ANALYTICAL METHODS APPLIED TO THE MEASUREMENTS OF DEFLECTIONS AND WAVE VELOCITIES ON HIGHWAY PAVEMENTS: PART 2, MEASUREMENTS OF WAVE VELOCITIES

by

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LIST OF SYMBOLS

c = phase velocity

λ, L = wavelength

$x = L/\pi H$

λ, μ = Lame's constants

β = phase velocity of shear waves

α = phase velocity of dilatational waves

u, v, w = displacements in the directions of the axes of the rectangular coordinates x, y, z.

ACKNOWLEDGEMENTS

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The opinions, findings, and conclusions expressed in this publication are those of the author and not necessarily those of the Bureau of Public Roads.

1. INTRODUCTION

The Shell vibrator has been used to investigate the behavior of a number of highway structures in the state of Texas. Measurements have been made of the wavelength at the free surface over a range of frequencies. It is found that the phse velocity of the wave is not constant but varies with the frequency. It is anticipated that the manner of the variation of the velocity relates to some of the physical properties of the structure. The object of the work was to determine what information can be obtained from the frequency-dispersion curves of phase velocity which have been obtained during the experimental work. Ultimately, it is desired to deduce, in numerical terms, values of the structural parameters, such as the interface depths of the layers composing the system and the elastic constants of the materials composing the layers.

In the following, empirical and semi-empirical methods of obtaining information from the results are discussed. It will be shown that some information can be obtained by simple calculations. These calculations are based on the assumption of a "free plate" response; this assumption neglects any loss of energy to the media underlying a layer which is acting as a free plate. The use of Jones's solutions (1) will then be discussed. These solutions provide a means of obtaining rapid results in a few special cases.

The general problem of a solution to the equations of propagation of waves in layered media will then be discussed, with the restriction that the particle motion is confined to a plane which is perpendicular to the free surface of the system, and which is oriented in the direction of propagation of the waves. Computer programs will be discussed for the solution of this problem, with the restriction that the radiation downwards of energy (into the subgrade of a highway pavement) can occur, but is limited to a finite depth; the case of propagation to infinite depth is not considered in this report.

2. THE APPLICATION OF THE FREE PLATE SOLUTIONS

Lamb's solutions for the propagation of SV (shear waves, in which the particle movement is perpendicular to the free surface) and P (dilational) waves in a plate of finite thickness moving freely in a vacuum has been given by Ewing, Jardetzky and Press (2). Two types of wave can exist under these conditions. The first, the symmetric, is a type of wave in which the motion of the particles of the material composing the plate is symmetrical about the plane of symmetry of the plate.* For waves which are long compared with the thickness of the plate, the phase velocity approaches a value given by

$$\frac{c^2}{\beta^2} = 4(1 - \frac{\beta^2}{\alpha^2})$$

When $\nu = 1/4$, $3\beta^2 = \alpha^2$, we have $c = 2\sqrt{2/3}\beta$. The wavelength and frequency of waves are related (Reference 2, Equation 6-12) and programs have been written which enable numerical results to be obtained.** Numerical results are not given explicity and the results are obtained after a number of cycles of calculation; the values of the solution sought are compared with the values given by the previous cycle, and when these are the same within sufficiently close limits that result is taken final. The results of a typical set of computations is given by Ewing (Reference 2,

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*The particle movements in the symmetric type of wave are shown in a diagram in reference (Reference 3, Figure 5.3, p. 129).

** Fortran program designated by WHC33E, and Wang programs designated by 681.09, 681.2 and 681.3.

Figure 6-1, Page 284). A series of results has been computed using the programs referred to. Some of these are shown in Figure 1. In this figure, the reciprocal wavelength is plotted against the frequency, with both quantities in dimensionless form. This single plot thus represents all dimensionally similar systems. The method of plotting which is adopted causes a succession of points to fall on or near a straight line. Two quantities (the intercept and the slope of the line) are available in order to assist the matching of a theoretical system to a given set of experimental results. On repeating the calculations using a range of values for Poisson's ratio, it is found that the plotted points are distributed in a relatively narrow band. The slope of the lines is close to unity, and the intercept on the axis representing the reciprocal of the wavelength varies by only a small amount. This intercept is shown plotted in Figure 2, with Poisson's ratio as the abscissa. There is some uncertainty (about 8%) in the value of the intercept. It depends on which of the points on the graph of reciprocal wavelength against the frequency are selected, as these points are closely but not exactly rectilinear. The result is intended to provide a practical means of analysis. As Poisson's ratio for highway materials is large, an average value of -0.72 was adopted for $1/\lambda$ at zero frequency. This is designated by $1/\lambda]_0$.

This method can be applied to the results of measurements of frequency and wavelength. The reciprocal of the slope of the line of points plotted as described gives the phase velocity of shear waves in the material of which the plate is composed. If the density ρ of the material is known,

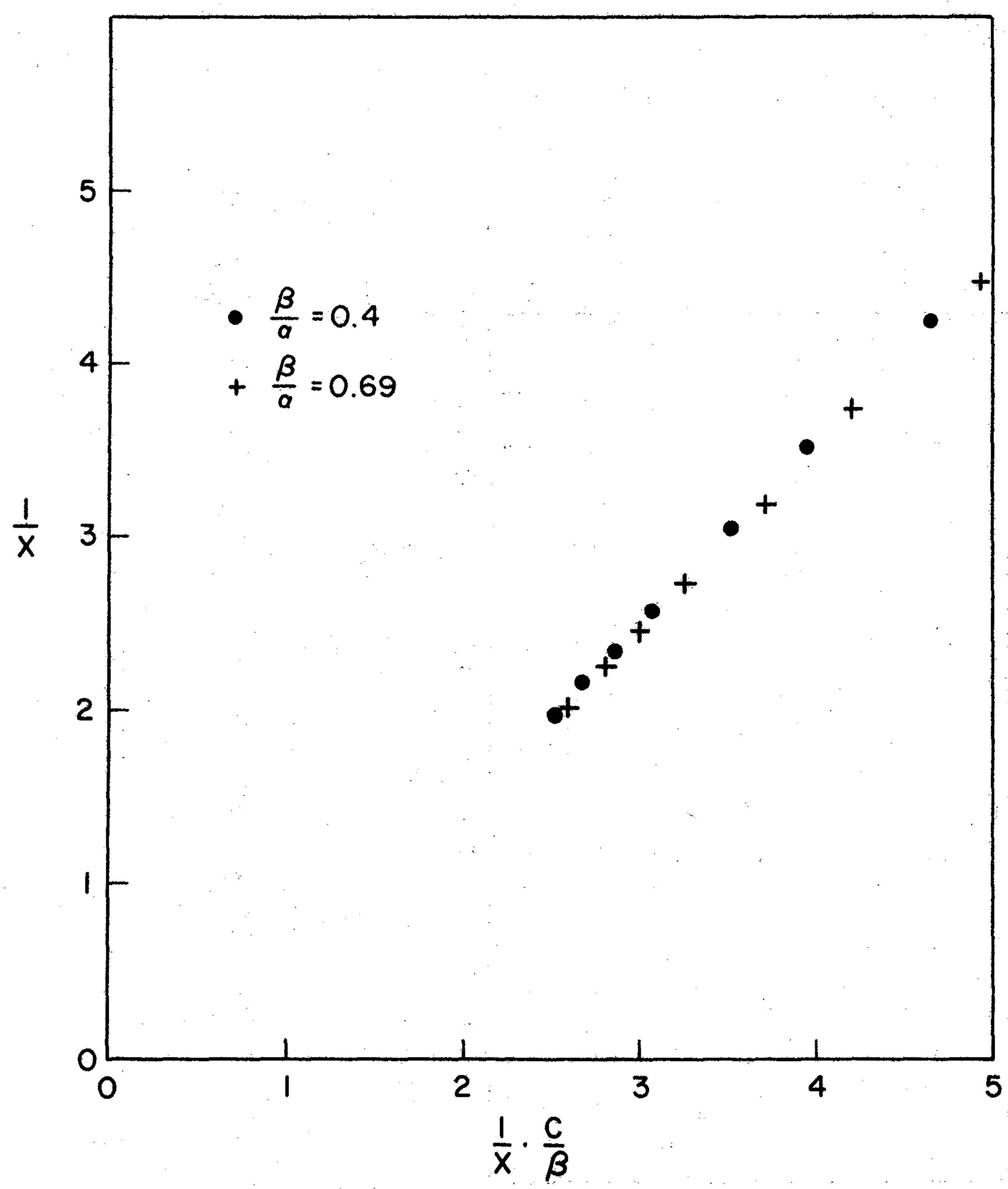

FIGURE 1 - Symmetric wave in a free plate. Plot of reciprocal wavelength against frequency.

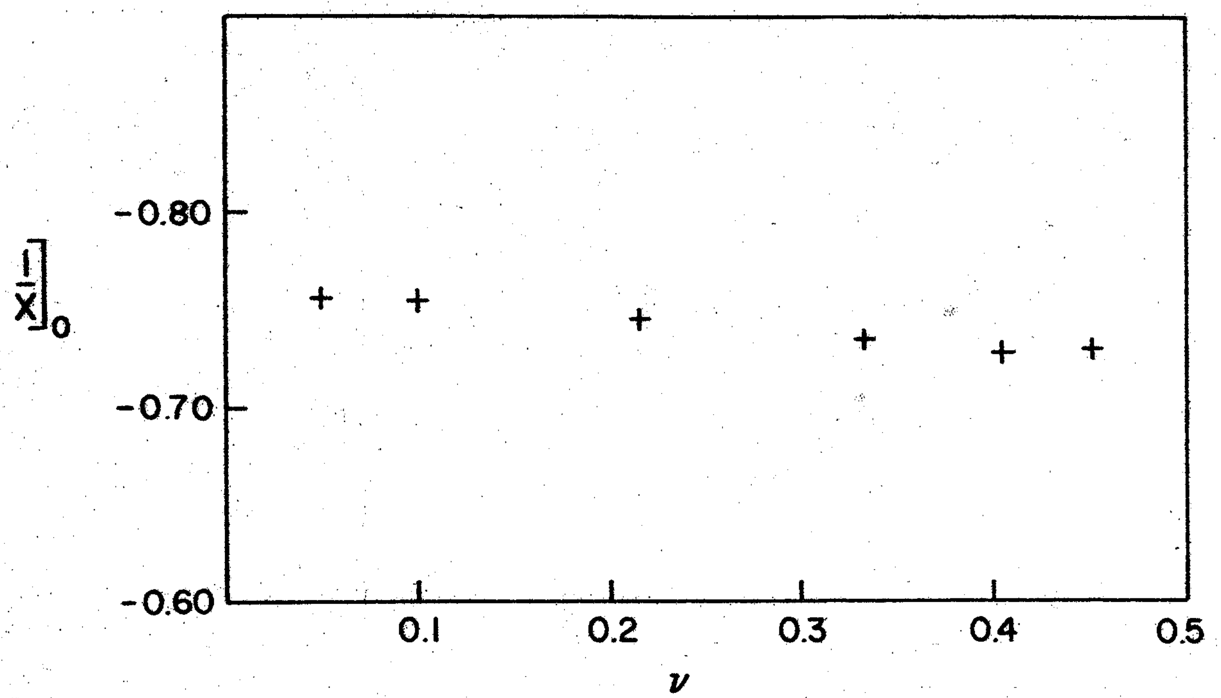

FIGURE 2 - Symmetric wave in a free plate. Plot of the intercept on the axis of reciprocal wavelength against Poisson's ratio.

the shear modulus μ can be calculated from the formula

$$\mu = \rho \cdot \beta^2$$

where β is the shear wave velocity. The modulus here is in gravitational units (poundals per square foot, if foot - pound - second units are employed).

The intercept on the reciprocal wavelength axis, $1/\lambda]_0$, is used to obtain an estimate of the thickness H of the plate. It is obtained from the formula

$$1/\lambda]_0 \pi H = -0.72.$$

Similar results are obtained for the case of the second type of wave, the asymmetric, for which the solution is given in Ewing, Jardetzky and Press (Reference 2, Equation 6-110, Page 283). This wave corresponds approximately with the Rayleigh surface wave (Reference 3, Page 50); the particle movements are smaller on one side of the plate than on the other. A typical set of the results of computations of the phase velocity at the surface of the plate over a range of wavelengths is given in Ewing, Jardetzky and Press (Reference 2, Figure 6-2, Page 286). At short wavelengths, the phase velocity of the waves approaches that of Rayleigh waves in a semi-infinite medium. The solution has been programmed and a typical set of results of the computations is shown in Figure 3.* As before, the reciprocal of the wavelength is plotted against the frequency; both the quantities are in dimensionless form. On repeating the calculations using a range of values of Poisson's ratio, it is found that the plotted points are again

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*The Fortran program is designated by WHC33A, and Wang programs by 681.08, 681.09 and 681.1.

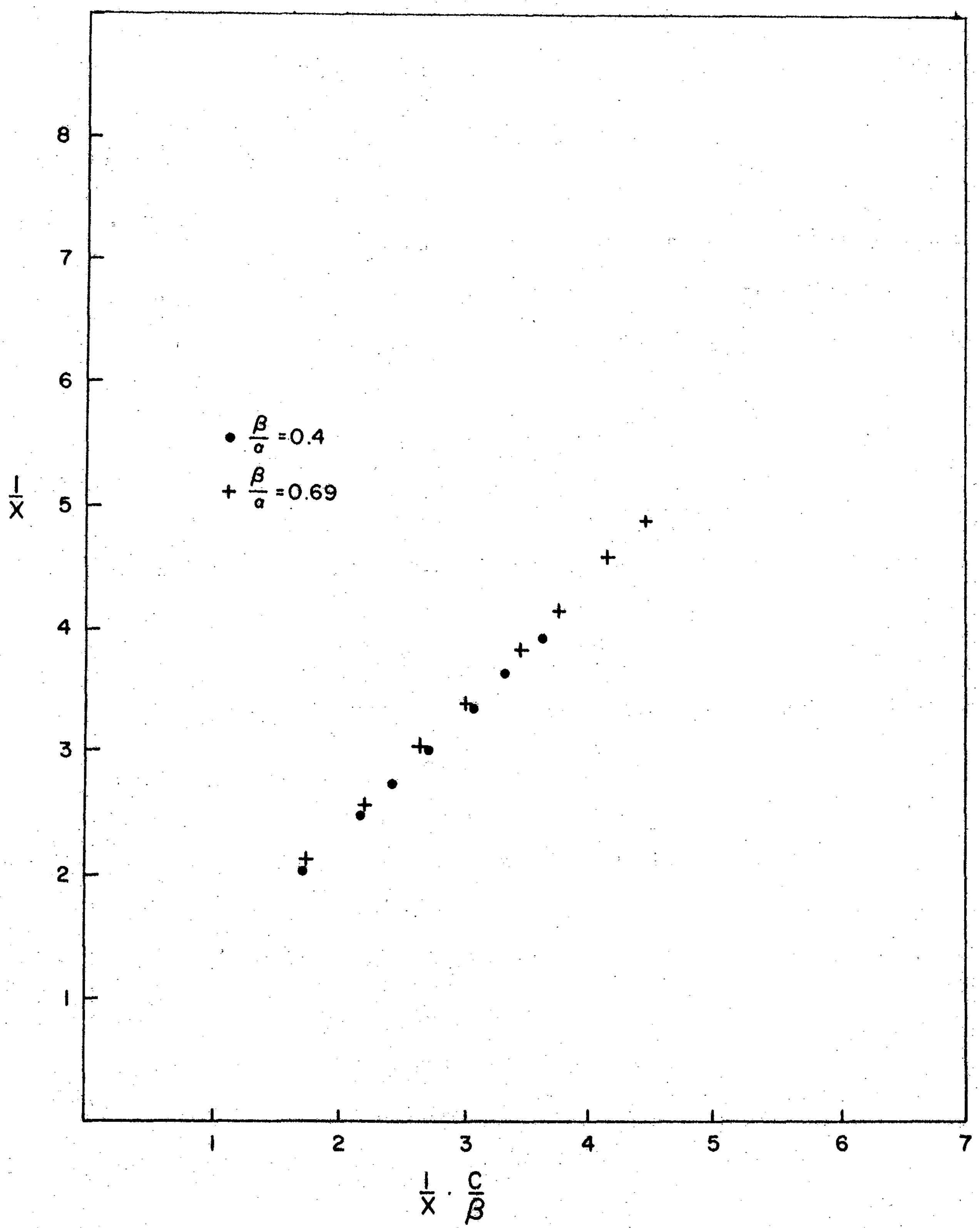


FIGURE 3 - Asymmetric wave in a free plate. Plot of the reciprocal wavelength against the frequency.

distributed in a relatively narrow band. Two parameters are available for matching a theoretical system to a set of experimental results, as in the previous case. The intercept in this case is on the positive side of the reciprocal wavelength axis. It is more dependent on the value of Poisson's ratio than is the intercept formed by a set of results obtained from the symmetric type of wave; its value is shown in Figure 4, and an average of 0.32 was adopted. The results can be applied in order to determine the shear modulus of the material composing the plate, and also the effective thickness of the plate. The method used is similar to the previous one. The reciprocal slope of the plot described yields the phase velocity of shear waves in the material, from which the shear modulus may be found in gravitational units. The intercept on the reciprocal wavelength axis can be used to determine the effective thickness as follows:

$$1/\lambda]_0 \pi H = 0.32$$

where H is the thickness of the plate.

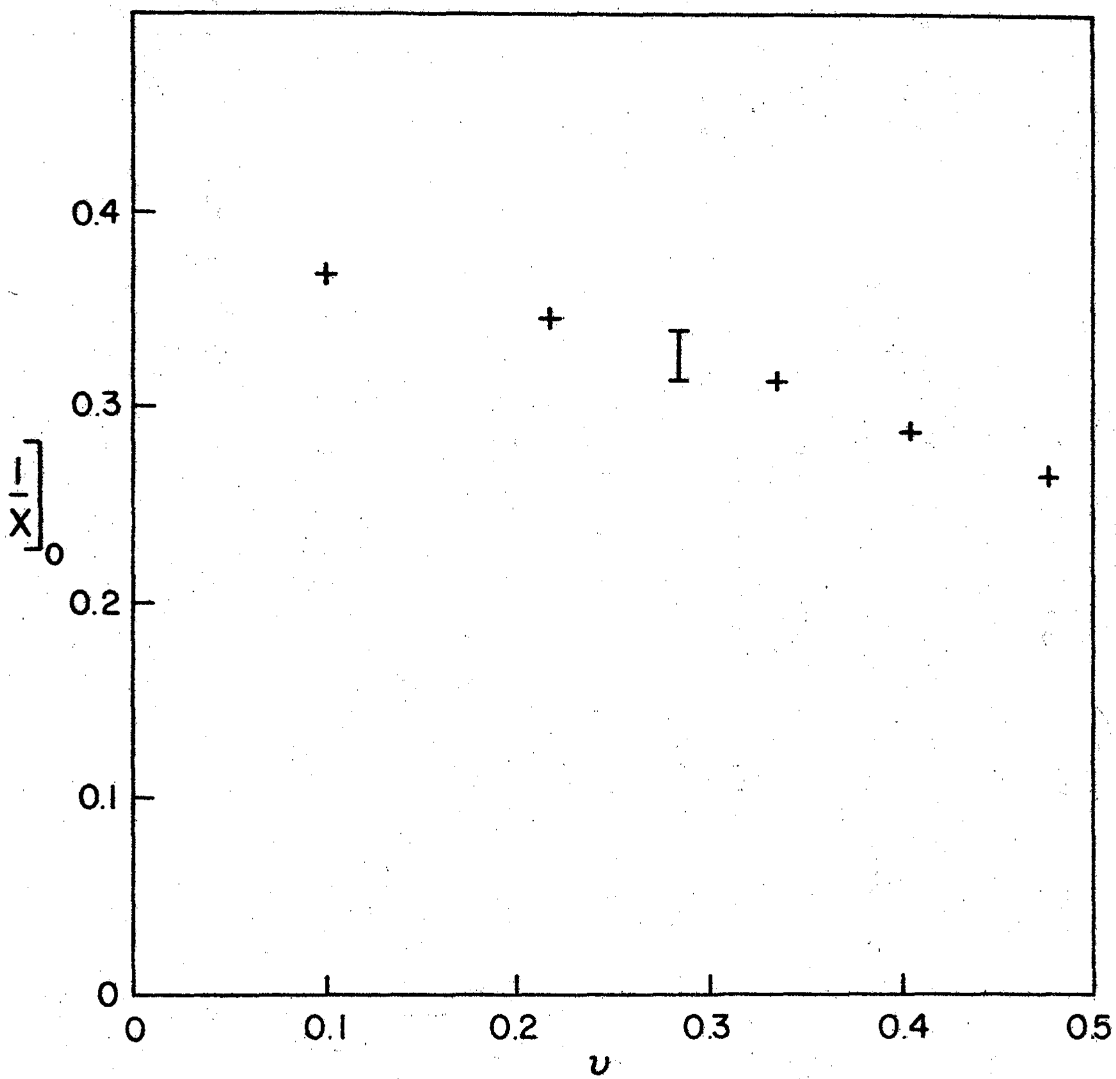


FIGURE 4 - Asymmetric wave in a free plate. Plot of the intercept on the axis of reciprocal wavelength against Poisson's ratio.

2.1 - The Application of these Results to some Particular Cases

The results of a set of measurements made by means of the Shell vibrator may be plotted as shown in Figures 8-15. These figures show the wavelength plotted as functions of frequency of the excitation; the wavelengths are those which are measured at the surface of the ground. Alternatively, the results may be plotted as shown in the same figures, where the reciprocal of the wavelength is plotted against the frequency. These plots can be represented, approximately at least, by a succession of straight lines, suggesting that they may be interpreted by the methods described in the preceding section. The results of interpretation by this method for all the sections referred to in Figure 6, Part 1 are shown in Table 1. The values of Young's modulus in this table are based on an assumed value of 0.45 for Poisson's ratio in order to give a comparison with the results given in Figure 6, Part 1 of this report. The results are plotted in Figure 5. This figure shows the thicknessess of a number of free plates, and the elastic moduli of the materials of which they are composed; these free plates are those which yield frequency dispersion curves of phase velocities which match parts of the frequency dispersion curves of the structures shown.

The results indicate that there is little agreement between the values of the Young's moduli determined by means of the Dynaflect and those found on the basis of the assumptions employed here. The method is an indication only, and can be applied without using computing facilities.

TABLE 1 - RESULTS OBTAINED ASSUMING THAT THE HIGHWAY STRUCTURES EACH VIBRATE AS A SUCCESSION OF FREE PLATES. DATA FROM A&M TEST FACILITY, SECTIONS 1 THROUGH 8

Section	Intercept $1/\lambda_e$ (Ft^{-1})	Thickness of Equivalent Free Plate (Ft.)	Slope of Points Assumed to be Shear Wave Velocity β (Ft/Sec)	Shear Modulus $\mu = \rho\beta^2$ ρ=125 lb/ft^3 (Lb/In^2)	Young's Modulus E, Assuming Poisson's Ratio ν = 0.45 (lb/in^2 x 1000)
1	0.49	0.2	17,000	8,000,000	23,000
	0.05	0.6	3,030	250,000	730
	0.05	0.6	1,852	94,000	270
2	0.04	2.3	1,110	33,000	96
	-0.16	1.4	1,110	33,000	96
	0.25	0.4	1,890	96,000	280
3	0.08	1.2	1,110	33,000	96
	-0.24	1.2	1,220	41,000	120
	0.35	0.3	2,080	120,000	350
4	0.24	0.4	2,000	110,000	320
	0.06	1.6	1,330	49,000	142
5	0.14	0.7	3,225	290,000	840
	0.05	1.9	1,695	78,000	230
	-0.02	11	560	8,000	23
6	0.04	2.4	1,143	35,000	102
	-0.06	3.8	990	26,500	77
	-0.05	4.6	1,087	32,000	93
	0.75	0.1	3,330	300,000	870

TABLE 1 - RESULTS OBTAINED ASSUMING THAT THE HIGHWAY STRUCTURES EACH VIBRATE AS A SUCCESSION OF FREE PLATES. DATA FROM A&M TEST FACILITY, SECTIONS 1 THROUGH 8 (Continued)

Section	Intercept $1/\lambda]_0$ (Ft^{-1})	Thickness of Equivalent Free Plate (Ft.)	Slope of Points Assumed to be Shear Wave Velocity β (Ft/Sec)	Shear Modulus $\mu = \rho\beta^2$ ρ-125 lb/ft^3 (Lb/In^2)	Young's Modulus E, Assuming Poisson's Ratio $\nu = 0.45$ (lb/in^2 x 1000)
7	0.23	0.4	1,350	50,000	145
	0.02	4.8	1,000	27,000	78
	0.02	4.8	770	16,000	47
8	0.02	4.8	833	19,000	55
	-0.015	15	1,099	33,000	96
	0.27	0.4	1,176	37,000	107
	0.02	4.8	1,053	30,000	87

3. THE APPLICATION OF THE JONES' SOLUTIONS

Jones (1) applied the solutions of the wave equation for three distinct systems composed of layered isotropic media. The solutions are intended for waves in which the particle movement is in the form of an ellipse, the plane of which is vertical and oriented in the direction of propagation of the wave. The three systems are:

(1) A solid layer of finite thickness overlying a semi-infinite liquid medium.

(2) As in (1) but with an intermediate layer in which the compressional wave velocity is less than that in the underlying liquid medium.

(3) As in (2) but with an intermediate layer in which the velocities of compressional and shear waves are similar to those in the surface layer but considerably greater than those in the underlying medium.

Jones has given examples of the application of the solutions which he discussed. In Jones' paper (Reference 1, Figure 2, Page 23) the figure shows the phase velocity-wavelength relationship for a hypothetical case (1) above, In the same paper (Figure 4, Page 25) theoretical curves are shown for a hypothetical case (3) above; this figure is reproduced, see Figure 6, showing the designations of the programs which have been prepared in order to obtain numerical values of the solutions discussed by Jones. The writer was unable to find a set of data which satisfies all the conditions of Jones' case (3). Only a few of the sets of results were considered capable of partial interpretation by this means. The restrictions on the values of the parameters of the layers render the solutions valid for only a small proportion of cases occurring in practice. The application of this case to a particular set of results obtained by means of the Shell Vibrator is shown in Figure 7.

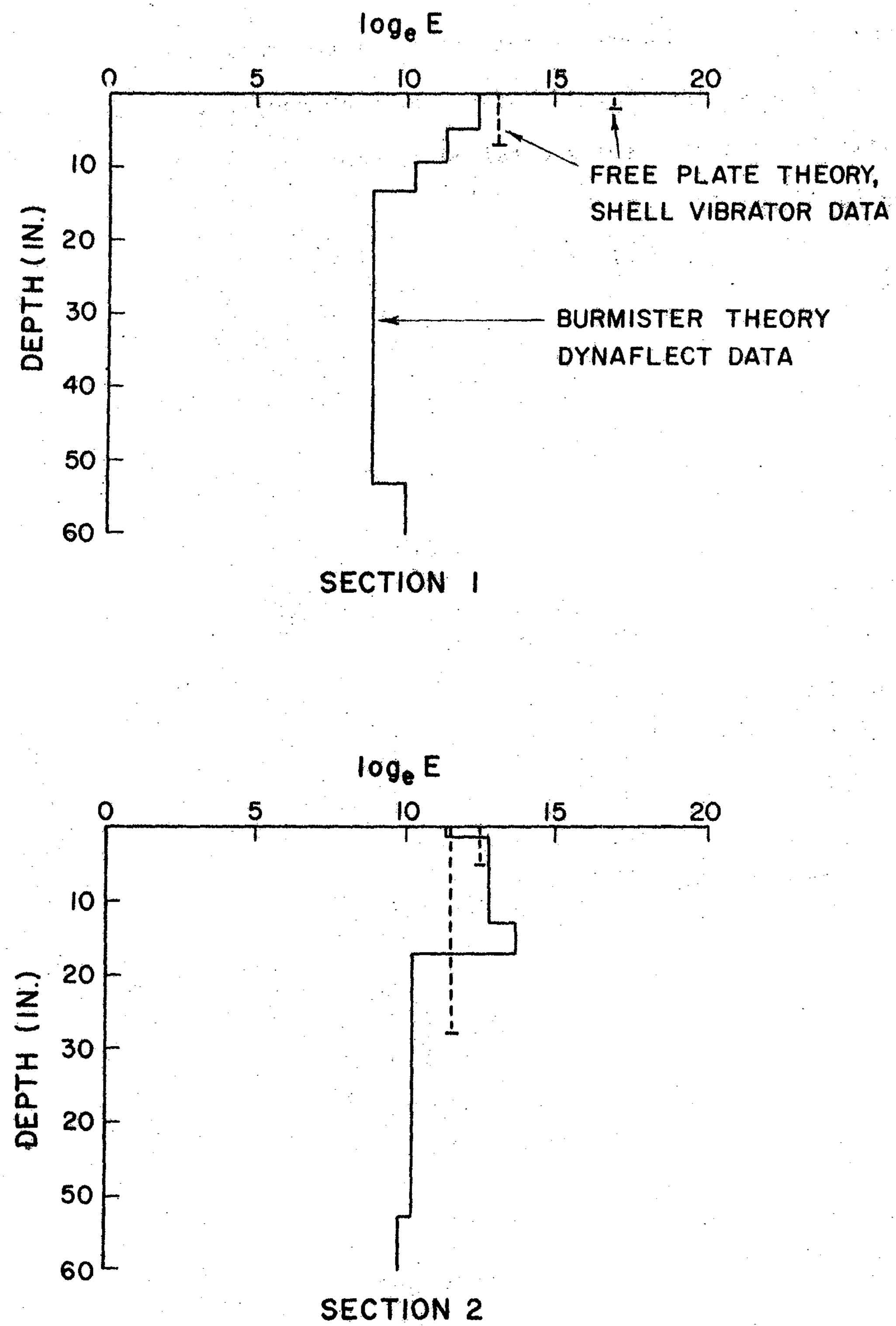


FIGURE 5 - The application of the free plate solutions to results obtained by means of the Shell Vibrator, Texas A&M Test Facility. Sections 1 and 2.

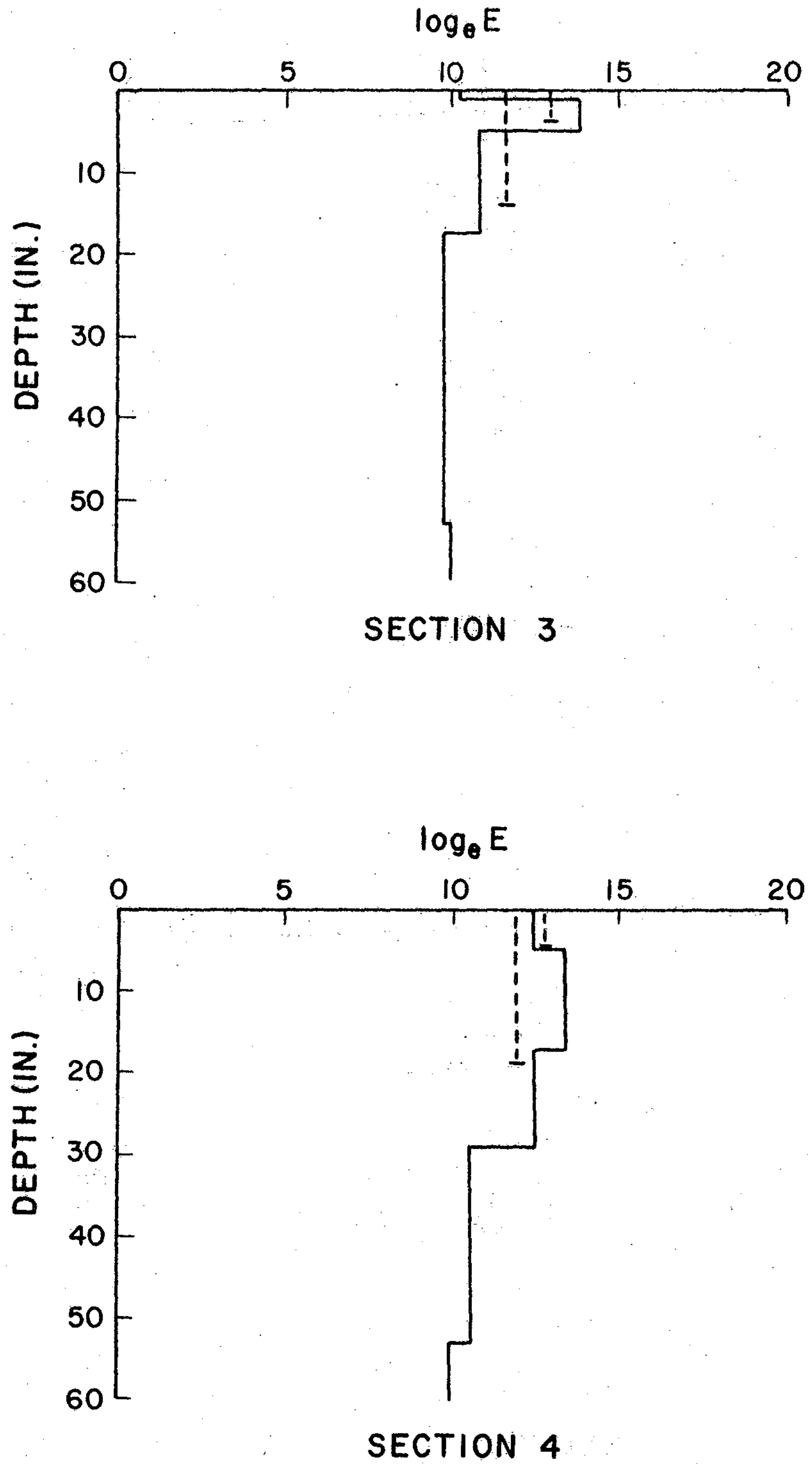
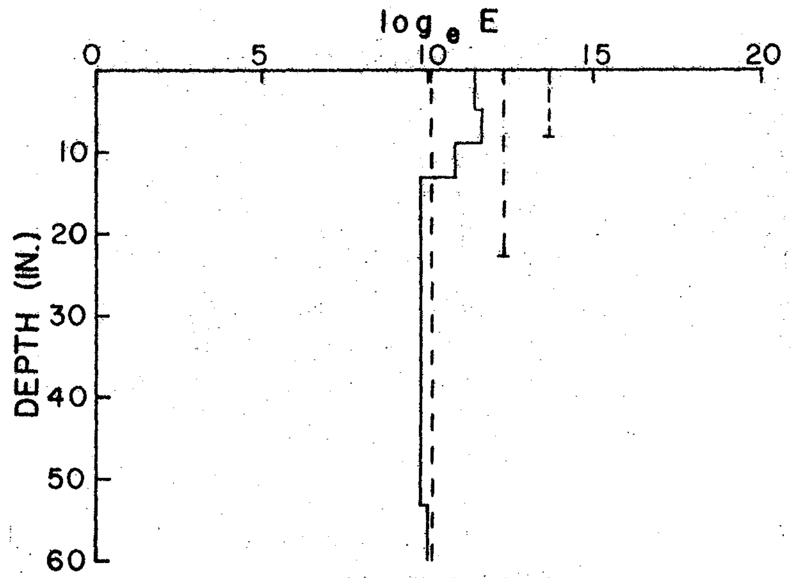
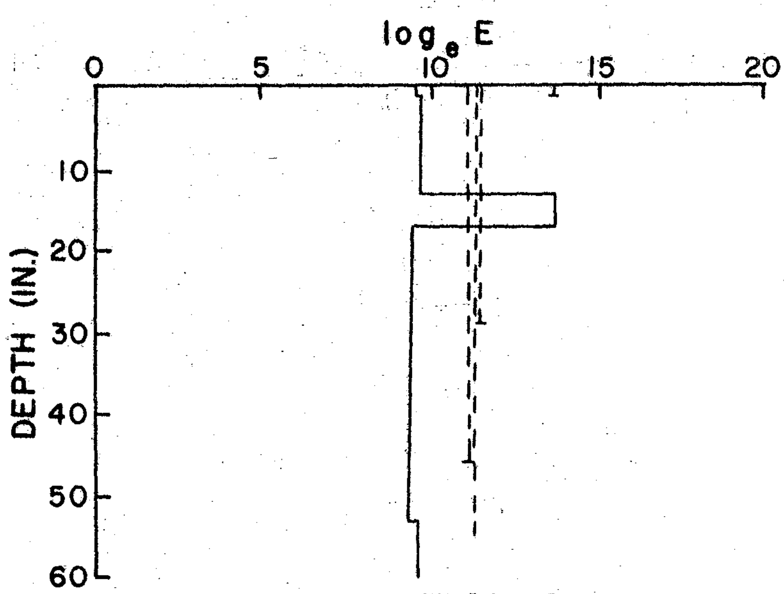


FIGURE 5 - The application of the free plate solutions to results obtained by means of the Shell Vibrator, Texas A&M Test Facility. Sections 3 and 4.

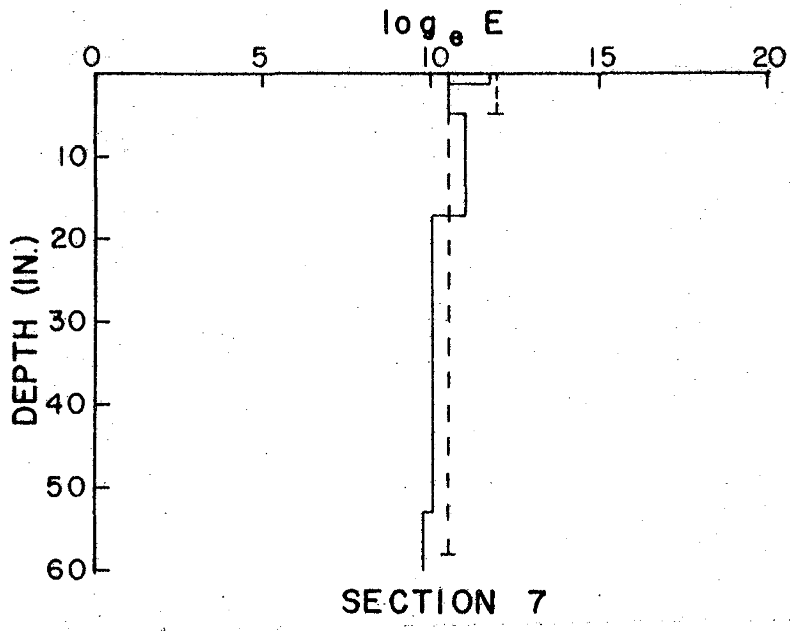


SECTION 5



SECTION 6

FIGURE 5 – The application of the free plate solutions to results obtained by means of the Shell Vibrator, Texas A&M Test Facility. Sections 5 and 6.



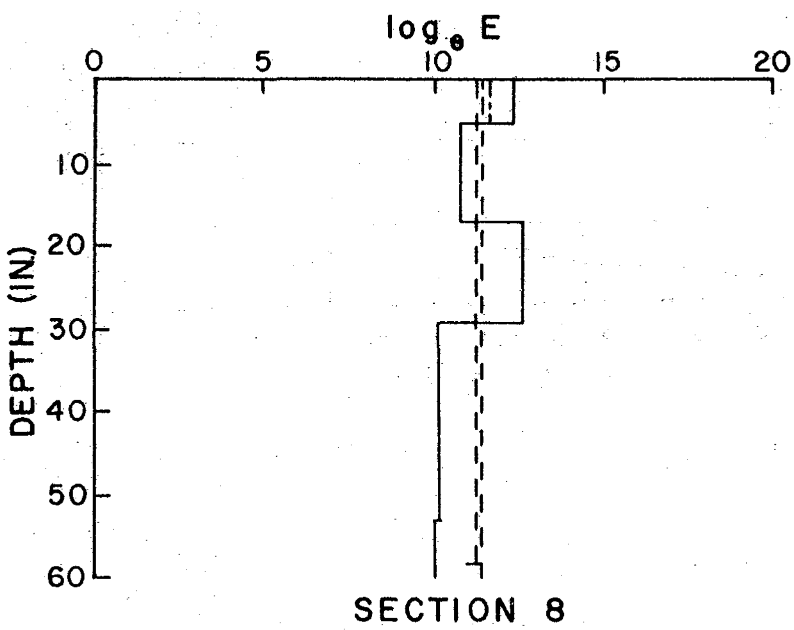


FIGURE 5 - The application of the free plate solutions to results obtained by means of the Shell Vibrator, Texas A&M Test Facility. Sections 7 and 8.

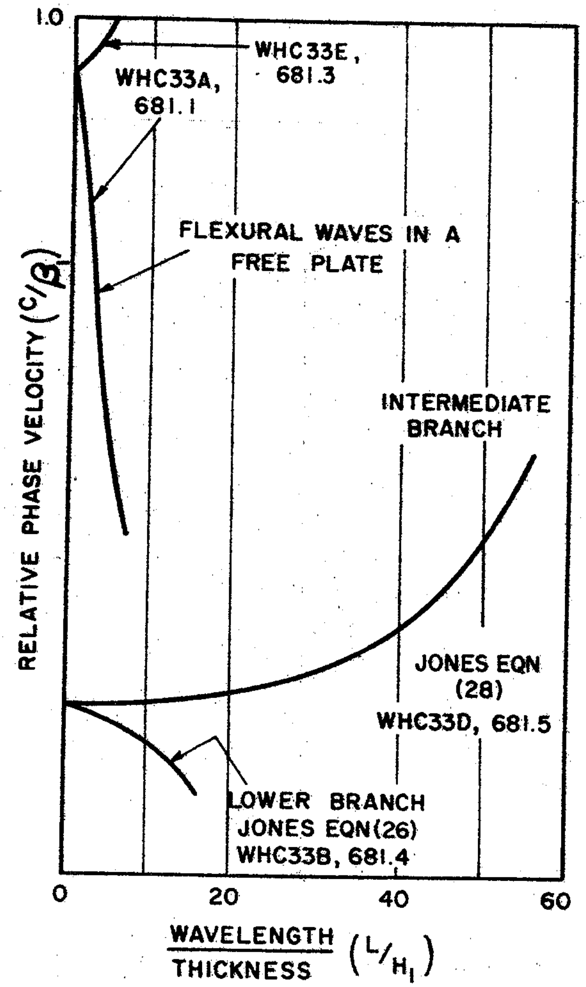


FIGURE 6 - The solutions for Jones' Case (3), showing the designations of the computer programs written in order to compute numerical results.

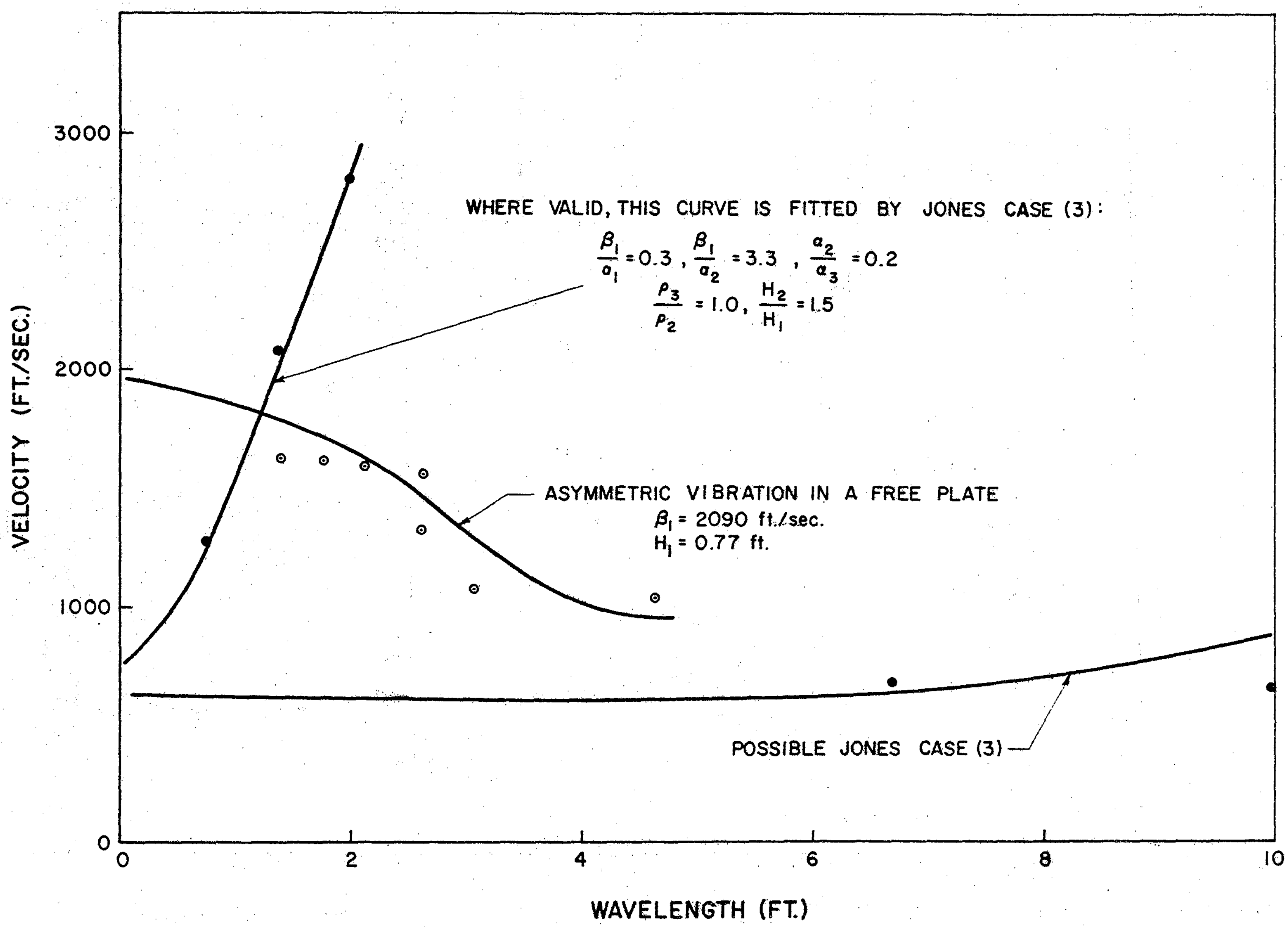
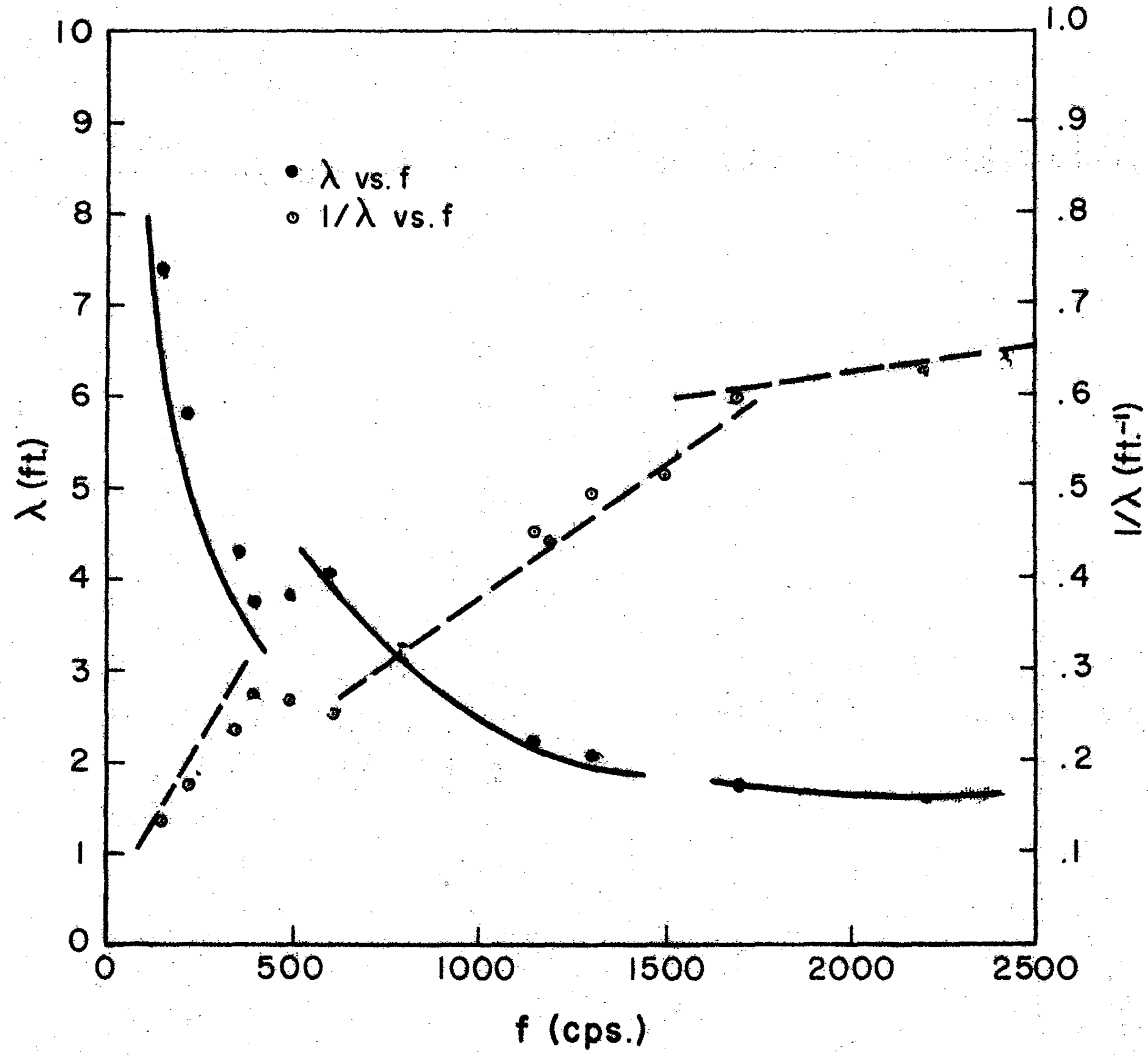


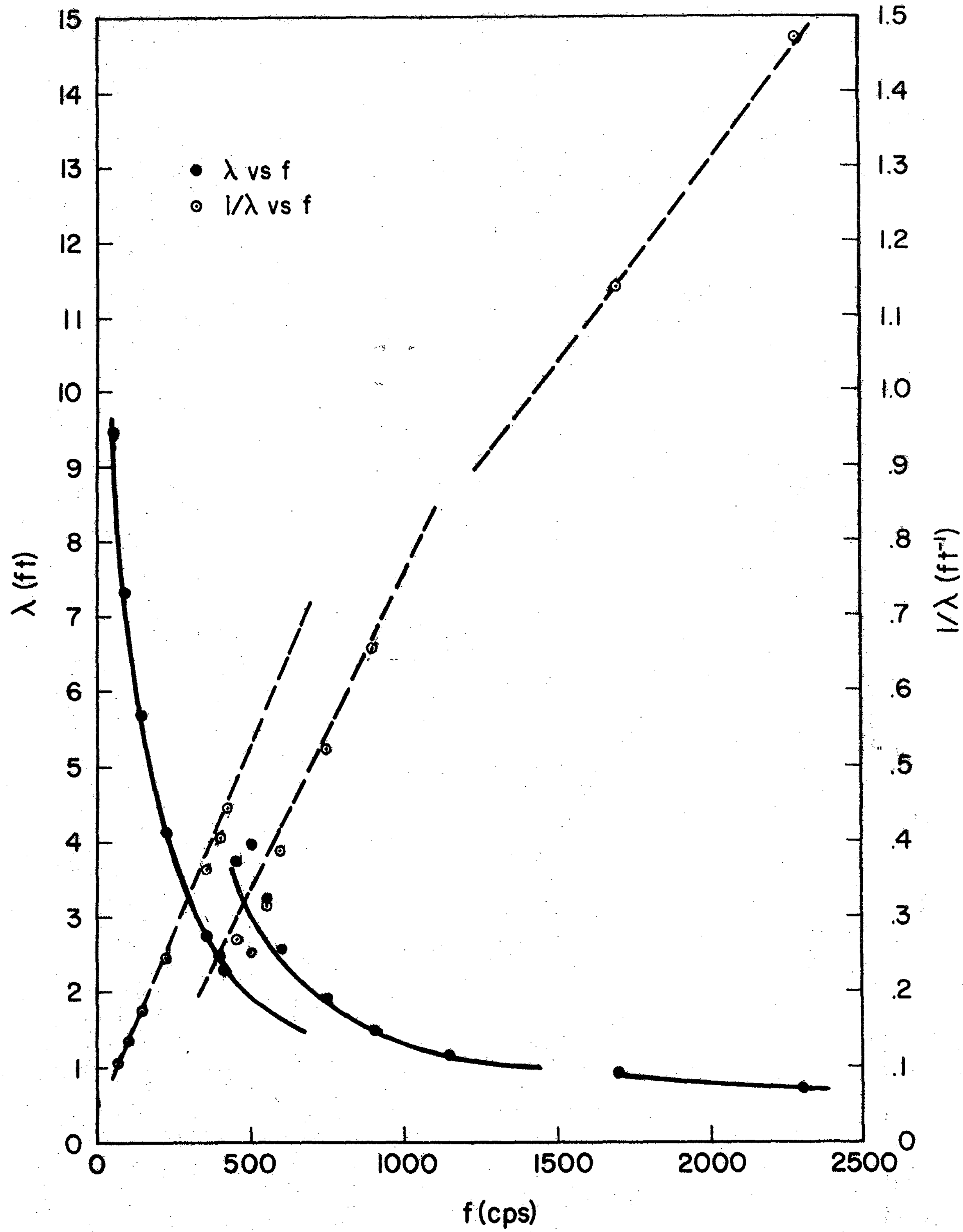
FIGURE 7 - The application of Jones' Case (3) to Section 4-67-1-1 (date 4-27-66).



SECTION NO. I

FIGURE 8 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 1.

The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).



SECTION NO. 2

FIGURE 9 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 2.
The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).

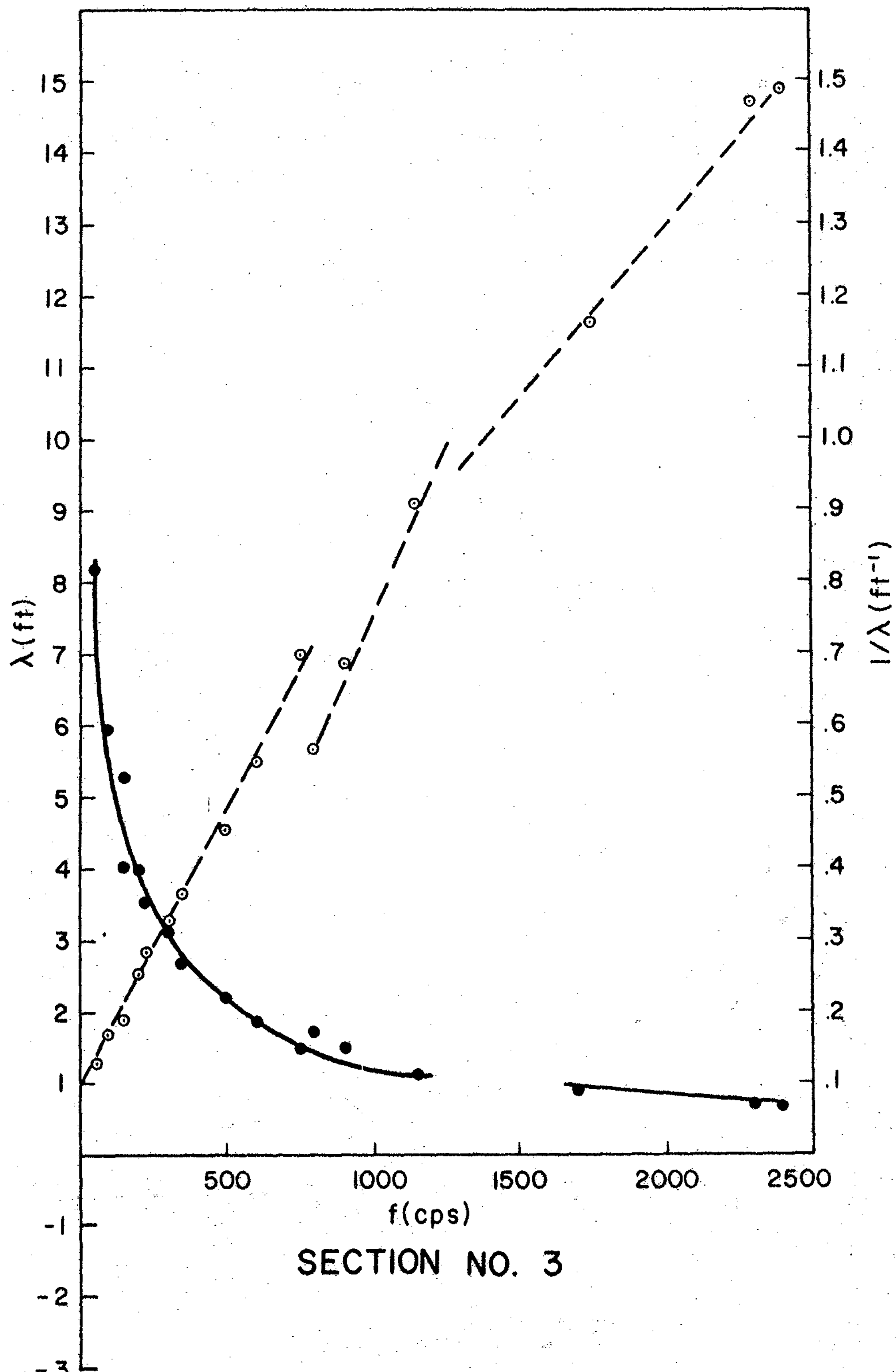
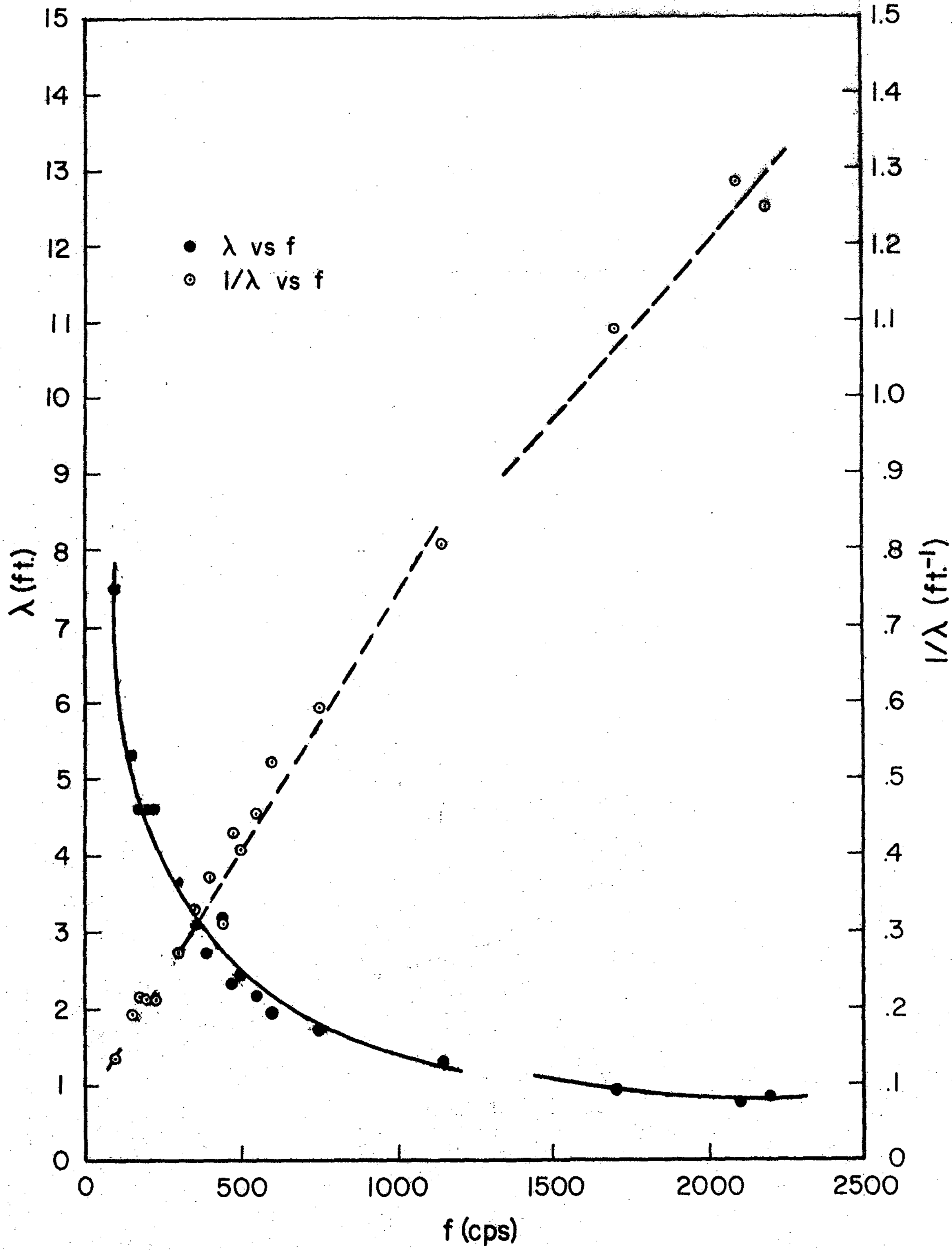


FIGURE 10 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 3.

The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).



SECTION NO. 4

FIGURE 11 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 4.

The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).

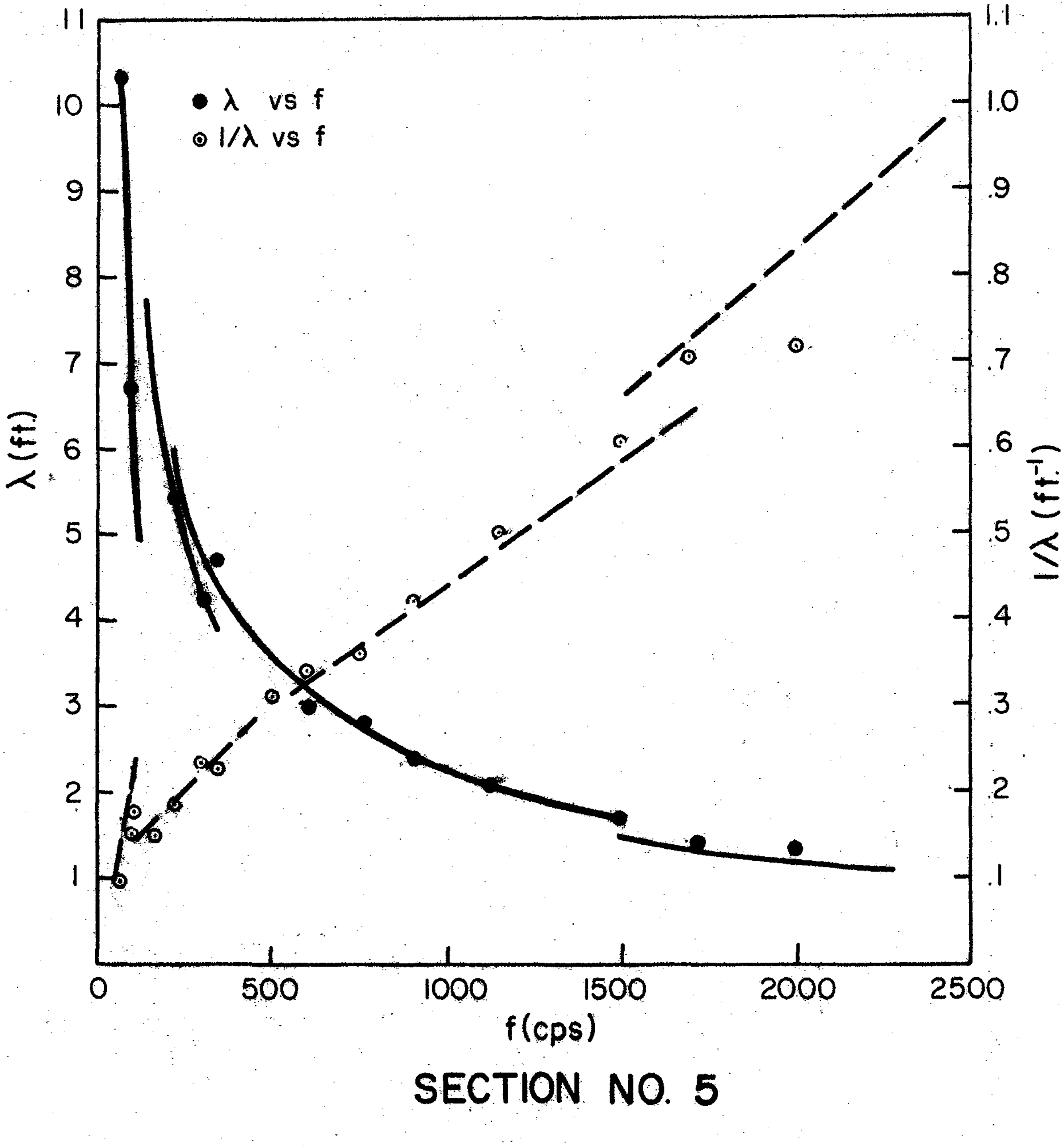
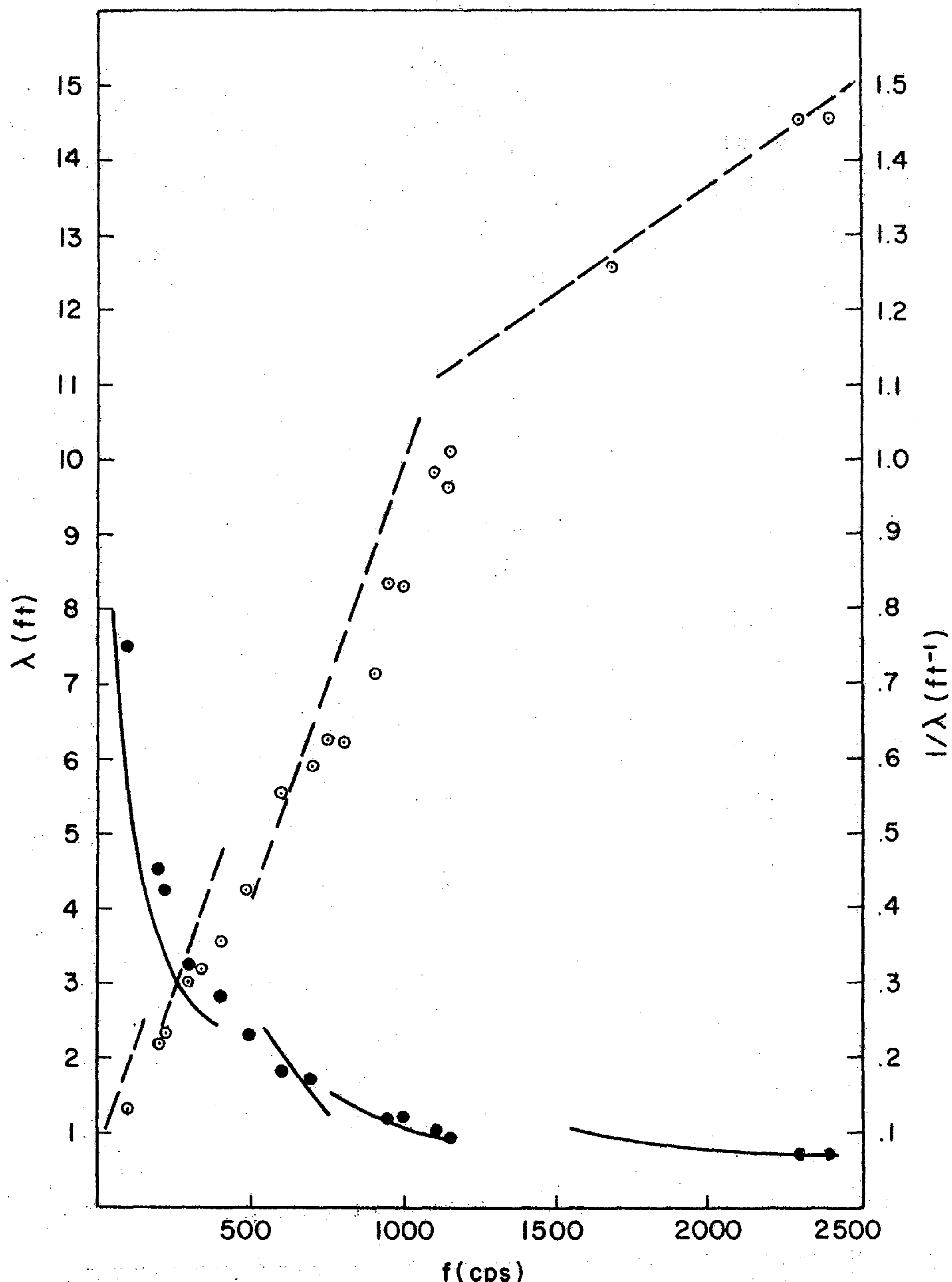


FIGURE 12 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 5.

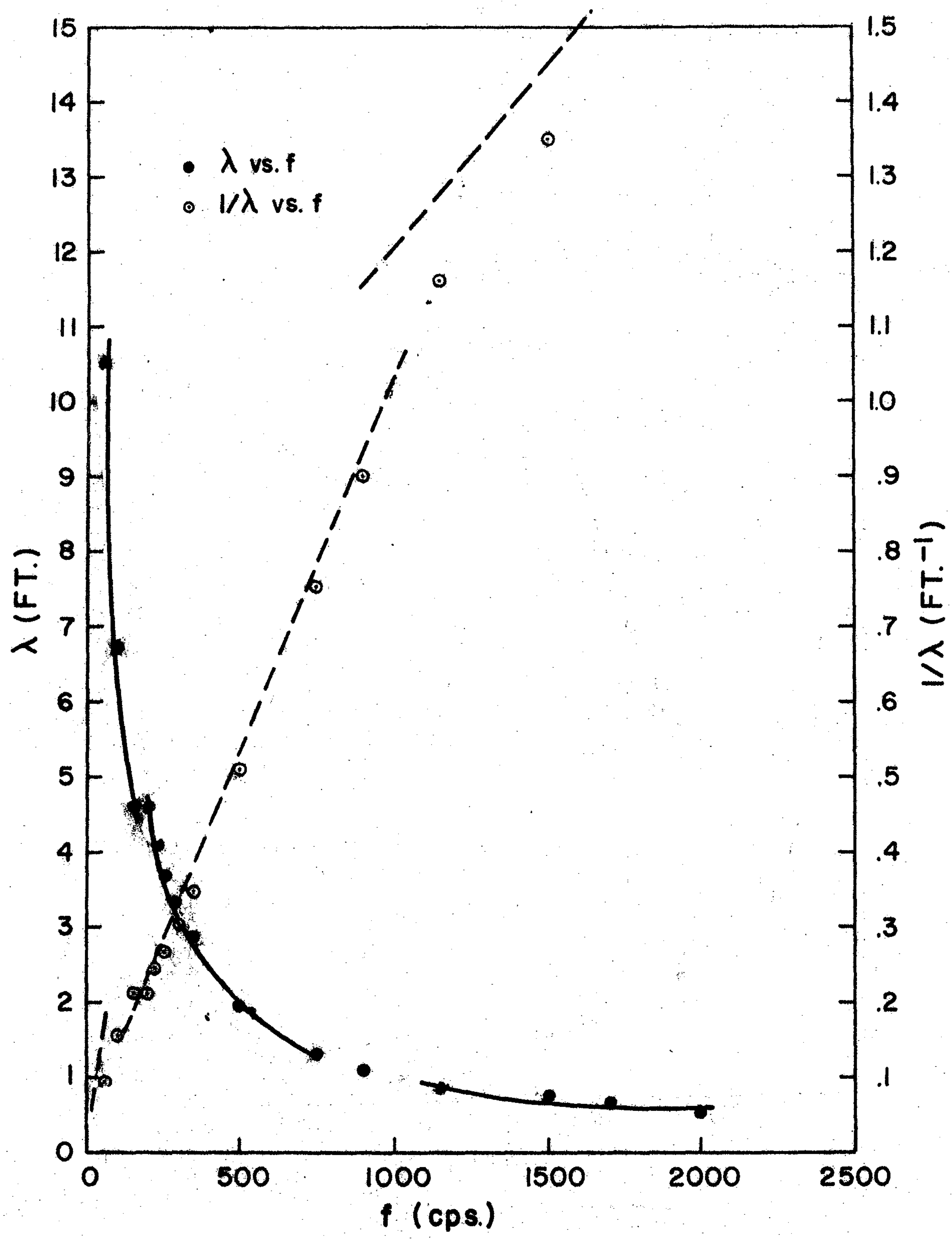
The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).



SECTION NO. 6

FIGURE 13 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 6.

The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).



SECTION NO. 7

FIGURE 14 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 7.

The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).

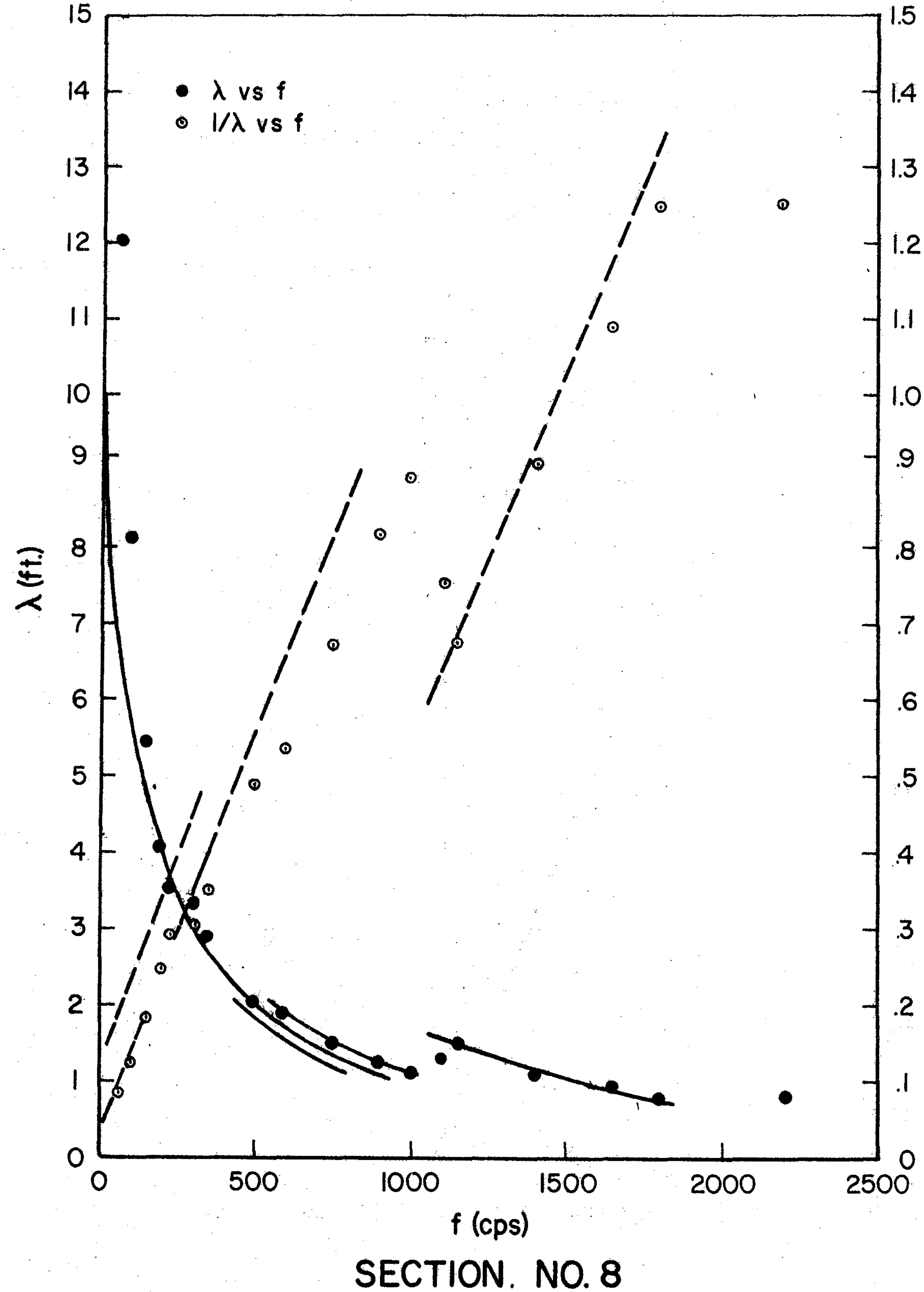


FIGURE 15 - Plot of wavelength and reciprocal wavelength against frequency, Texas A&M Test Facility - Section 8.

The curves show the calculated results for a compound free plate (Programs: WHC37, WHC38).

4. THE SOLUTION OF THE WAVE EQUATION FOR THE TRANSMISSION OF RAYLEIGH TYPE WAVES IN A MULTILAYERED SYSTEM - THOMSON-HASKELL MATRICES

We shall write the differential equations governing the motion of the particles of the materials composing the layers of a highway structure as follows:

$$\nabla^2\phi_o = \frac{1}{\alpha^2} \cdot \frac{\partial^2\phi_o}{\partial t^2} \quad \text{and} \tag{1}$$

$$\nabla^2\psi_o = \frac{1}{\beta^2} \cdot \frac{\partial^2\psi_o}{\partial t^2} \quad . \tag{2}$$

Within each layer, the displacements u and v and the stresses σ_y and τ_{xy} can be derived from the potentials ϕ_o and ψ_o. If ϕ_o and ψ_o are assumed periodic both in time and along the x-axis, the potentials can be written

$$\phi_o = \phi(y)e^{i(\omega t-kx)}$$

$$\psi_o = \psi(y)e^{i(\omega t-kx)}$$

and (1) and (2) become

$$\frac{d^2\phi}{dy^2} = r^2\phi \tag{3}$$

and

$$\frac{d^2\psi}{dy^2} = s^2\psi \tag{4}$$

Following Thrower (4), take the solutions of (3) and (4) as

$$\phi = \left[\frac{b_1}{\mu} \cosh ry - \frac{b_3}{\mu} \sinh ry\right] e^{i(\omega t - kx)} \qquad (5)*$$

$$\psi = \left[\frac{ib_2}{\mu} \cosh sy - \frac{ib_4}{\mu} \sinh sy\right] e^{i(\omega t - kx)} \qquad (6)$$

The constants b_j are to be determined for each layer from the conditions at its interfaces. The stresses σ_y and τ_{xy}, and the displacements u and v are given by

$$\sigma_y = \lambda \nabla^2 \phi + 2\mu \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \quad ,$$

$$\tau_{xy} = \mu \left[2 \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right], \qquad (7)$$

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \quad ,$$

$$v = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \quad .$$

Equations (5), (6), and (7) enable the stresses (σ_x, τ_{xy}) and the displacements (u,v) in any one medium to be expressed in terms of the four constants b, which appear in the expressions for the potential in that medium. The relationship is a simple linear one in the b_j's. It may be written in the following form:

---

*Here k may be either real or complex; if the phase velocity c is real, the amplitudes of the potentials remain the same for all distances x from the origin. If k is real, it implies that the phase velocity c at the free surface is real. If k (and therefore the phase velocity) is complex, the amplitude diminishes with increasing x. Such behavior is termed a "leaking mode," because of the leakage of energy into the underlying layers of the system. The relative values of σ_y, τ_{xy}, u and v are not affected by the choice of k. Therefore, in what follows it does not matter whether k is real or complex.

$$w_1 = q_{11}b_1 + \ldots + q_{14}b_4$$
$$w_2 = q_{21}b_1 + \ldots + q_{24}b_4 \qquad (8)$$
$$w_3 = q_{31}b_1 + \ldots + q_{34}b_4$$
$$w_4 = q_{41}b_1 + \ldots + q_{44}b_4$$

where the vector w_i denotes the quantities $(\sigma_y, \tau_{xy}/i, \mu_1 v, \mu_1 u/i)$. The q_{ij}'s are the elements of the Q-matrix (see Reference (4) Equation (8), Page 214). They are obtained by substituting equations (5) and (6) in (7). We can write one such set of equations for each layer. If the q's are known at the n^{th} interface, their values at the $(n-1)^{th}$ interface can be determined directly, because the manner of their variation with depth is governed by the potentials, Equations (5) and (6).

If $w_{m,\,m-1}$ represents the value of w within the m^{th} layer and at the $(m - 1)^{th}$ interface, then we can write

$$(w_{m,\,m-1})_1 = c_{11}b_1 + \ldots + c_{14}b_4$$
$$(w_{m,\,m-1})_2 = c_{21}b_1 + \ldots + c_{24}b_4 \qquad (9)$$
$$(w_{m,\,m-1})_3 = c_{31}b_1 + \ldots + c_{34}b_4$$
$$(w_{m,\,m-1})_4 = c_{41}b_1 + \ldots + c_{44}b_4$$

where the c's are the q's with a y-coordinate corresponding with the $(m-1)^{th}$ interface.* The c's are the elements of Thrower's C-matrix (Reference 4, Equation 9). Similarly, the stress/displacement elements at the m^{th} interface

*As each pair of potentials ϕ, ψ is true only within a single layer, a separate origin of coordinates can be chosen for each layer. If the origin is chosen in the top interface, then the y-coordinate of that interface is zero; the y-coordinate of the lower interface is H_m, the thickness of the n^{th} layer.

can be written

$$
\begin{aligned}
(w_{m,m})_1 &= d_{11}b_1 + \dots + d_{14}b_4 \\
(w_{m,m})_2 &= d_{21}b_1 + \dots + d_{24}b_4 \\
(w_{m,m})_3 &= d_{31}b_1 + \dots + d_{34}b_4 \\
(w_{m,m})_4 &= d_{41}b_1 + \dots + d_{44}b_4
\end{aligned}
\qquad (10)
$$

where the d's are obtained from the q's by putting $y = H_m$, the thickness of the m^{th} layer.

It is possible to solve equations (9) for the b's in the m^{th} layer in terms of the $(w_{m,m-1})$'s, and we obtain

$$
\begin{aligned}
b_1 &= \bar{c}_{11}(w_{m,m-1})_1 + \dots + \bar{c}_{14}(w_{m,m-1})_4 \\
b_2 &= \bar{c}_{21}(w_{m,m-1})_1 + \dots + \bar{c}_{24}(w_{m,m-1})_4 \\
b_3 &= \bar{c}_{31}(w_{m,m-1})_1 + \dots + \bar{c}_{34}(w_{m,m-1})_4 \\
b_4 &= \bar{c}_{41}(w_{m,m-1})_1 + \dots + \bar{c}_{44}(w_{m,m-1})_4
\end{aligned}
\qquad (10.1)
$$

where the c^-'s are derived from the c's.

Using Equations (10) makes it possible to write

$$
\begin{aligned}
(w_{m,m})_1 &= e_{11}(w_{m,m-1})_1 + \dots + e_{14}(w_{m,m-1})_4 \\
(w_{m,m})_2 &= e_{21}(w_{m,m-1})_1 + \dots + e_{24}(w_{m,m-1})_4 \\
(w_{m,m})_3 &= e_{31}(w_{m,m-1})_1 + \dots + e_{34}(w_{m,m-1})_4 \\
(w_{m,m})_4 &= e_{41}(w_{m,m-1})_1 + \dots + e_{44}(w_{m,m-1})_4
\end{aligned}
$$

where

$$e_{ij} = \bar{e}_{i1} \cdot d_{1j} + \bar{c}_{i2} \cdot d_{2j} + c_{i3} \cdot d_{3j} + \bar{c}_{i4} \cdot d_{4j} \ .$$

Assuming that no slip occurs at the interfaces (5), $w_{m,m-1} = w_{m-1,m-1}$ and

$$\begin{aligned}
(w_{m,m})_1 &= e_{11}(w_{m-1,m-1})_1 + \ldots + e_{14}(w_{m-1,m-1})_4 \\
(w_{m,m})_2 &= e_{21}(w_{m-1,m-1})_1 + \ldots + e_{24}(w_{m-1,m-1})_4 \\
(w_{m,m})_3 &= e_{31}(w_{m-1,m-1})_1 + \ldots + e_{34}(w_{m-1,m-1})_4 \\
(w_{m,m})_4 &= e_{41}(w_{m-1,m-1})_1 + \ldots + e_{44}(w_{m-1,m-1})_4
\end{aligned} \tag{11}$$

Equations (11) relate the stress/displacement elements at the bottom of the m^{th} layer to those at the top of the same layer. Using the condition of continuity at each interface, (11) can be applied to successive layers, starting with the top layer. This yields

$$\begin{aligned}
(w_{n,n})_1 &= f_{11}(w_{1,0})_1 + \ldots + f_{14}(w_{1,0})_4 \\
(w_{n,n})_2 &= f_{21}(w_{1,0})_1 + \ldots + f_{24}(w_{1,0})_4 \\
(w_{n,n})_3 &= f_{31}(w_{1,0})_1 + \ldots + f_{34}(w_{1,0})_4 \\
(w_{n,n})_4 &= f_{41}(w_{1,0})_1 + \ldots + f_{44}(w_{1,0})_4
\end{aligned} \tag{12}$$

where f_{ij} is a function of the e_{ij}'s which is tedious to express in full, although the individual values of f_{ij} are easy to compute as the w's (for successive layers upwards from the n^{th} layer) are substituted in equations (11) written for the n^{th} layer.

The operations needed in the foregoing can be performed using the methods of linear algebra as indicated. However, it is very much easier to use the equivalent manipulations of matrices composed of the elements c, d, e, and f; this applies particularly to the operation of obtaining the f's from the e's. The use of these matrices appears to have been proposed by Thomson (6) and has been discussed by Haskell (7).

4.1 The Dispersion Equation for a Compound Free Plate

If the structure consists of a layered free plate, with no underlying semi-infinite medium, the stresses σ_y, τ_{xy} are zero on the two free surfaces. Equations (12) become

$$
\begin{aligned}
0 &= f_{11} \cdot 0 + f_{12} \cdot 0 + f_{13}(\mu_1 v)_{1,0} + f_{14}(\mu_1 u/i)_{1,0} \\
0 &= f_{21} \cdot 0 + f_{22} \cdot 0 + f_{23}(\mu_1 v)_{1,0} + f_{24}(\mu_1 u/i)_{1,0} \\
(\mu_1 v)_{n,n} &= f_{31} \cdot 0 + f_{32} \cdot 0 + f_{33}(\mu_1 v)_{1,0} + f_{34}(\mu_1 u/i)_{1,0} \\
(\mu_1 u/i)_{n,n} &= f_{41} \cdot 0 + f_{42} \cdot 0 + f_{43}(\mu_1 v)_{1,0} + f_{44}(\mu_1 u/i)_{1,0}
\end{aligned} \tag{13}
$$

The first two of equations (12) can be solved for $(\mu_1 v)_{1,0}$ and $(\mu_1 u/i)_{1,0}$, the displacements on the upper free surface. The second pair can be used to obtain the displacements on the lower free surface. Thus

$$
\begin{aligned}
f_{13}(\mu_1 v)_{1,0} + f_{14}(\mu_1 u/i)_{1,0} &= 0 \\
f_{23}(\mu_1 v)_{1,0} + f_{24}(\mu_1 u/i)_{1,0} &= 0
\end{aligned} \tag{14}
$$

and

$$
\begin{aligned}
f_{33}(\mu_1 v)_{1,0} + f_{34}(\mu_1 u/i)_{1,0} &= (\mu_1 v)_{n,n} \\
f_{43}(\mu_1 v)_{1,0} + f_{44}(\mu_1 u/i)_{1,0} &= (\mu_1 u/i)_{n,n}
\end{aligned} \tag{15}
$$

From equations (15), the displacements on the upper free surface can be non-zero only if

$$
\begin{vmatrix} f_{13} & f_{14} \\ f_{23} & f_{24} \end{vmatrix} = 0 \tag{16}
$$

This is the frequency equation of the system; it contains the phase velocity c as an implicit function of the wavelength L and can be solved by some repetitive method.

4.1.1 The Displacements in a Compound Free Plate

When a pair of values (c,L) have been found which make the determinant zero in equation (16), the displacement expressed by $(\mu_1 u/i)_{1,0}$ can be found in terms of $(\mu_1 v)_{1,0}$ from

$$(\mu_1 u/i)_{1,0} = -\frac{f_{13}}{f_{14}} \cdot (\mu_1 v)_{1,0}$$

$$= -\frac{f_{23}}{f_{24}} \cdot (\mu_1 v)_{1,0} \ .$$

The particle displacement in the x direction (the direction of propagation) thus lags by 90° on the vertical displacement.

The displacements on the lower face may be found from (15), and the intermediate displacements from (8).

4.2 The Dispersion Equation for a Layered Half Space

In order to write the dispersion equation for a layered system which rests on a semi-infinite medium, we denote the semi-infinite medium by the suffix (n + 1), and put m = n + 1 in equation (10.1). This yields

$$
\begin{aligned}
(b_1)_{n+1} &= \bar{c}_{11}\cdot(w_{n+1,n})_1 + \dots + \bar{c}_{14}(w_{n+1,n})_4 \\
(b_2)_{n+1} &= \bar{c}_{21}\cdot(w_{n+1,n})_1 + \dots + \bar{c}_{24}(w_{n+1,n})_4 \\
(b_3)_{n+1} &= \bar{c}_{31}\cdot(w_{n+1,n})_1 + \dots + \bar{c}_{34}(w_{n+1,n})_4 \\
(b_4)_{n+1} &= \bar{c}_{41}\cdot(w_{n+1,n})_1 + \dots + \bar{c}_{44}(w_{n+1,n})_4
\end{aligned}
\qquad (16.1)
$$

where the $\bar{c}$'s are derived from the c's for the $(n+1)^{th}$ layer; the c's in turn are obtained from the expressions (7) for the stresses and displacements in terms of the potentials ϕ and ψ in the semi-infinite medium, using (5) and (6).

The stresses and displacements are continuous at the interface between layer n and the semi-infinite medium, that is

$$(w_{n+1,n}) = (w_{n,n})$$

at the interface. Using this condition and the expressions (12) for $(w_{n,n})$, (16.1) can be written

$$
\begin{aligned}
(b_1)_{n+1} &= j_{11}(w_{1,0})_1 + \dots + j_{14}(w_{1,0})_4 \\
(b_2)_{n+1} &= j_{21}(w_{1,0})_1 + \dots + j_{24}(w_{1,0})_4 \\
(b_3)_{n+1} &= j_{31}(w_{1,0})_1 + \dots + j_{34}(w_{1,0})_4 \\
(b_4)_{n+1} &= j_{41}(w_{1,0})_1 + \dots + j_{44}(w_{1,0})_4
\end{aligned}
\qquad (17)
$$

where

$$j_{ij} = c^-_{i1} \cdot f_{1j} + c^-_{i2} \cdot f_{2j} + c_{i3} \cdot f_{3j} + c^-_{i4} \cdot f_{4j}$$

in which the c^-'s are calculated for the semi-infinite medium as in (16.1). Following Thrower, it is necessary that $b_1 = b_3$, $b_2 = b_4$ in order that ϕ and ψ may be expressible in terms of exp(-ry) and exp(-sy); r and s may be complex.

Rewriting (17), and substituting for $(w_{1,0})$,

$$(b_1)_{n+1} = j_{11} \cdot 0 + j_{12} \cdot 0 + j_{13}(\mu_1 v)_{1,0} + j_{14}(\mu_1 u/i)_{1,0}$$
$$(b_2)_{n+1} = j_{21} \cdot 0 + j_{22} \cdot 0 + j_{23}(\mu_1 v)_{1,0} + j_{24}(\mu_1 u/i)_{1,0}$$
$$(b_1)_{n+1} = j_{31} \cdot 0 + j_{32} \cdot 0 + j_{33}(\mu_1 v)_{1,0} + j_{34}(\mu_1 u/i)_{1,0}$$
$$(b_2)_{n+1} = j_{41} \cdot 0 + j_{42} \cdot 0 + j_{43}(\mu_1 v)_{1,0} + j_{44}(\mu_1 u/i)_{1,0} .$$

This yields, on subtracting the third from the first and the fourth from the second

$$0 = (j_{13} - j_{23})(\mu_1 v)_{1,0} + (j_{14} - j_{34})(\mu_1 u/i)_{1,0}$$
$$0 = (j_{23} - j_{43})(\mu_1 v)_{1,0} + (j_{24} - j_{44})(\mu_1 u/i)_{1,0} .$$

As before, the displacements can be non-zero only if

$$\begin{vmatrix} j_{13} - j_{23} & j_{14} - j_{34} \\ j_{23} - j_{43} & j_{24} - j_{44} \end{vmatrix} = 0 \qquad (18)$$

When (18) has been solved, the displacements at the surface are related by

$$(\mu_1 u/i)_{1,0} = - \frac{j_{13} - j_{23}}{j_{14} - j_{24}} (\mu_1 v)_{1,0} = - \frac{j_{23} - j_{43}}{j_{24} - j_{44}} (\mu_1 v)_{1,0} .$$

In the next chapter, the program for obtaining the zeros of equation (16) will be discussed.

5. THE COMPUTATION OF THE REAL ROOTS OF THE DISPERSION EQUATION

5.1 Limitation of the Solution Involving Only Real Roots

In determining the numerical values of the wavelengths and phase velocities at the free surface in a layered system, it seemed advisable to investigate the results obtainable for a compound free plate. This solution involves purely real values of phase velocity. The search for the zeros of the frequency equation (16) involved some points of programming which require consideration.

5.2 Description of the Computation Procedure

The main object of the program is to generate the determinant (16). This is accomplished by first calculating the e's for each layer, in the subroutine EMATRX.* The f's for the combined system can then be calculated. This is done in the subroutine PROMAT by means of the DO loop terminating at statement 25 (see Figure 20).**

After the cycle at which the determinant (16) changes sign, the two most recent values having opposite sign are retained, together with their corresponding phase velocities. A Newton-Raphson interpolation is used to improve the value of the phase velocity CB1 by making the determinant of the frequency equation VALUE successively closer to zero. The precision of the final CB1 is set by the constant in the IF statement at statement 140 + 0002 in the subroutine TRAVEL. Once the Newton-Raphson interpolation routine has been entered, it is accessed on all subsequent cycles of computation until successive values of CB1 satisfy the precision criterion.

Following Thrower, the possibility was considered that the e's may introduce precision difficulties owing to the hyperbolic sine and cosine terms; these terms become nearly equal for large arguments, and the terms involving their differences may be swamped by other terms which do not contain such differences. The program was, therefore, rewritten, using

---

*The sets of e's are each 4 x 4 matrices and their continued product yields a 4 x 4 matrix; however only a 2 x 2 determinant of this product is required to form the frequency equation. As shown by Thrower [Ref. 4, equation (19)], this determinant can be obtained starting with only the two final columns of e's for the top layer. Subsequent product matrices are, therefore, 4 (row) x 2 (column). Half of the resulting product matrix is used to form the determinant required.

**Listings of the programs, designated by WHC37 and WHC38, are available from the Pavement Design Department, Texas Transportation Institute, Texas A&M University, College Station, Texas 77843.

compound (or delta) matrices instead of the normal matrices for the e's (8). It was found, however, that the use of double precision throughout enabled results to be achieved by means of the normal matrices which are identical with those obtained by the use of compound matrices. The use of double precision entails no additional work when programming by means of Fortran IV, and the execution time is affected only marginally.

6. THE APPLICATION OF THE DISPERSION PROGRAM FOR A COMPOUND FREE PLATE TO THE MEASUREMENTS MADE BY MEANS OF THE SHELL VIBRATOR ON SECTIONS 1 THROUGH 8 OF THE A&M TEST FACILITY

In order to show that the theory presented here is at least a plausible one, and in order to gain experience with the use of a dispersion program of this kind, the computations were applied to eight structures within the A&M Test Facility of experimental highway pavements. The method of measurement has been described elsewhere (9) and the results were used in the form of the plots shown in Figures 8 through 15. The theoretical relationships are shown on the overlaid plots. In these figures, both the wavelength and the reciprocal wavelength are plotted on a common base of frequency, each providing a sensitive parameter where the other fails to do so. Further, the closely rectilinear form of the plot when the reciprocal wavelength is used is helpful in determining when the successive branches of the dispersion curve are entered.

The program was supplied with the experimental points as trial values; the elastic parameters were obtained from the measurements made by means of the Dynaflect, assuming a Poissón's ratio of 0.45 as described in Part 1 of this report. The densities were obtained from nuclear measurements both during and after construction, and their relative values are probably within 5% of the true relative values. The scale factor for converting the dimensionless output from the computations to actual values is determined by the modulus established for the material composing the top layer.*

*The subgrade thickness was set at 5.0 feet. Increasing the subgrade thickness from 3.0 to 5.0 feet decreased the calculated frequencies corresponding to the roots of the frequency equation by 3%.

The writer unfortunately knows of no grounds on which the use of Dynaflect results for this purpose can be justified--the expected frequency effect on the elastic parameters of such granular materials has not been investigated. There was no alternative but to use these results, however. The further use of elastic parameters obtained by means of readings from the Dynaflect can be justified, partly at least, by the need for relative values only,

As the zeros of the frequency equation for the system are very closely spaced, the agreement between the theoretical and experimental values indicates only that the theory is a plausible one. The program selected the nearest zero to the experimental value, supplied in dimensionless form. Neighboring modes are shown in a few cases, indicating the order of the discrepancy which may be expected.

7. CONCLUSIONS

While the agreement between the theory and the experimental results is by no means decisive, the writer feels that much information may be obtained from the present work.

Comparisons between results obtained for elastic moduli by wave methods and those found using observations made by means of the Dynaflect are open to suspicion. The Dynaflect measurements are made at a loading frequency of eight cycles per second, while the wave measurements are made at frequencies of hundreds or even thousands of cycles per second. The values of the elastic parameters may change appreciably between these limits of frequency. The writer feels that this point has not been investigated sufficiently under highway conditions to enable conclusions to be drawn. The elastic parameters obtained using measurements obtained by means of the Dynaflect have been used as a guide only, and are not intended for direct comparison.

As indicated in Part 1 of this report, the values of the elastic parameters obtained using observations made by means of the Dynaflect may be incorrect due to errors inherent in the Dynaflect.

The assumption that the whole or part of a highway pavement vibrates as if it were a free plate is suggested by the fact that the plots of reciprocal wavelength against the frequency form sets of straight lines. This is the form taken by similar plots obtained from the measurements made on a free plate. As shown in Chapter 2, it leads to results which are qualitatively correct and which may be used for the purpose of ranking within a group of materials.

Some engineers believe that the velocity at the free surface of a layered structure represents the phase velocity of waves (presumably shear waves) at a depth equal to half of the wavelength. This hypothesis can be tested by plotting the results in Figures 8-15 on the basis of phase velocity against wavelength. It is seen that the hypothesis holds, if at all, in isolated cases only.

The writer wishes to emphasize that the theoretical frequency-dispersion curves of wavelength (Figures 8-15) were obtained using the values of the Young's moduli found by means of the Dynaflect. There is no suggestion that the reverse operation can be performed: the elastic parameters of the materials cannot be determined by this means from the wave measurements which were made. The reverse operation is not possible because there is a large variety of structures which correspond (theoretically at least) with a given dispersion curve.

7.1 Applications of the Simple Free Plate Solutions

The theory used for the analysis of free plates is very well established and deserves application where the physical conditions justify its use. Information concerning the elastic parameters and the thicknesses of reinforced concrete bridge decks may be determined by this means. The wavelengths used should equal or be less than the thickness of the deck. The frequency of excitation should, therefore, exceed ten kilocycles per second; fifty kilocycles per second is the lowest desirable upper limit. Another application is in determining the thickness, and secondarily the extent, of delamination in decks and pavements; the elastic parameters of delaminated material can be found by this means.

7.2 The Further Application of the Jones' Solutions

There may be an advantage in applying the solutions discussed by Jones, particularly where clay subgrades are prevalent. Extensive computations could be used, in graphical form, to fit a given frequency dispersion curve by comparison, thus deducing the associated structure. Examples of such applications are given by Vidale (10).

7.3 Further Investigation of the Roots of the Dispersion Equation

An investigation should be made of the complex roots of the dispersion equation for a layered half space. When the behavior of these is known, it should be possible to investigate the displacements within the media. The next aim should be to determine the cause of the change in the phase velocity, well known to the operators of the field equipment, which often occurs as the pick-up is moved radially from the source of vibration. This change may be related to the relative damping of the various modes of response of the structure.

7.4 A Note on Computation which is the Reverse of that Performed by the dispersion program for a compound free plate

The dispersion programs WHC37 and WHC38 are not adequate for engineering purposes. The problem which engineers face is as follows: given the frequency dispersion of the phase velocities, as in Figures 8-15, find the elastic parameters of the materials and the thicknesses of the layers composing the structures. Owing to the close spacings of the zeros of Equation (16), the solution is not unique and, therefore, a large number of structural parameters may exist which satisfy the equation.

If indeed it is possible to perform the reverse calculations of those performed in the programs mentioned, the problem which engineers face will still not be solved. This is particularly true in view of the assumption of a compound free plate, which is an inaccurate model of a highway structure.

APPENDIX A - A NOTE ON THE ARBITRARY CONSTANTS INVOLVED IN THE SOLUTION OF THE DISPERSION EQUATION IN A LAYERED HALF-SPACE

Equations (5), (6) and (7) enable the stresses (σ_y, τ_{xy}) and the displacements (u, v) in any one medium to be expressed in terms of the four constants b_j in the expressions for the potentials for that medium.

Consider a system composed of n layers, each of finite known thickness, overlying a semi-infinite medium; all the elastic parameters are assumed known, leaving the b_j's as the only unknowns. There are 4n such unknowns introduced by the n layers, and four more due to the semi-infinite medium, a total of 4 (n+1) unknowns. Equations for these unknowns are established as follows:

	No. of Equations
Free Surface: Stresses (σ_y, τ_{xy}) are zero	2
Interfaces: There are (n-1) layer/layer interfaces. There is 1 layer/semi-infinite medium interface. The stresses (σ_y, τ_{xy}) and the displacements (u,v) are continuous, yielding 4 equations for each interface.	4(n-1) + 4
Semi-infinite Medium: Potentials ϕ and ψ vanish as y becomes large	2
Total	4 (n+1)

This number of equations is sufficient to solve for the arbitrary constants b_j.

However, there is no need to determine all the arbitrary constants and much algebraic work is avoided by not solving for the b's but by adopting a fresh approach. Start by considering the stresses and displacements in the semi-infinite medium. As these must vanish for infinite values of y, they are governed by only two arbitrary constants and are relatively simple in form. Using the condition of continuity of stresses and displacements over an interface, we can write expressions for these quantities in the n^{th} layer, putting $y_n = h_n$, and equate them to the expressions for the same quantities in the semi-infinite medium, putting $y_{n+1} = 0$*. A similar procedure can be adopted, relating the quantities in the $(n-1)^{th}$ layer (putting $y_{n-1} = h_{n-1}$) with those in the n^{th} layer (putting $y_n = 0$), and in turn with those within the semi-infinite medium.

Continuing upwards, an expression for the stresses and displacements at the free surface is found. If the displacements at the free surface are to be non-zero, a sub-matrix of the product matrix formed (by following the stresses and displacements upwards from the semi-infinite medium) must be zero. This is the frequency equation of the system. Its zeros are found by trial and error.

The displacements can be found, if required, as described in Chapter 4.

---

*As the expressions for each pair of potentials ϕ, ψ hold true only within the medium for which they are intended, it is correct and much easier to establish origins of the y-coordinate in the upper face of each layer. It is correct to do this because the y for each layer is never required outside that layer; it is easier because we are saved the inconvenience of keeping a tally of layer thickness in specifying a y coordinate within a multi-layer structure.

APPENDIX B - FORTRAN PROGRAMS WRITTEN FOR THIS WORK

The following programs have been written for use with a Fortran compiler, using the language Fortran IV (G-level). They operate also on the Watfor compiler. Execution is much faster in almost all cases if the latter compiler is used. Basic language has been used as far as possible, in order to minimize the changes necessary for any other Fortran compiler.

In the following, "f" denotes a floating point field in a punched card. It should in general contain a decimal point. For example, "ffffffffff" denotes a ten-column data field, starting in column zero of the card, and including one column in which there is a decimal point. Similarly, "b" denotes a blank column, and "n" denotes a right-justified field for fixed-point data.

The notation is as follows:

CB, CB1	The phase velocity of waves at the free surface
XK	The wave number, expressed as 2π/wavelength
HH, H	A layer thickness
ALPHA	Phase velocity of compression waves in a medium
BETA	Phase velocity of shear waves in a medium
V, SIGMA	Poisson's ratio
XLMDA, XMU	Lame's constants
RHO	Density of the medium (Mass per unit volume)
EM	Young's modulus

Suffixes or subscripts (1, 2,...) denote a particular medium within the structure. The surface layer is medium #1.

Almost all the inputs can be expressed in terms of relative rather than absolute values; this permits the programs to be used with a variety of systems of units. The following quantities in the programs are independent of the unit system:

RHO (the density); EM (Young's modulus); HH (layer thickness); SIGMA (Poisson's ratio).

The wavelength (WLNGTH) must be expressed in the same units as the layer thickness, however. The phase velocity at the free surface (CB1) is the ratio c/β_1, where β_1 is the phase velocity of shear waves in the material composing the first (the top) layer and c is the phase velocity at the free surface. In the output from the programs, the frequency (FREQ) is the dimensionless phase velocity divided by the wavelength WLNGTH, also in dimensionless form.

B.1 - Fortran Programs Written for the Free Plate Solutions

The free plate solutions of Lamb (see Chapter 1) have been programmed for use with the Fortran IV (G-level) or the Watfor compiler.

B.1.1 Program Which Computes the Solution to the Anti-Symmetric Mode

The data cards for the program to compute the solution for the anti-symmetric mode (WHC33A) are as follows:

First card type

Column 1-10 f....f

Column 11-80 b....b

where fffffffff is a ten-column field containing β/α.

Second card type

Column 1-80 f....f

where the f's are seven fields of ten columns each containing the values of c/β for which results are required. If more than seven fields of c/β are needed, a 999. should be punched in columns 72-80 and the subsequent velocity values punched on the next card; otherwise columns 72-80 should be left blank.

If a zero is read in a field where a value of c/β is expected, the program expects to read a card of the first type as the subsequent card. The program can be terminated by making the values of β/α in the first card a negative quantity. The flow diagram of the program is shown in Figure 16.

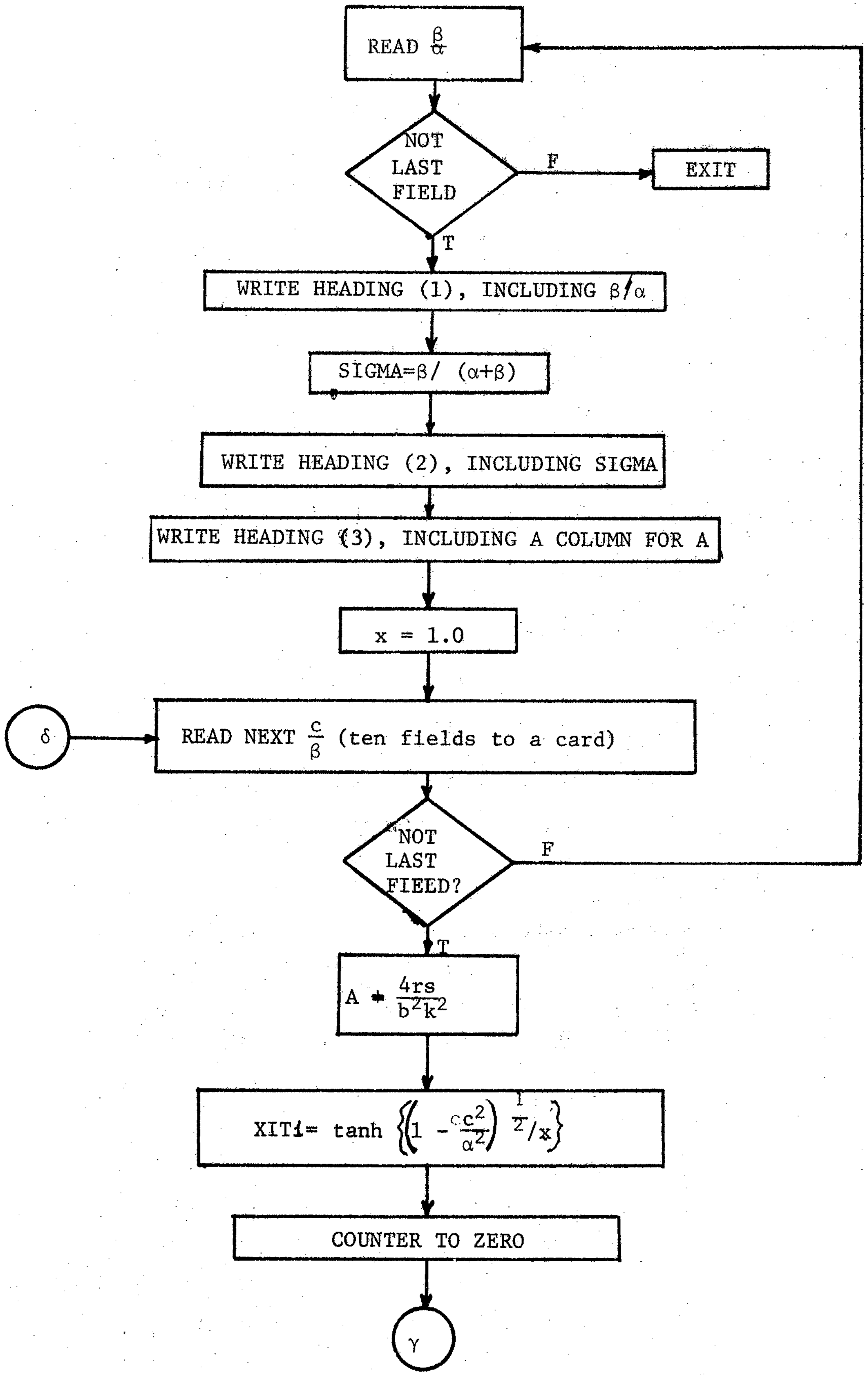


FIGURE 16 - Anti-symmetric mode of free plate - flow diagrom of program.

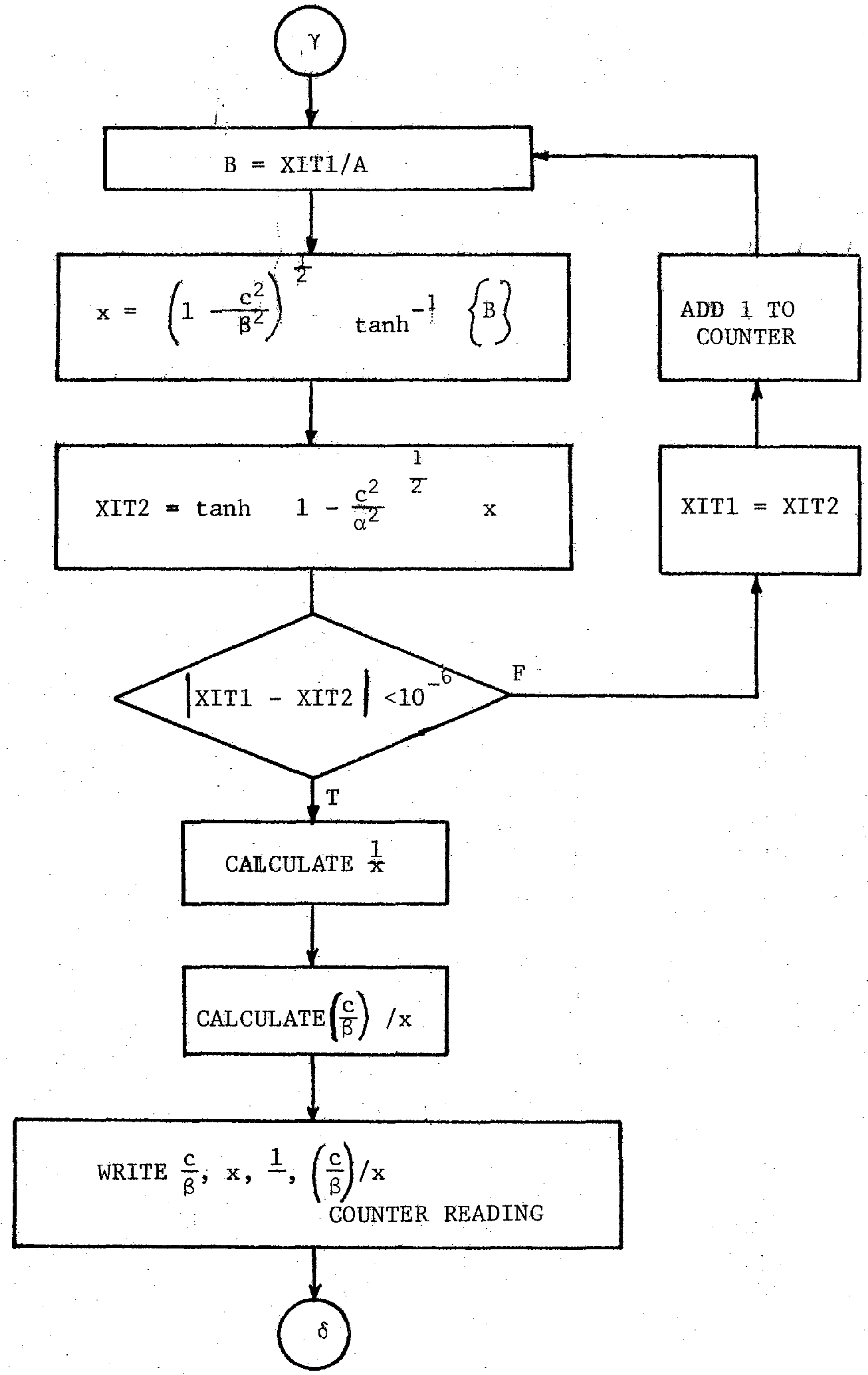


FIGURE 16 - Anti-symmetric mode of free plate - flow diagram of program (continued).

B.1.2 <u>Program which Computes the Solution to the Symmetric Mode</u>

The data cards for the program which computes the solution to the symmetric mode (WHC33E) are as follows:

<u>First card type</u>

Column 1-10 f....f

Column 11-80 b....b

where ffffffffff is ten-column field containing β/α.

<u>Second card type</u>

Column 1-80 f....f

where the f's are fields of ten columns each, containing the values of c/β for which results are required. If less than eight fields are required, the last field should contain not zeros but 999.; the program then expects the next card to contain a new value of β/α. If the latter is zero the program terminates.

NOTE: Values of c/β in this program are greater than unity.

The flow diagram for the program is shown in Figure 17.

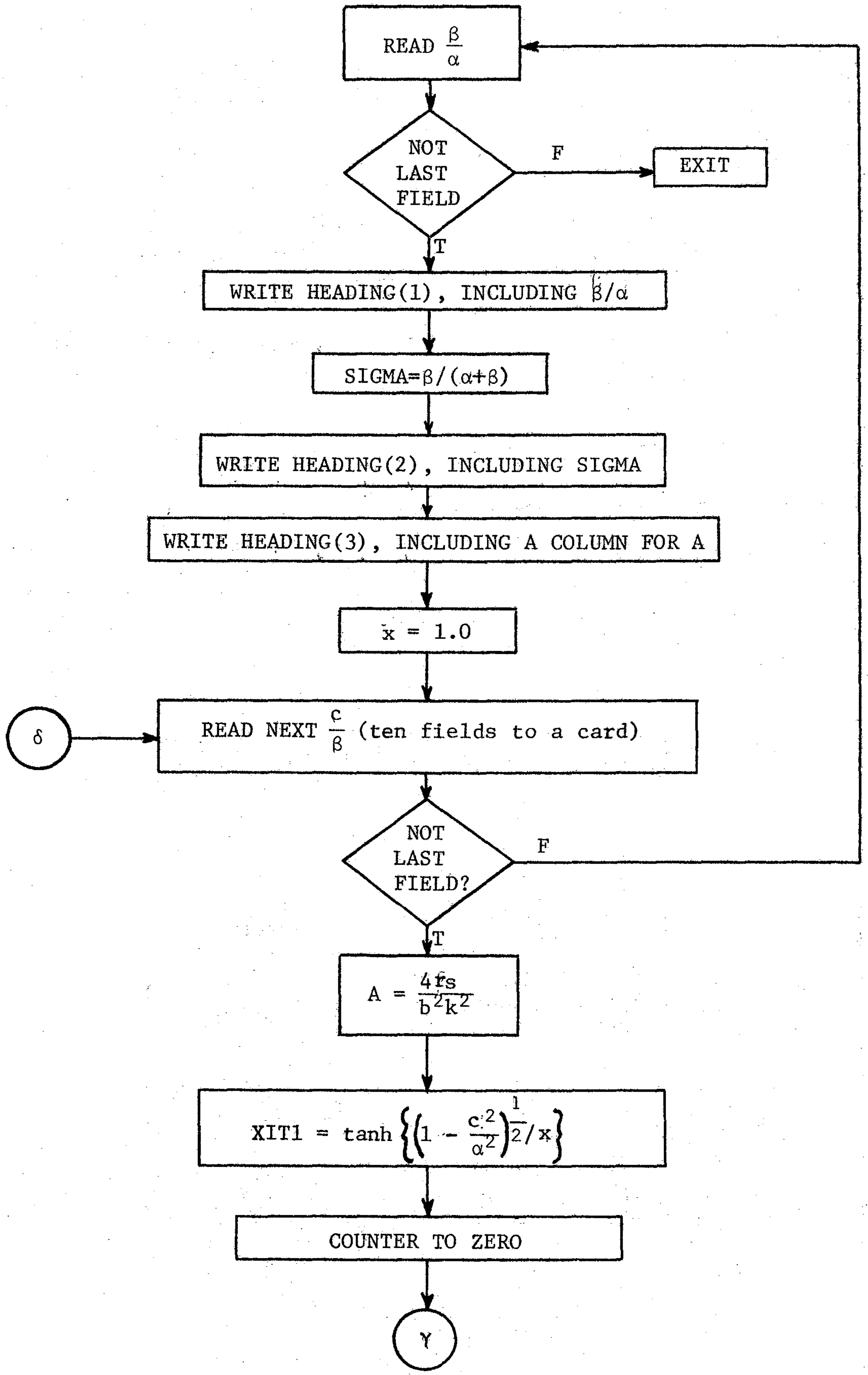


FIGURE 17 - Symmetric mode of free plate - flow diagram of program.

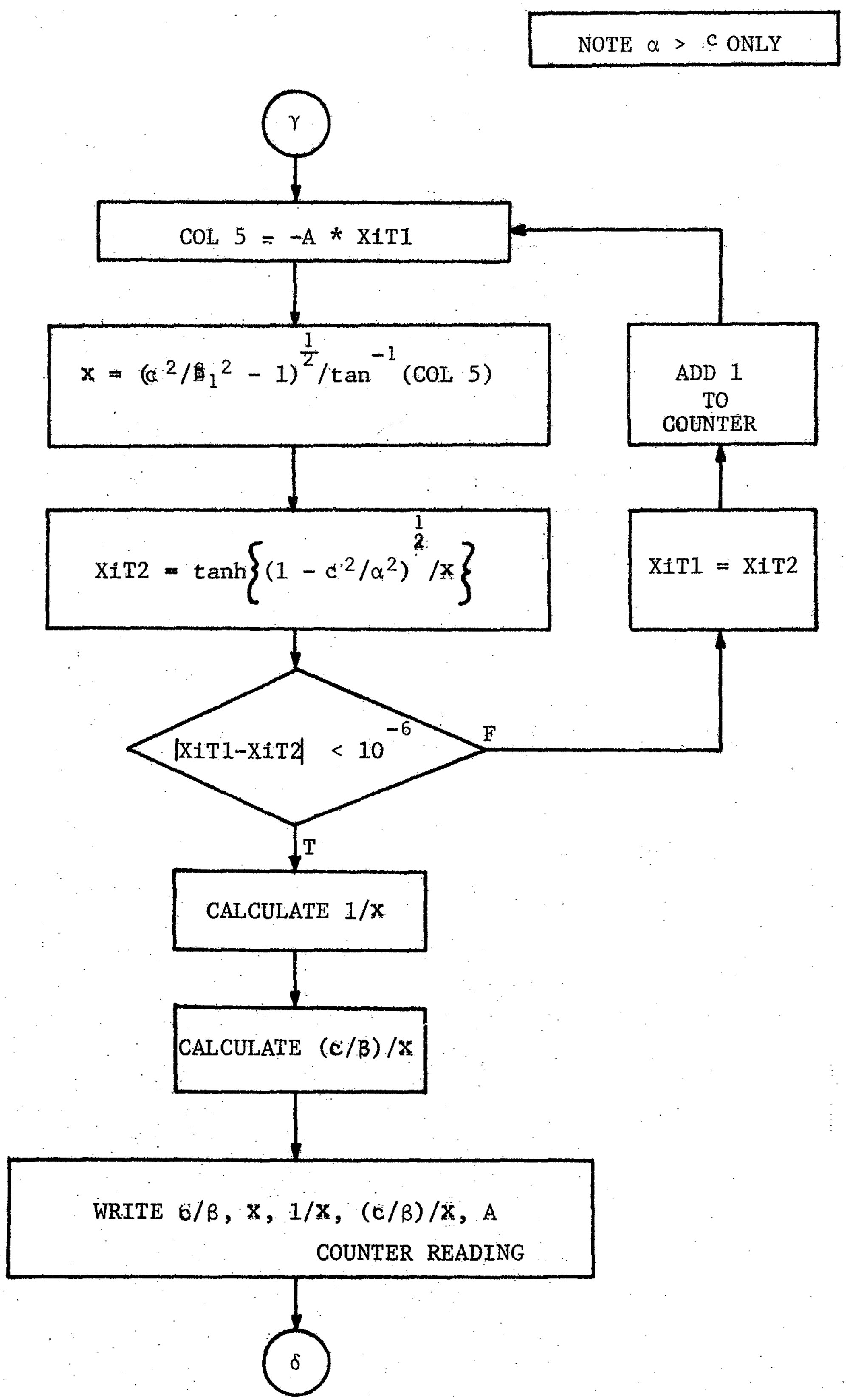


FIGURE 17 - Symmetric mode of free plate - flow diagram of program (continued).

B.2 Fortran Programs Written for the Jones' Solutions

B.2.1 Program Written for the Computation of the Solution of Jones' Equation (26), Reference (1).

This program (WHC33B) computes the numerical values of the solutions to Jones' equation (26), reference (1). The wavelength, wave number and frequency are computed for input values of the phase velocity c/β_1.

First card type

The first card is punched as follows:

Columns 1-60 f....f

Columns 61-80 b....b

where the f's are six fields of ten columns each. They contain the following in order: β/α, ${}^{\alpha_1}/\alpha_2$, ${}^{\alpha_1}/\alpha_3$, ${}^{\rho_3}/\rho_2$, ${}^{\rho_3}/\rho_1$, ${}^{H_2}/H_1$

Second card type

The second card is punched as follows:

Columns 1-80 f....f

where the f's are eight fields of ten columns each. They contain the values of c/β_1 for which results are required. Any field may contain zeros, and if it does the program will read the subsequent card as a card containing values of c/β_1. If 999. is punched in any field which is expected to contain a value of c/β_1, the next card read will be data of the type contained in cards of the first type; the program terminates if it encounters a blank or a zero in the first field of a card of the first type.

The flow diagram of the program is shown in Figure 18.

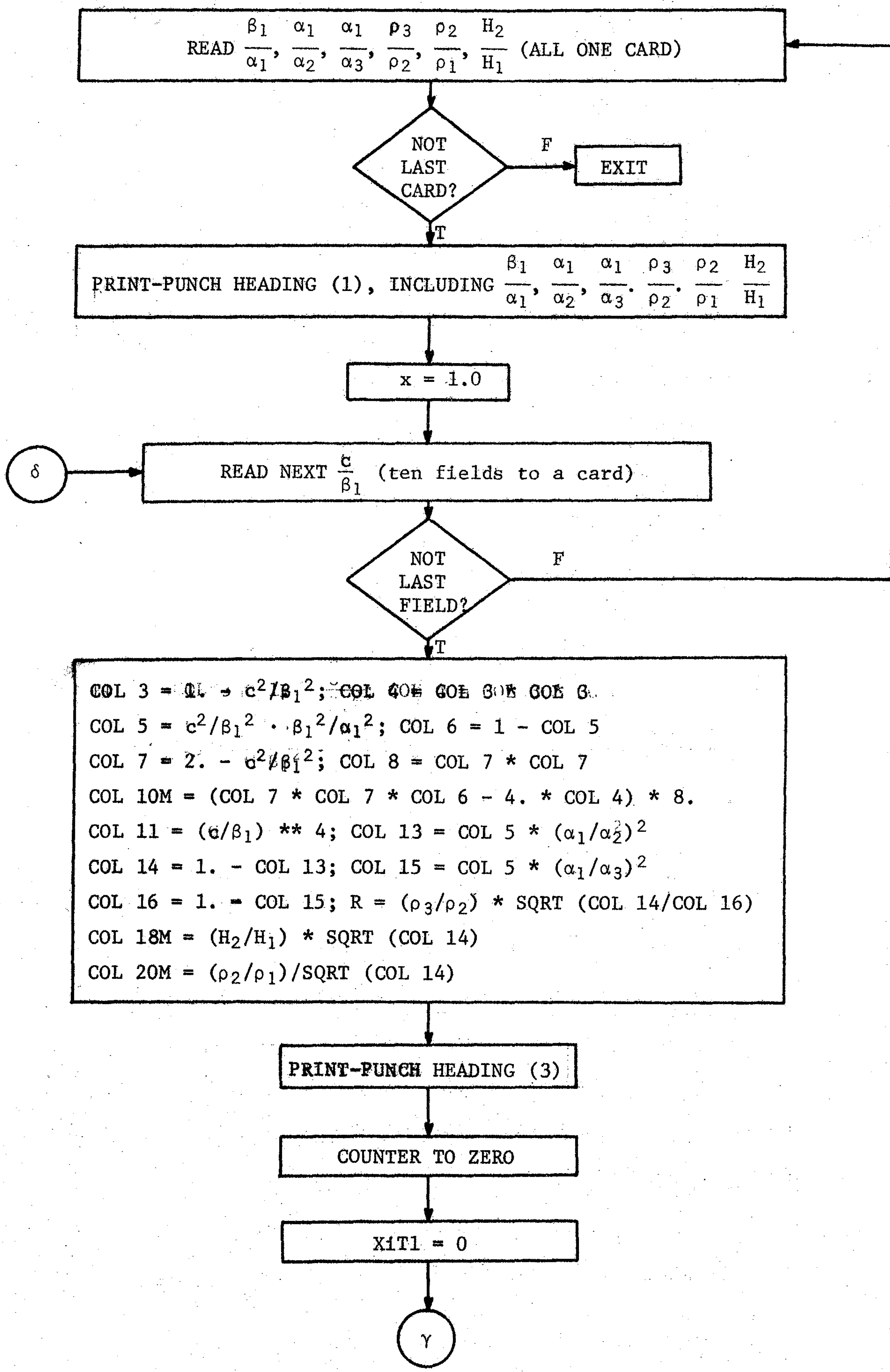


FIGURE 18 - Jones' Equation (26) Lower Branch - flow diagram of program.

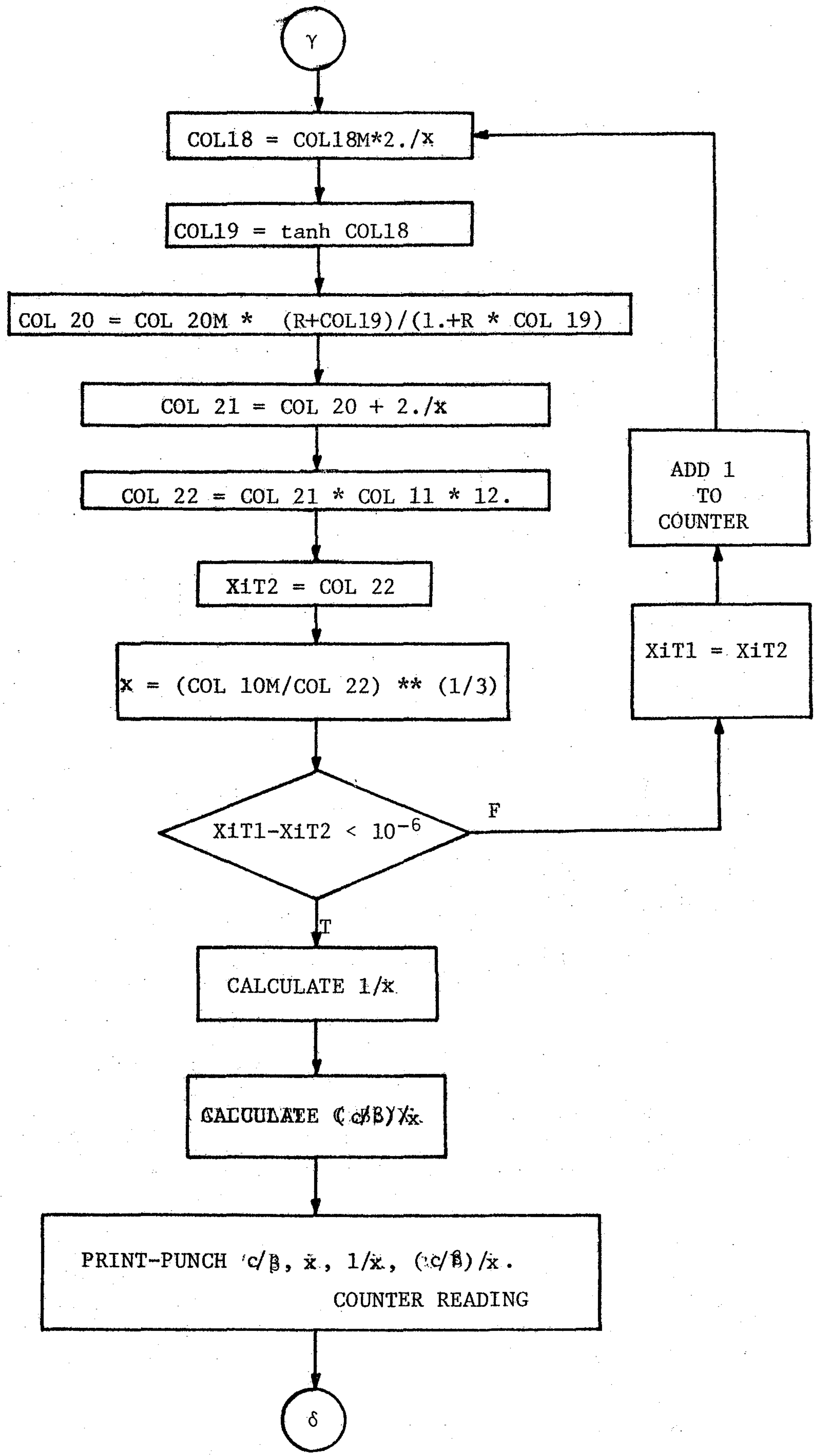


FIGURE 18 - Jones' Equation (26) Lower Branch - flow diagram of program (continued).

B.2.2 Program Written for the Computation of the Solution of Jones' Equation (28), Reference (1)

This program (WHC33D) computes the numerical values of the solutions to Jones' equation (28), reference (1). The wavelength, wave number and frequency are computed for input values of the phase velocity c/β_1.

The data cards for the program are as follows:

First card type

Columns 1-70 f....f

Columns 71-80 b....b

where the f's are ten-column fields containing the following input parameters: $^{\beta_1}/\alpha_1$, $^{\alpha_1}/\alpha_2$, $^{\alpha_1}/\alpha_3$, $^{\rho_3}/\rho_2$, $^{\rho_2}/\rho_1$, $^{H_2}/H_1$, mode number required (1., 2.,...)

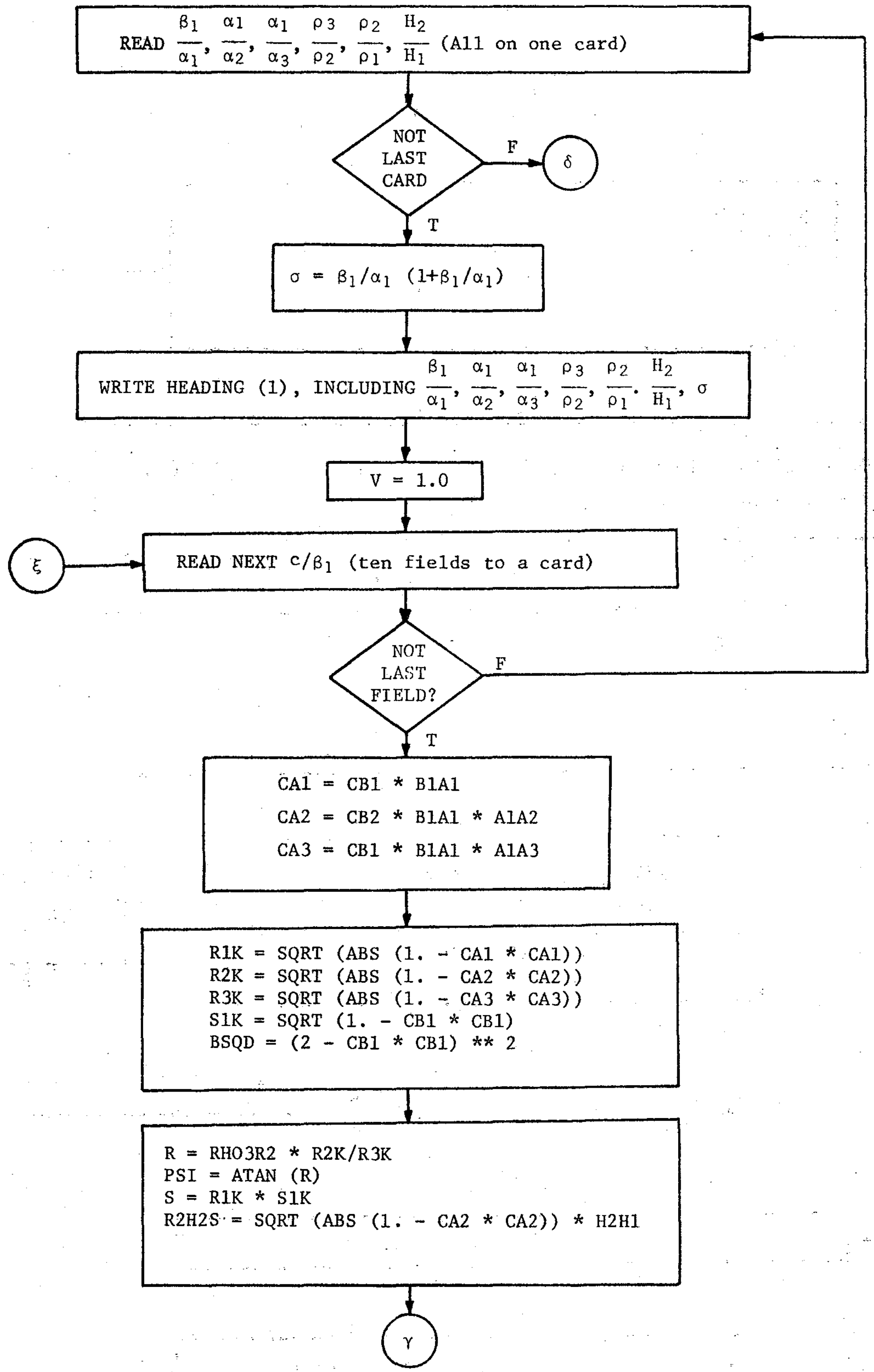
Second card type

Columns 1-80 f....f

where the f's are ten-column fields, each containing a value of c/β_1 for which results are required. If a field is blank or zero, the program will read a further card of the second type. If the field contains 999. the program will read the subsequent card as one of the first type.

The flow diagram of the program is shown in Figure 19.

FIGURE 19 - Jones' Equation (28) Intermediate Branch - flow diagram of program.



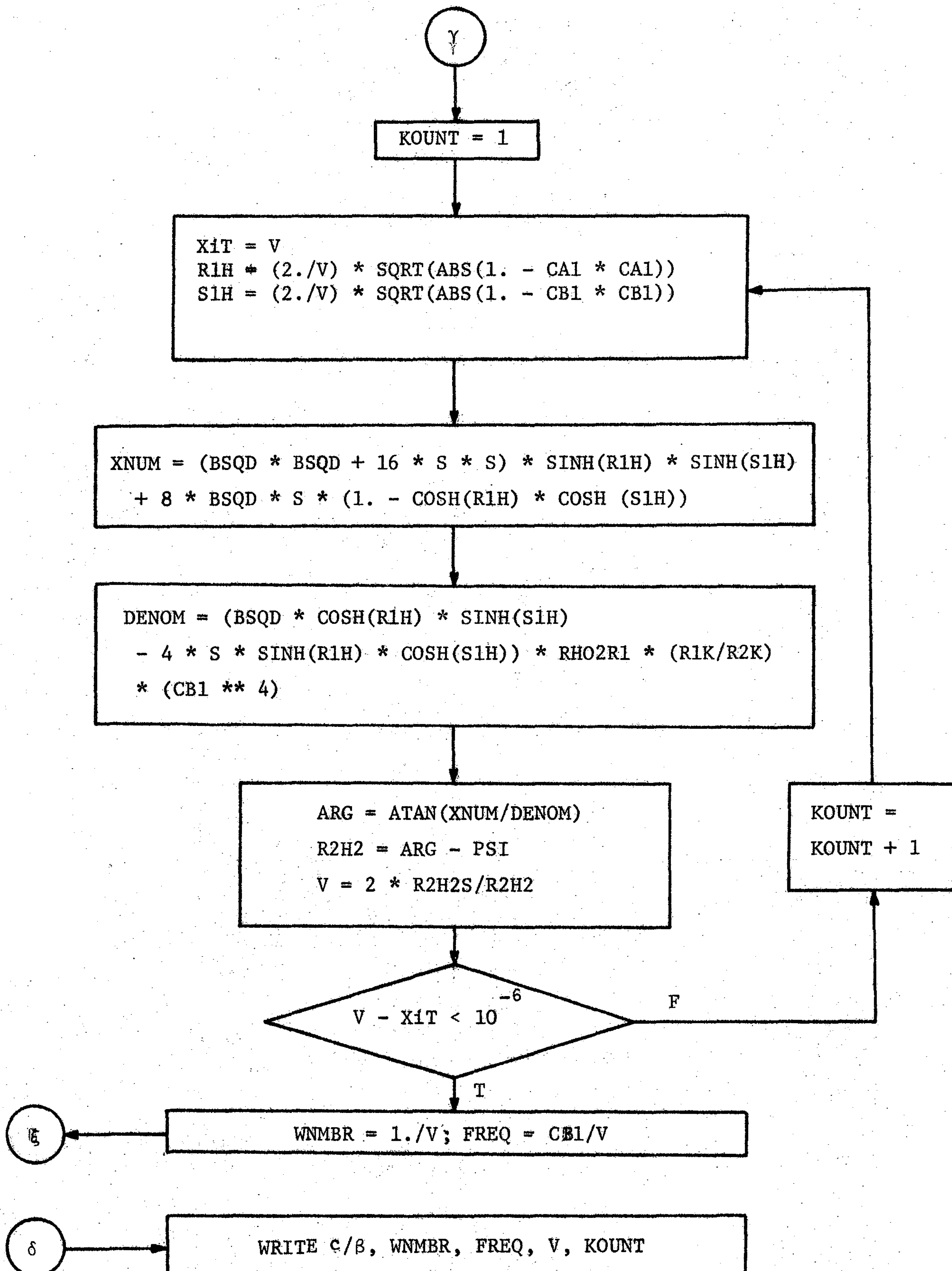


FIGURE 19 - Jones' Equation (28) Intermediate Branch - flow diagram of program (continued).

B.3 Fortran Program Written for the Computation of the Real Roots of the Frequency Equation for a Compound Free Plate (Program Designated by WHC37)

B.3.1 Data Cards

First card type

Columns 1-72; Title and description of data.

Columns 73-80: Floating Point Constant "WSWTCH"

If WSWTCH = 0.0 "VALUE" is not printed

If WSWTCH = 1.0 "VALUE" is printed with the final table of output.

If WSWTCH = 2.0 "VALUE" and its corresponding trial velocity "CB1" are printed after each cycle of calculation.

Second card type

Columns 1-2: not read

Column 3: # of layers of finite thickness plus unity

Columns 4-6: not read

Columns 7-78: Twelve floating point fields of six columns each containing, in order, EM(1), V(1), EM(2), V(2), ..., EM(6), V(6). The program will read only the number of pairs given in Column 3.

Third card type

Columns 1-6: not read

Columns 7-42: six floating point fields of six columns each, containing RHO(1), RHO(2), RHO(3),..., RHO(6). The program will read only the # of values given in Column 3 of the preceding card of the second type.

Fourth card type

Columns 1-36: Six floating point fields of six columns each, containing HH(1), HH(2),..., HH(6).

Fifth card type

Columns 1-6: The wavelength for which results are required, in six-column floating point form

Columns 7-18: The trial value of phase velocity at which the search for a root is to start, and the increments to this value for the initial sweep; both are in six-column floating point fields, and both are expressed as a fraction of BETA(1).

The data set may include any number of cards of the fifth type in order to obtain the results required. If the program expects a card of the sixth type and encounters a blank, the subsequent card will be read as a new title card (card of the first type); if the expected title card is supplied, the printer will advance one page. This permits termination without an error record on the page containing the computed output.

All cards except those of the fifth type must be in sequence.

B.3.2 Description of the Program

The program consists of a main program, five subroutines and an arithmetic function; the program involving compound matrices has one less subroutine.* A flow diagram of the program is shown in Figure 20. The operations performed by the routines are as follows.

MAIN : MAST

This routine performs the input/output operatings and does some calculation of parameters.

TRAVEL ; TRASTR

This subroutine calculates a pair of values (CB1, WLNGTH) which satisfies the frequency equation (16). Its main function is to calculate the e's for each layer. This is performed in the subroutine EMATRX : ESTAR which is accessed through TRAVEL. The subroutine PROMAT : PROMST is also accessed** The TRAVEL ; TRASTR carries out a Newton-Raphson interpolation which approximates the root of the frequency equation CB1 to the accuracy specified. The accuracy is controlled by the IF statement at statement 140 + 0002; this statement contains a constant specifying half of the greatest difference between successive values of CB1 which is acceptable for the result to be printed as final.

*The program in which compound matrices have been used has been renamed in order to avoid its mistaken use. The names of the routines have been altered as follows: MAIN : MAST; TRAVEL : TRASTR; EMATRX : ESTAR; (GMATRX : GSTAR); PROMAT : PROMST. Owing to the resulting economy of programming, the subroutine corresponding with CHECK is not required in the program employing compound matrices.

**The subroutine GMATRX : GSTAR has been included to allow for future development.

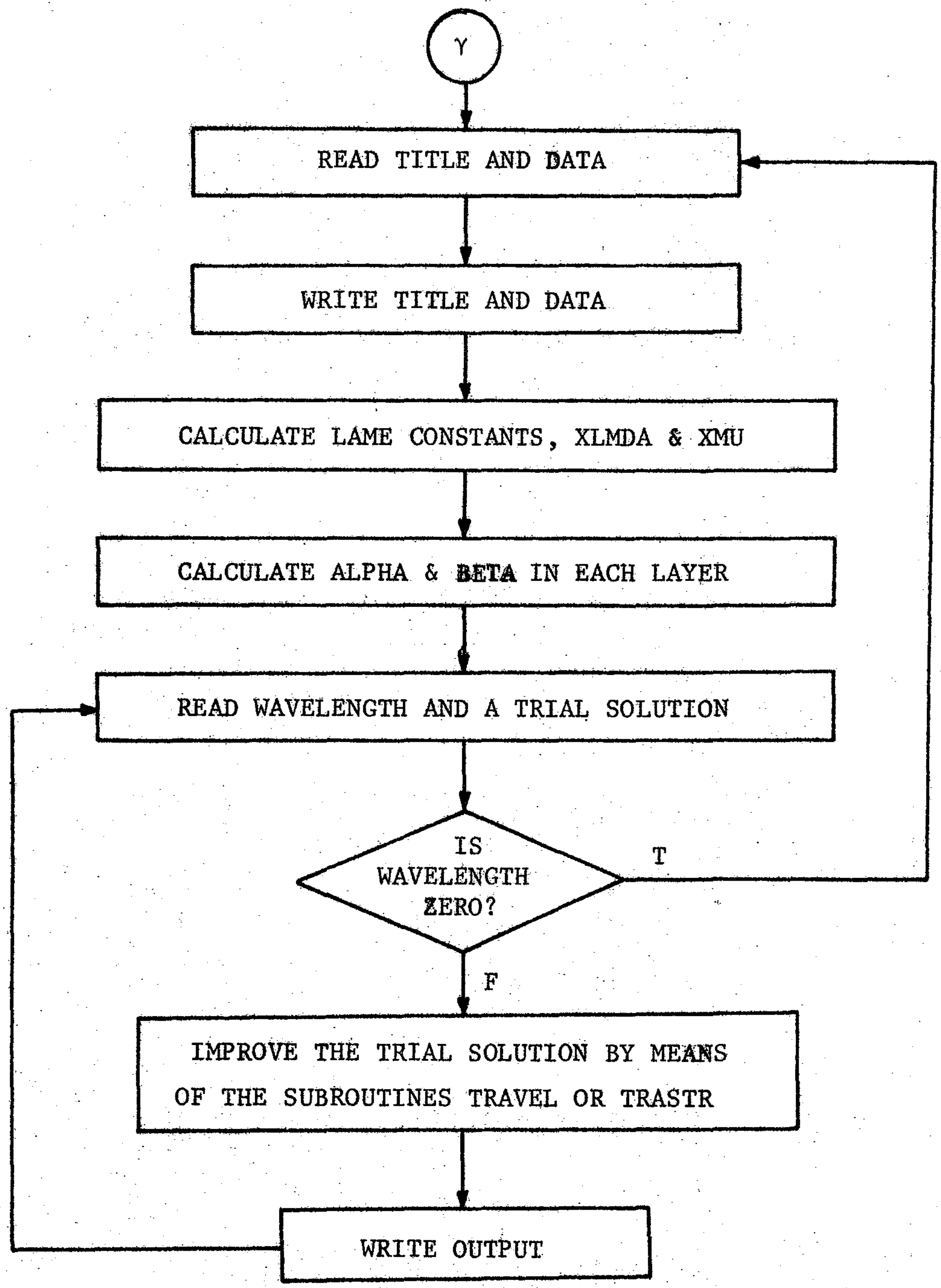
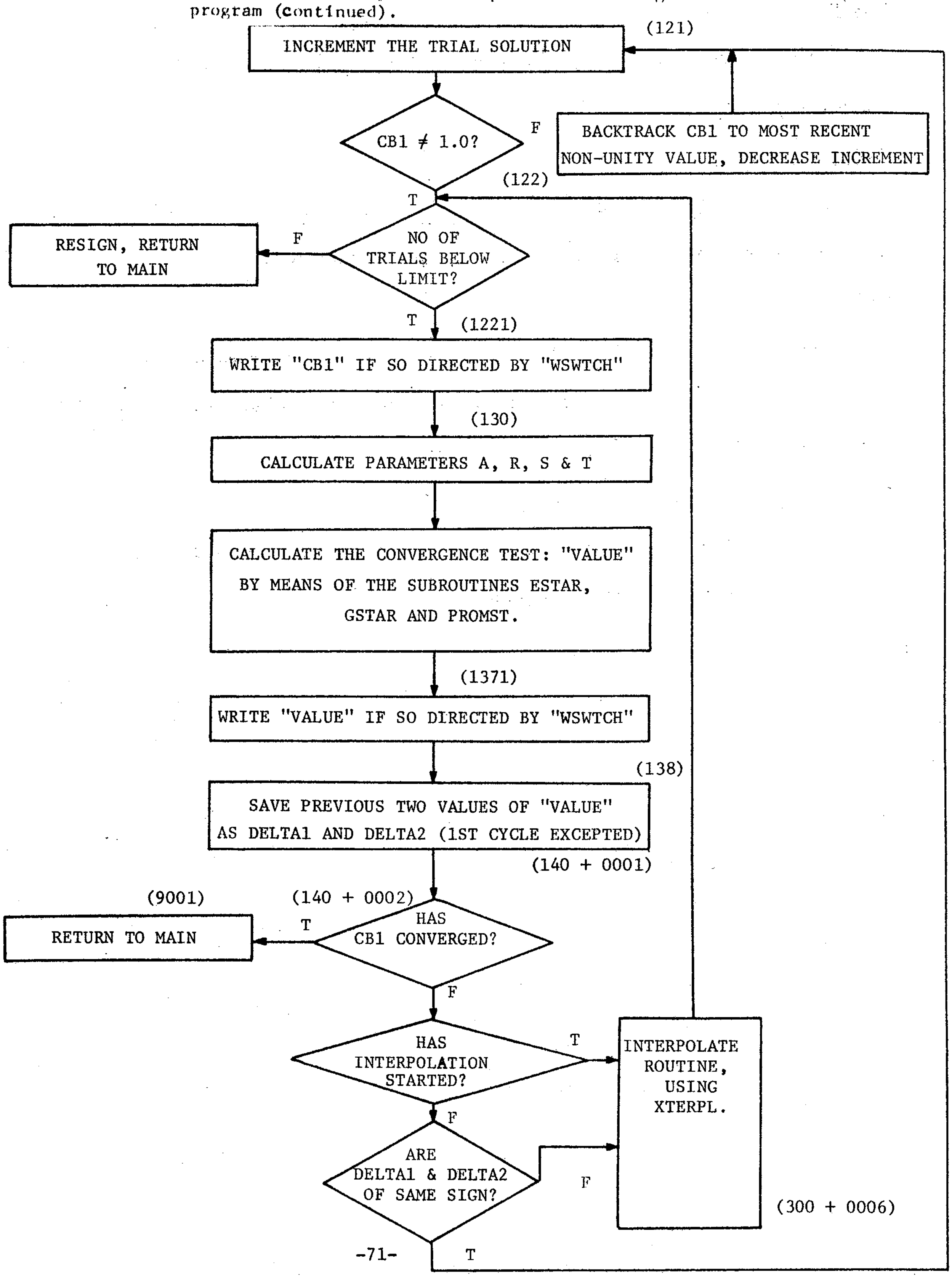


FIGURE 20 - Program for determining the real roots of the frequency equation for a compound free plate - flow diagram of program.

FIGURE 20 - Program for determining the real roots of the frequency equation for a compound free plate - flow diagram of program (continued).



EMATRX : ESTAR

This subroutine calculates the values of the e's (or the corresponding compound matrix).

PROMAT : PROMST

This subroutine calculates the values of the f 's and the elements of the determinant which is used as the frequency equation.

CHECK

This subroutine calculates the value of the determinant used as the frequency equation. The result is returned to the subroutine TRAVEL : TRASTR for testing.

The arithmetic function XTERPL

This function performs the Newton-Raphson interpolation of a value of the determinant of the frequency equation, and the corresponding phase velocities.

APPENDIX C - WANG PROGRAMS WRITTEN FOR THIS WORK

C.1 A Description of the Wang Computer

The Wang Model 360 computer is a desk instrument with four storage registers and two addition/subtraction registers. It can be programmed by means of a machine language code of up to 80 steps punched on a standard IBM card. All calculations are made in floating point with fourteen-digit accuracy, although only the ten most significant are displayed; the indication of an error due to floating point overflow occurs when the number registered exceeds 10^{10}. Hardware subroutines are available for the following operations: add, subtract, multiply, divide, square, square root, natural logarithm and exponential. A library of sub-programs is available for routines such as trigonometric functions and statistical calculations.

This computer was used in the present work for the following:

(1) Programming the Jones' solutions (1); it was found practicable to solve these equations by iteration, as the comparative slowness of the Wang computer was outweighed by its accessibility relative to an IBM360.

(2) Statistical calculations such as the determination of mean square errors and linear regression coefficients.

(3) Calculations of compression and shear wave velocities in elastic materials as functions of the Lame constants; relationships between the Lame constants, Poisson's ratio and Young's modulus.

C.2 Notes on the Programs Written for the Wang Computer

The set of programs consists of cards which are described in this Appendix. The following information is available in TTI files.

(1) Details of the statements in each program; the comments on these sheets are intended to permit each program to be followed step by step when the cards are punched afresh or if a card is damaged. Detailed operating instructions are included.

(2) Flow diagrams, showing which cards are to be combined in order to perform the operations intended, and the sequence in which they are to used. Abbreviated operating instructions are given on the diagrams in order that an operator accustomed to the use of the programs need refer only to the block diagrams for directions and not to the detailed sheets giving the actual statements.

(3) Tests are given for each group of programs, which include the intermediate readings and the final results for typical values of the parameters.

Symbols used:

$SR_{0,1,2,3}$ indicates "store in storage register 0, 1, 2, or 3"

$+A_L$, $-A_L$, $+A_R$, $-A_R$ indicates "add, subtract to adder left, right"

The contents of the storage registers are shown enclosed in a box thus,

d	3
c	2
b	1
a	0

indicates that the numbers represented by the symbols a, b, c, d are held in registers 0, 1, 2, 3 respectively.

APPENDIX D - LIST OF FORTRAN PROGRAMS WRITTEN FOR THIS WORK (Available in Texas Transportation Institute Files)

Designation	Description
WHC33A	Antisymmetric waves in a simple free plate. Lamb Solution. Q = 0.
WHC33B	Jones (1) Lower branch, Equation (26).
WHC33C	Jones (1) Intermediate branch, Equation (23), short wavelengths only.
WHC33D	Jones (1) Intermediate branch, Equation (28), long wavelengths.
WHC33E	Symmetric waves in a simple free plate. Lamb solution. P = 0.
WHC33F	Rayleigh-type waves in a structure consisting of a solid layer overlying a solid semi-infinite medium Ewing, Jardetzky and Press (2) Equation (4-202)
WHC35	Data checking program for WHC36
WHC36	Program for finding the values the Young's moduli of the materials composing the layers of a highway structure, using the deflection basin as input.
WHC37	Program for calculating the points on the frequency dispersion curve of phase velocities for a compound free plate.
WHC38	Same as WHC37, employing compound (or delta) matrices.

APPENDIX E - WANG PROGRAMS
(Available in Texas Transportation Institute Files)
x = wavelength/πH_1

Designation	Description
681.08	Lamb solution Q = 0. Set trial XITI in R_o.
681.09	Lamb solution Q = 0. Set parameters in registers R_1, R_2, R_3 (R_o is unaffected)
681.1	Lamb solution Q = 0. Iterate x.
COMMENT:	681.08, 681.09 and 681.1 form the set for iterating the Lamb solution Q = 0.
681.2	Lamb solution P = 0. Find tan 1/2sH.
681.3	Lamb solution P = 0. Complete the iteration cycle for determining x.
COMMENT:	681.09, 681.2 and 681.3 form the set for iterating the Lamb solution P = 0. 681.3 contains an arctan routine, the result being available at the first stop (step 68); the argument in radians must be in adder left at the start.
681.41	Jones Equation (26), for two surface layers overlying a semi-infinite medium. Lower branch. Set up parameters.
681.42	Jones Equation (26), for two surface layers overlying a semi-infinite medium. Lower branch. Iterate 2/x.
COMMENT:	681.41 and 681.42 form the set for iterating Jones Equation (26).
681.50	Jones Equation (28), for two surface layers overlying a semi-infinite medium. Intermediate branch. Determine parapmeters which may be required for checking manually the corresponding Fortran program.
681.51	Jones Equation (28), for two surface layers overlying a semi-infinite medium. Intermediate branch. Calculate multipliers.
681.52	Jones Equation (28), for two surface layers overlying a semi-infinite medium. Intermediate branch. Iterate 2/x (1st card)

Designation	Description
681.53	Jones Equation (28), for two surface layers overlying a semi-infinite medium. Intermediate branch. Iterate 2/x (2nd card)
681.54	Jones Equation (28), for two surface layers overlying a semi-infinite medium. Intermediate branch. Iterate 2/x (3rd card)

COMMENT: 681.51, 681.52, 681.53, and 681.54 form the set for iterating Jones Equation (28).

681.61	Jones Equation (23), for two surface layers overlying a semi-infinite medium. Intermediate branch: short wavelengths. Calculate parameters.
681.62	Jones Equation (23), for two surface layers overlying a semi-infinite medium. Intermediate branch: short wavelengths. Determine tan (r_2H_2).
681.63	Jones Equation (23), for two surface layers overlying a semi-infinite medium. Intermediate branch: short wavelengths. Determine 2/x.

COMMENT: 681.61, 681.62 and 681.63 form the set for solving Jones Equation (23).

681.7	Determine Poisson's ratio, given the ratio β/α.
681.71	Jones Equation (24), for two surface layers overlying a semi-infinite medium. Lower branch: short wavelengths. Card 1.
681.72	Jones Equation (24), for two surface layers overlying a semi-infinite medium. Lower branch: short wavelengths. Card 2.
681.73	Jones Equation (24), for two surface layers overlying a semi-infinite medium. Lower branch: short wavelengths. Card 3. Determine 2/x.

COMMENT: 681.71, 681.72 and 681.73 form the set for solving Jones Equation (24)

682	Inverse Chevron. Linear law for log (y). Determine parameters.

Designation	Description
684	Inverse Chevron. Square law for log(y). Determine parameters used as input for WHC36, Fortran program.

COMMENT: 682 and 684 require Chevron calculations, with the E's for the layers spaced factors of two apart. Three Chevron calculations are needed for each Wang program input.

ADDENDUM - PROGRAM FOR COMPUTING THE DISPERSION CURVE IN A LAYERED HALF SPACE

A program was written with the object of computing the frequency dispersion curve of phase velocity in a layered half space (WHC39). The stiffness of the materials composing the layers is entirely arbitrary.

The data cards are the same as those for the programs which compute the dispersion curves in a compound free plate, except for the cards which provide the starting points for the trial solutions. The format of these cards is as follows:

Cols. 1-6, 7-12	The real and imaginary parts of the value of WLNGTH for which a solution is required; the imaginary part may be left blank if desired, and the program will compute it in such a way as to yield a purely real value of the frequency FREQ.
Cols. 13-18, 19-24	The real and imaginary parts of the trial value of the phase velocity CB1 at which the search for a root is to start.
Cols. 25-30, 31-36	The real and imaginary parts of the increment to the trial value of the phase velocity; this increment will be used in the initial search for a solution.

RESULTS

The writer has not succeeded in finding the true zeros of the dispersion Equation (18) by means of this program. An initial search was carried as far as a zero on the real axis of CB1; this was followed by a search for a zero (in the imaginary part of the determinant representing the frequency equation) along the negative imaginary axis of CB1. The point reached in the imaginary plane of CB1 was used as the center of a spiral of decreasing radius, restricted to negative values of the imaginary part of CB1. Whenever a small increment along the spiral produced simultaneous changes in sign of both parts of the solution criterion (the determinant of Equation 18), the center of the spiral was moved in that direction.

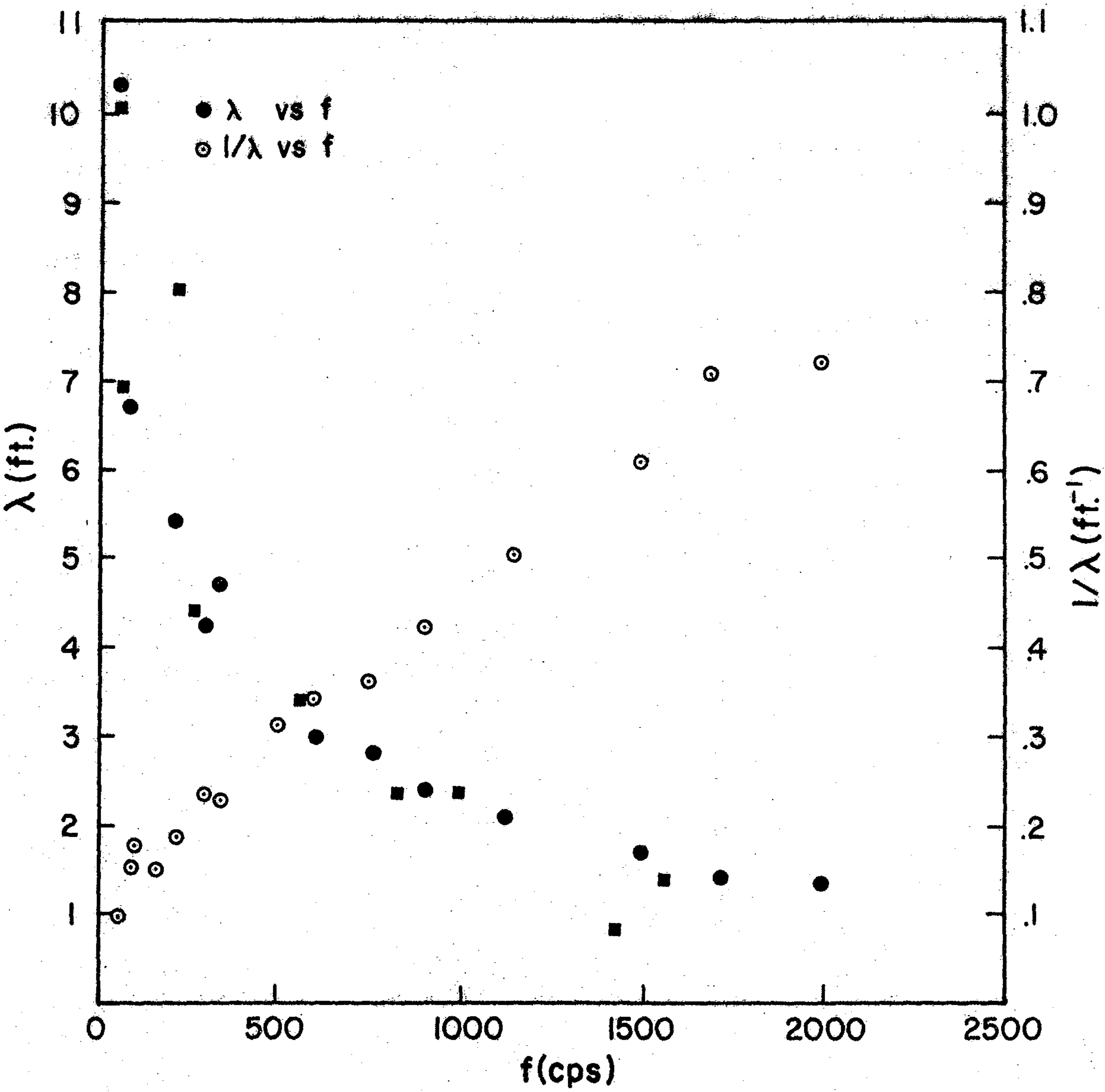
The impression was gained that roots exist in the vicinity of points shown in Figure 21. There appear to be fewer roots of the frequency equation in this case than in the case of the equation relating to a compound free plate. An approximate fit of the experimental points is obtained if the phase velocity of shear waves in the top layer is taken as 2500 feet per second. Assuming a value of 0.45 for Poisson's ratio, the value of Young's modulus for the material composing this layer is 600,000 lb./inch, about twice that obtained as the result of observations made by means of the Dynaflect.

LIMITATIONS

1. The program requires values of the Young's moduli and the densities of the materials composing the structure as data. It does not perform the reverse operation required by engineers of calculating the elastic parameters of the materials composing the structure, using the dispersion curve of phase velocities as data.

2. There is no certainty that roots exist in the vicinity of the sign changes which are used here as a means of improving the trial phase velocity CB1.

3. The program has not been fully tested and may contain significant malfunctions.



SECTION NO. 5

FIGURE 21 - Plot of wavelength against frequency, Texas A&M Test Facility - Section 5.

The squares show the calculated results for a layered half space (Program: WHC39).

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