ , of $\lambda$ that satisfies the determinantal equation,

$$\begin{vmatrix} W_{11} - \lambda & W_{12} & \cdots & W_{1p} \\ W_{21} & W_{22} - \lambda & & W_{2p} \\ \vdots & & & \\ W_{p1} & W_{p2} & & W_{pp} - \lambda \end{vmatrix} = 0.$$

(Note that this determinant is formed by subtracting $\lambda$ from the diagonal elements of the determinant formed in step 4).

6. Solve the following matrix equation for the p - 1 ratios, $B_j/B_p$ (j = 1, ...., p - 1):

$$\begin{bmatrix} W_{11} - \hat{\lambda} & W_{12} & \cdots & W_{1,p-1} \\ W_{21} & W_{22} - \hat{\lambda} & \cdots & W_{2,p-1} \\ \vdots & & & \\ W_{p-1,1} & W_{p-1,2} & \cdots & W_{p-1,p-1} - \hat{\lambda} \end{bmatrix} \times \begin{bmatrix} B_1/B_p \\ B_2/B_p \\ \vdots \\ B_{p-1}/B_p \end{bmatrix} = \begin{bmatrix} -W_{1p} \\ -W_{2p} \\ \\ -W_{p-1,p} \end{bmatrix} .$$

7. Find the estimates, $\hat{A}_j$, of the coefficients, $A_j$, (j=1, ..., p) from

$$\hat{A}_j = \left( \frac{B_j}{B_p} \right) M_j \quad .$$

8. Find the estimate, $\hat{A}_o$, of the intercept, $A_o$, from

$$\hat{A}_o = -\sum_{j=1}^{p} \hat{A}_j \, \bar{X}_j$$

If it is desired to force the regression plane through the origin ($A_o$ arbitrarily made zero), the procedure is the same as that given above with the following two exceptions:

(a) Change step 2 to read as follows:

2. Compute p values of $\sum_{i=1}^{n} V_{ij}^2$ (j=1, ...., p) from

$$\sum_{i=1}^{n} V_{ij}^2 = \sum_{i=1}^{n} X_{ij}^2$$

(b) Eliminate step 8.

(Note that the value of $M_j$, computed in step 3, is not affected by the change in the definition of $V_{ij}$, while the matrix element, $W_{jk}$, computed in step 4, is affected.)

## 8C A NUMERICAL EXAMPLE

To illustrate the effect of variations in the quality ratios, $Q_j/Q_p$, on the regression coefficients, consider the following numerical example involving only two variables (p = 2), and hence only one quality ratio, $Q_1/Q_2$. The "data" (artifically contrived to emphasize certain features of the multiple error technique), are given in Table B-1.

TABLE

Data for Example

| i | Measured Values $X_1$ | $X_2$ |
|---|---|---|
| 1 | 1 | 5 |
| 2 | 4 | 2 |
| 3 | 6 | 7 |
| 4 | 8 | 12 |
| 5 | 11 | 9 |

Using the multiple error method, five analyses of the data were performed, each for a different quality ratio, $Q_1/Q_2$. The results are given in Table 2C and are plotted, together with the data, in Figure 1C.

Comparisons with results given by the classical method can be made at extreme values of $Q_1/Q_2$. For example, if $Q_1/Q_2$ is made very small, as in Analysis 1 of Table 2C, the coefficients given by the multiple error method approach those computed by the classical procedure when $X_1$ is regressed on $X_2$. If $Q_1/Q_2$ is made very large, as in Analysis 5, the coefficients approach those given by the classical method when $X_2$ is regressed on $X_1$.

The result, clearly illustrated in Figure 1C, of making the quality of both variables the same ($Q_1/Q_2 = 1$), is a regression line that follows the visible trend of the data, and bisects the angle between the two lines obtained by the classical method. Also apparent from the figure is the fact that all possible regression lines lie between the extremes given by the classical method.

This two-variable example hopefully will confirm for the reader certain conclusions reached by the writers regarding the multiple error regression technique. These are the following:

(1) The multiple error technique is general in the sense that it includes the classical method as a special case.

(2) If measurement errors exist in more than one of the variables entering into an experiment, estimates of the regression coefficients made by the classical method will be biased. Resort to the multiple error method (if estimates of the quality ratios can be made) may lead to better estimates of the coefficients.

(3) Though the multiple error method (not necessarily under that name) has been discussed in the literature; it has not, to our knowledge, come into general use. It should.

TABLE 2C

Effect of Quality Ratio on Analysis

Model: $A_o + A_1X_1 + X_2 = 0.$

| | Multiple Error Method | | | Classical Method | | | |
|---|---|---|---|---|---|---|---|
| Analysis No. | $Q_1/Q_2$ | $A_o$ | $A_1$ | $Q_1/Q_2$ | $A_o$ | $A_1$ | Dependent Variable |
| 1 | $10^{-6}$ | 1.70000 | -1.44500 | 0 | 1.70000 | -1.44500 | $X_1$ |
| 2 | 0.20 | 0.87435 | -1.31239 | -- | -- | -- | -- |
| 3 | 1.00 | -1.00000 | -1.00000 | -- | -- | -- | -- |
| 4 | 5.00 | -2.42820 | -0.76197 | -- | -- | -- | -- |
| 5 | $10^6$ | -2.86207 | -0.68966 | $\infty$ | -2.86207 | -0.68966 | $X_2$ |

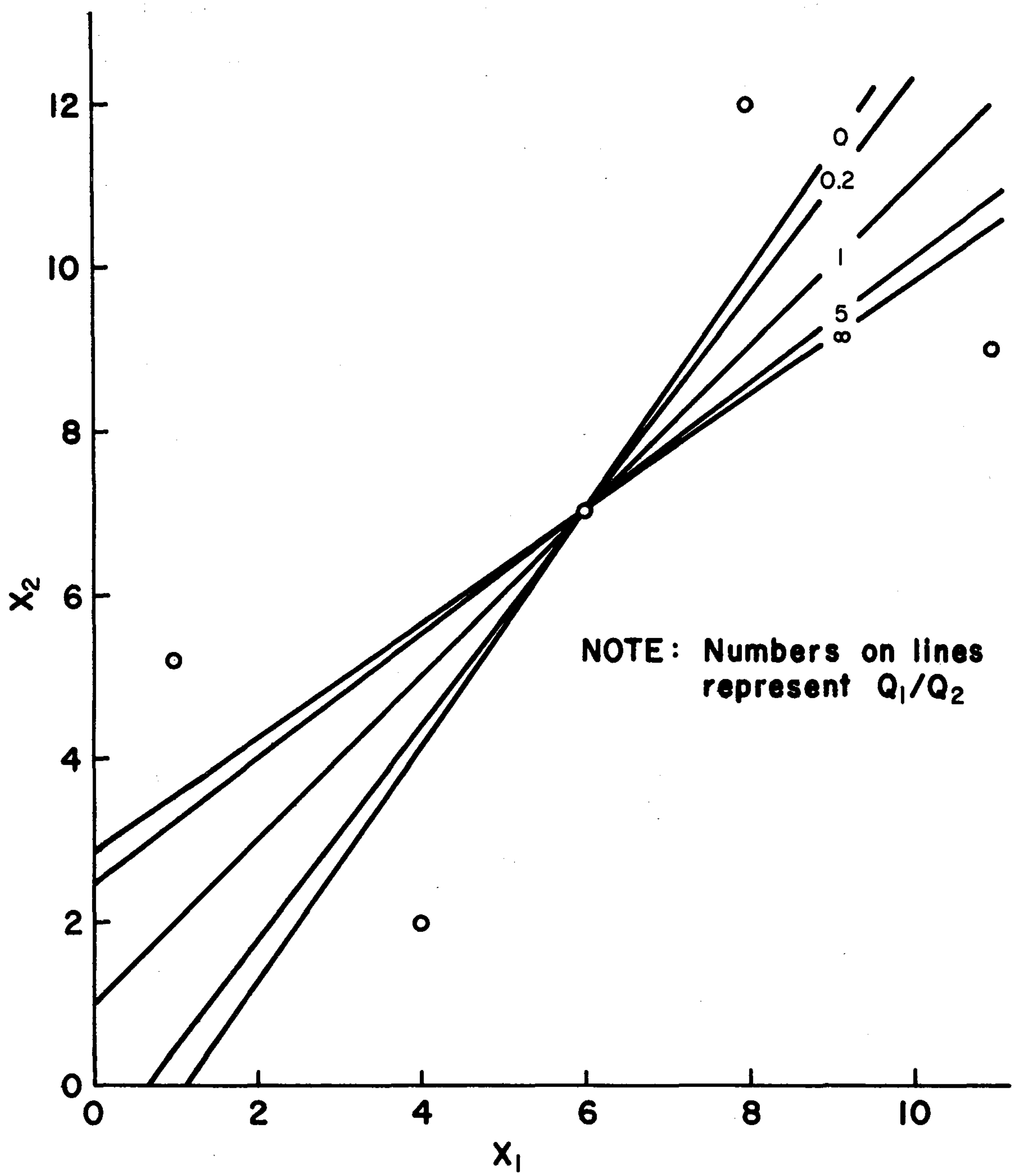



Figure 1C: Effect on the regression line of varying the quality ratio, $Q_1/Q_2$, in a two-variable analysis. The five circled points represent the data to which the model $A_o + A_1X_1 + X_2 = 0$, was fitted.

When the multiple error method was used in the analysis described in article 11 and 12, it was assumed that the two variables involved (pavement performance and design index ) were of equal quality. For this case the model was

$$A_o + A_1 X_1 + X_2 = 0 \quad , \tag{28}$$

where $X_1$ = Texas Design Index , and

$X_2$ = Performance (estimated from deflections)
Estimates of the constants are given by

$$A_1 = \pm \left[ \sum_{i=1}^{n} V_{i2}^2 \Big/ \sum_{i=1}^{n} V_{i1}^2 \right]^{1/2} \quad \text{, and} \tag{29}$$

$$\hat{A}_o = -\hat{A}_1 \overline{X_1} - \overline{X_2} \quad , \tag{30}$$

where the sign of $A_1$ is opposite to the sign of $\sum_{i=1}^{n} V_{i1} V_{i2}$.

The correlation coefficient, R, is given by

$$R = -\sum_{i=1}^{n} V_{i1} V_{i2} \Big/ \left[ \sum_{i=1}^{n} V_{i1}^2 \sum_{i=1}^{n} V_{i2}^2 \right]^{1/2} \quad , \tag{31}$$

and the root-mean-square-residual, rmsr, by

$$\text{rmsr} = \left[ \frac{2 \left(\hat{A}_1 \sum_{i=1}^{n} V_{i1} V_{i2} + \sum_{i=1}^{n} V_{i2}^2 \right)}{n-2} \right]^{1/2} .$$

The symbols $V_{i1}$ and $V_{i2}$ were defined at the bottom of page 6C.