

PEAK-PERIOD ANALYSIS AND CONTROL  
OF A FREEWAY SYSTEM

by

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## PEAK-PERIOD ANALYSIS AND CONTROL OF A FREEWAY SYSTEM

### Introduction

Peak-period control of urban freeways is receiving increasing attention as a possible means of reducing congestion on these facilities. Control on freeway entrance ramps during the morning peak periods consists of routine procedures on the Gulf Freeway Surveillance and Control Research Project in Houston as well as at other locations in the country. The Houston Research Project is conducted by the Texas Transportation Institute and sponsored by the Texas Highway Department and the U.S. Bureau of Public Roads.

There are several approaches to the philosophy of freeway controls and this paper presents one of them.

The ultimate goal of peak-period freeway control is to allow the entire automobile transportation system, of which the freeways are a part, to accommodate the same number of trips with reduced total travel time. In other words, the same origin-destination demand for trips would be accommodated in the system but these same trips, in aggregate, would require less travel time. This would be accomplished by more efficient use of the freeway and possibly the arterial street system.

The diseconomies of freeway congestion are reasonably well documented. It has been shown that for an oversaturated traffic system with a fixed demand-time function or input-time function, the system travel time can be reduced only by increasing the output rate of the system at some time.<sup>1,2</sup> It has, in fact, been shown for a system with a fixed input-time function, that minimizing the system travel time in any time period  $t_1$  to  $t_2$  is equivalent to maximizing  $\int_{t_1}^{t_2} O(t) dt$  where  $O(t)$  is the cumulative system output.<sup>2</sup> The output rate of a freeway system\* is maximized before congestion develops, that is, when the demand at each bottleneck in the system equals but does not exceed its capacity. When the demand on the freeway system increases, congestion sets in at one or more bottlenecks and, if the congestion becomes severe, the output rate of the freeway system is decreased. As explained in Reference 3, the decrease in output rate stems from two sources, (1) congestion at a bottleneck can decrease the flow rate below its capacity level and (2) congestion or queues forming at a bottleneck can be propagated upstream past exit ramps, thereby decreasing the output rates on these exits. Thus, the desirable operation of the system is to maintain capacity flow rates at each critical bottleneck without allowing congestion to develop.

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\*In this report freeway system refers to a one-directional length of freeway.

Several control techniques are available for use in a peak-period freeway control system and each has its advantages and disadvantages. The most positive control of vehicles can be accomplished as they are entering the freeway rather than after they are on the freeway. Therefore, entrance ramp metering or one of its special cases, entrance ramp closure, appears to be the most promising technique for controlling a large freeway system. (Entrance ramp metering means the placing of an upper limit on the flow rate on a ramp by controlling the time headways of entering vehicles.)

Since traffic conditions change during the day, the controls must be capable of being changed. The controls must be started at the beginning of the peak period and ended when it is over, the metering rates must be determined and (perhaps) ramps must be closed and reopened. There are, of course, many means available for operating the control system and, again, each has its own advantages. The simplest operational scheme is the so-called fixed time scheme under which closure and opening of ramps, changes in metering rates on the ramps, etc., are all initiated at predetermined times.

For additional flexibility, traffic measurements can be used to operate the control system, i.e., the metering rates and closure times would be determined by measurements and/or observations of traffic on the freeway or surrounding facilities. One technique would be to detect a single variable such as speed, density, or lane occupancy on the freeway near an entrance ramp to determine the control required at that ramp. This technique has been applied in Chicago<sup>4</sup> for metering one ramp. Another method would involve the detection of gaps in the freeway lane adjacent to the entrance ramp and upstream of the ramp and releasing vehicles from the ramp when an "acceptable gap" was detected. This method has been suggested by Drew<sup>5</sup> and May.<sup>6</sup>

A fourth, somewhat similar, control philosophy would be to maintain the sum of the flow rates on the entrance ramp and the adjacent freeway lane less than or equal to the single lane merging capacity. Under this scheme, detection in the adjacent lane would provide the flow rate there and the metering rate on the ramp would be set at the difference between the merging capacity (one lane) and the flow rate in the adjacent freeway lane.

This paper presents some of the possible applications and advantages of another means of operating a peak-period metering (and closure) control system. The method involves the use of total flow (in one direction) which must be maintained at levels less than or equal to the capacity at each freeway bottleneck.<sup>2, 7</sup> This control philosophy has greater potential than those previously discussed because the use of total flow across all freeway lanes permits use to be made of the continuity characteristics of traffic flow, thereby making system considerations possible. The above-mentioned techniques can necessarily be concerned only with the operation of individual merging areas and each can (theoretically) maintain a smooth merging operation at each entrance ramp.

However, in many instances bottlenecks occur at locations between ramps, frequently caused by grades, and the methods mentioned earlier may not be too well adapted to these situations. Also, the operation of a given entrance ramp is not an independent consideration. Each ramp is one member of a system and the operation on one affects the operation of the system. It is desirable that the control system take this interdependency into account.

The discussion of the control system is presented in three parts. The first concerns the operation at individual ramps. The second is the consideration of the operation of the entire (one directional freeway) system and the third is operation during reduced-capacity situations, such as occurrences of accidents, disabled vehicles, etc.

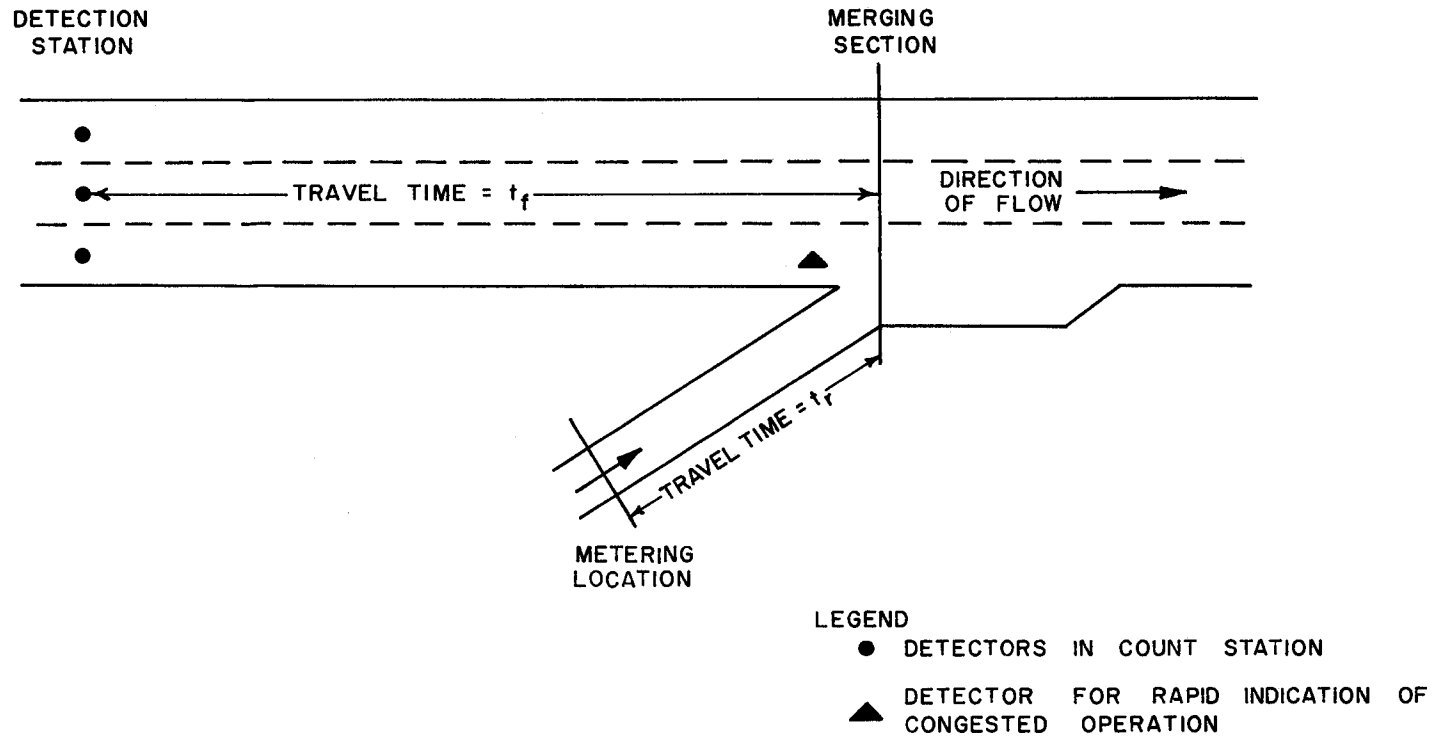
### CONTROL AT INDIVIDUAL ENTRANCE RAMPS

At each entrance ramp in the controlled freeway system the sum of the total directional flow on the freeway and the flow on the ramp must be maintained less than or equal to some desired merging rate. The desired merging rate will usually be one of the following: (1) the merging capacity of the freeway at the ramp, (2) the capacity of a bottleneck between the entrance ramp and the next downstream ramp, or (3) another rate based on the optimization of the total system operation and which will be discussed in the next section. Figure 1 is a schematic of a typical metering situation.

The detection station upstream of the ramp provides the flow rate approaching the ramp or the rate of flow of the traffic stream into which the ramp vehicles must merge. Criteria for the location of the metering station and detection station have been presented previously.<sup>2,7</sup> The relationship between critical times in the system is  $t_f > t_d + t_c + t_r$  where  $t_f$  is the travel time on the freeway between the detection station and the merging section,  $t_d$  is the detection time,  $t_c$  is the computation time or data storage time and  $t_r$  is the average travel time on the ramp between the metering location and the merging section.

The upstream flow rate will, of course, vary with time and the desired merging rate will usually be constant for fairly long periods. If  $fr(t)$  is the time function of upstream flow rate and  $MR$  is the desired merge rate,  $MR - fr(t)$  is the rate at which available "capacity" (using the term quite loosely) is approaching the entrance ramp. In order not to allow any unused capacity to pass by the entrance ramp, the  $i^{th}$  vehicle should be released from the

metering station at time  $t_i$  such that  $\int_{t_{i-1}}^{t_i} [MR - fr(t)] dt = 1$ , where  $t_{i-1}$  is the time of release of vehicle  $i-1$ . This assures that over a period of time the merging volume will (nearly) equal the desired merging volume if there is sufficient demand on the ramp.



SCHMATIC OF METERING AND DETECTION LOCATIONS

FIGURE 1



Figure 2 shows the relationships among several of the control variables. The flowrate is shown in digital form but it could equally well be shown in analog form and in practice the type would depend on the computing equipment used. In Figure 2 desired merge rate is a constant 95 veh/min.

One method of computing the running average flow rate is shown. In this manner, the average flow rate shown for the time period 7:00:15 - 7:00:16 is based on the fifteen-second count from 7:00:00 to 7:00:15, the 7:00:16 - 7:00:17 flow rate is based on the 7:00:01 - 7:00:16 count, etc. Thus,  $t_d$  equals 15 seconds and  $t_c$  equals 1 second. Therefore, proper location of the detector station would make  $t_f = t_r + 16$  seconds.

While prevention of congestion would probably be one major goal of a peak-period freeway ramp control system, no practical control system will achieve this goal. If the attempt is made to operate bottlenecks at or near their capacity (which is probable) some congestion will be expected at these locations. Also, the "unusual events" such as accidents, etc., occur all too frequently and can drastically reduce the capacity of the freeway. A method of prompt detection of congestion is, therefore, necessary if the congestion is to be cleared up as soon as possible.

The addition of one detector immediately upstream of the entrance ramp, as shown in Figure 1, would provide the rapid indication of congestion. Lane occupancy, speed or (calculated) density could be the variable used for this purpose and when a critical level (the proper level will be obtained by experience) is reached an override to the metering rate would be provided to clear the congestion. When the override is in effect, a fixed, low metering rate is called for. This is similar to a scheme which was tried in Chicago. <sup>6</sup>

Figure 3 is a schematic of the logic of the controls at a particular location. It can be seen from this figure and from the preceding discussions that, once the desired metering rate (MR) for the location is set by the system monitor and control computer (see next section), the individual location can operate independently of all other control locations.

## CONTROL OF THE SYSTEM

Each of the metering controllers at each entrance ramp should be operated in a manner which would result in optimal operation of the system of interest, at least when considered over a fairly long period of time. In the discussions which follow a central digital computer is envisioned in the role of monitoring the system operation and controlling the desired merge rates at each entrance ramp. While only one freeway is discussed here, a single computer could probably control several freeways.

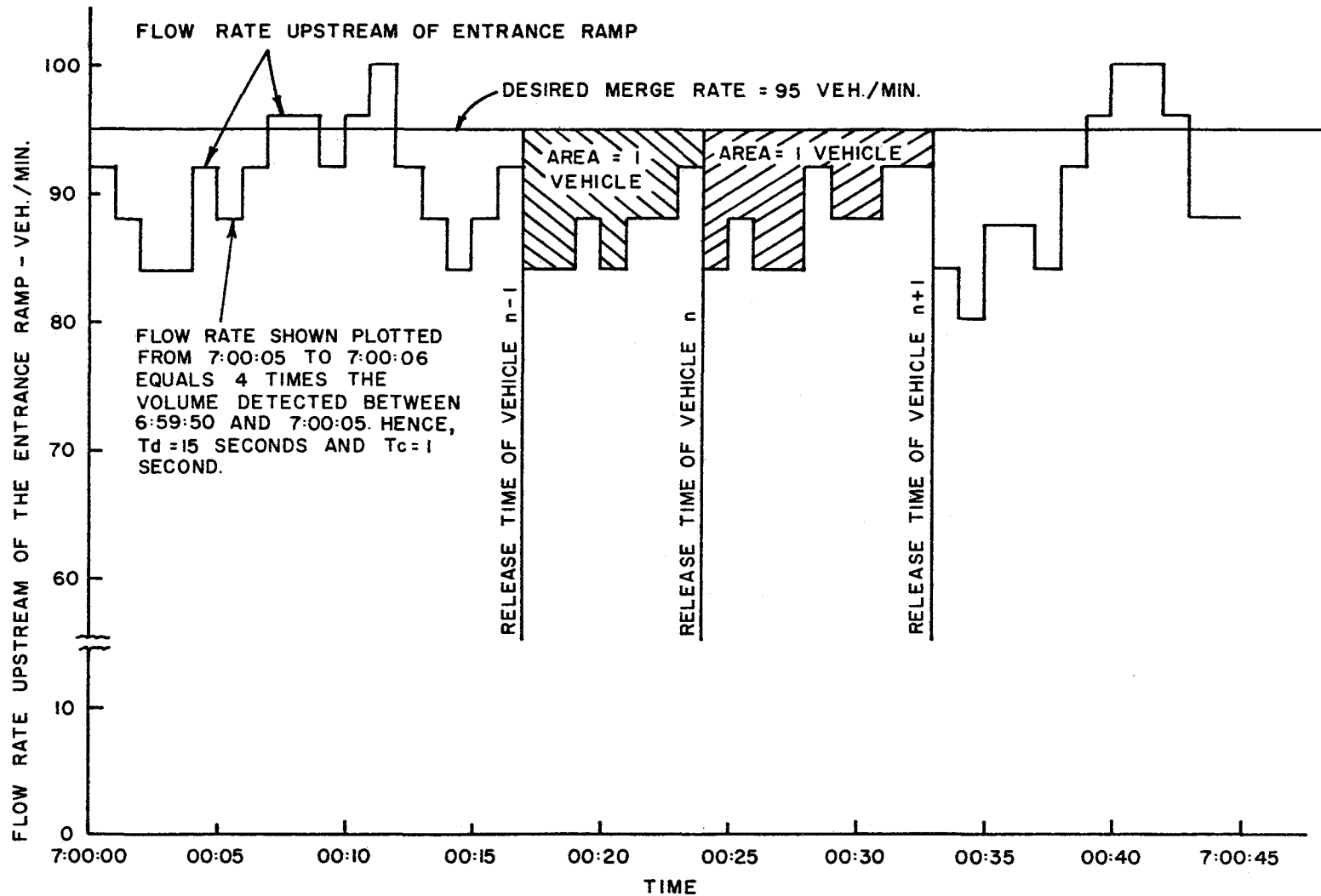


FIGURE 2 - RELATIONSHIPS AMONG DETECTED VOLUME, UPSTREAM FLOW RATE, DESIRED MERGE RATE AND RELEASE TIMES OF VEHICLES ON THE METERED RAMP

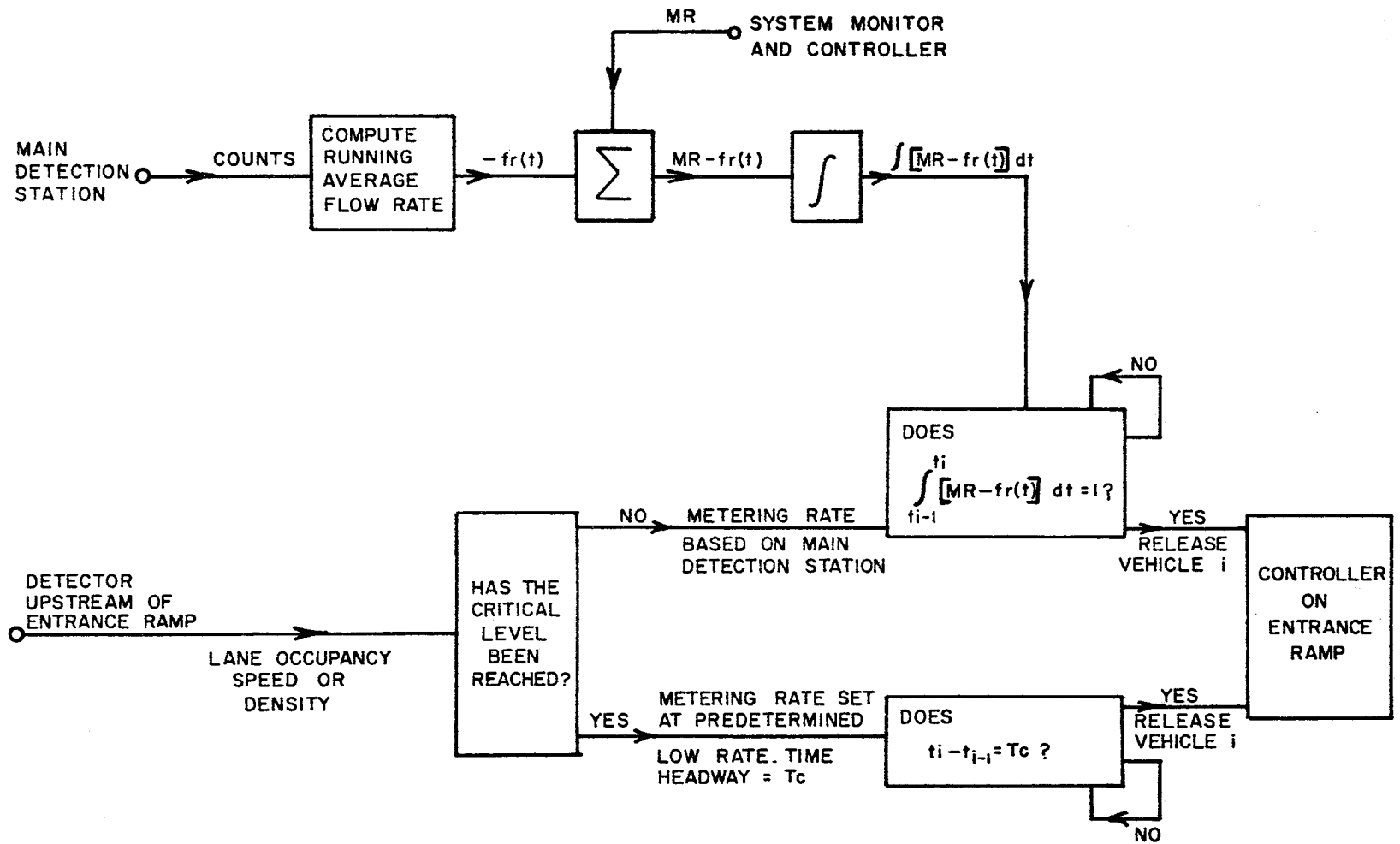


FIGURE 3—SCHEMATIC OF CONTROLS OF AN INDIVIDUAL ENTRANCE RAMP

In the steady-state analysis presented, the length of analysis period must be such that the traffic demand and origin-destination desires are relatively constant over the time period. For example, the inbound Gulf Freeway in Houston was found to be congested from about 7-8 a.m. Vehicles accumulated on the freeway (when demand on the system is greater than capacity) from about 7:00-7:30 a.m. and cleared from about 7:30-8:00 a.m.<sup>3</sup> Hence in this case one type of steady state operation might be applicable from 7:00-7:30 a.m. and another from 7:30-8:00 a.m. so each period would be analyzed individually. In cases in which congestion is more severe, the periods of analysis could be lengthened.

A linear programming model<sup>2,7</sup> is used for the control of a freeway system such as that in Figure 4. The simplest form of the model is

$$\begin{aligned} &\text{Maximize} && \sum_{j=1}^n X_j \\ &\text{subject to} && \sum_{j=1}^n A_{jk} X_j \leq B_k \quad k=1, \dots, m \\ &&& \text{and} \quad X_j \leq D_j \quad j=1, \dots, n \\ &&& \text{and} \quad X_j \geq 0 \quad j=1, \dots, n \end{aligned}$$

The  $X_j$  are the input volumes to the freeway system ( $j=1, \dots, n$ ), the  $A_{jk}$  are the decimal fraction of vehicles entering at input  $j$  which pass through section  $k$  ( $j=1, \dots, n$  and  $k=1, \dots, m$ ) the  $B_k$  are the capacities\* of the freeway sections and the  $D_j$  are the hourly demands at input  $j$ .

Briefly; this model maximizes the output\*\* of the freeway system subject to constraints which keep the demand less than the capacity at each section and which maintain the feasibility of the solution.

The solution vector contains the optimal inputs to the freeway as well as the values of the slack variables  $S_k$  in the active basis. The slack variables for the first set of (capacity) constraints place bounds on the

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\*If the level of service concept<sup>8</sup> is used for the system operation instead of the capacity concept, the  $B_k$  would represent service volumes.

\*\*Due to the first set of constraints, the output (very nearly) equals the input.<sup>2,7</sup>

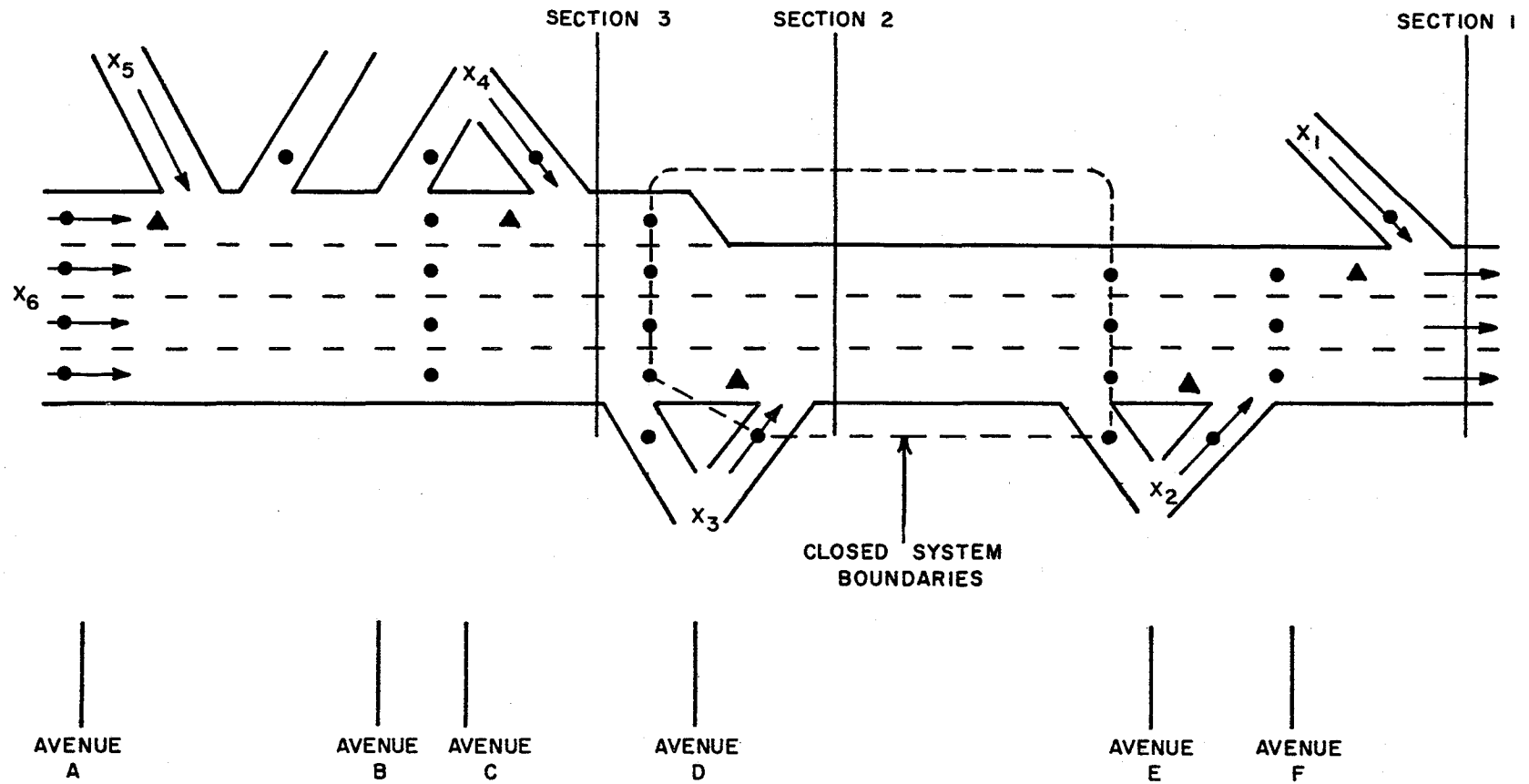


FIGURE 4 — SCHEMATIC OF FREEWAY SUBSYSTEM USED IN THE LINEAR PROGRAMMING MODEL.

desired merge rates. The desired merge rate at section K conforms to the following relationship:

$B_k \geq MR_k \geq B_k - S_k$ . For minimizing the queue at the upstream entrance ramp,

$MR_k = B_k$ . However, a higher level of service would prevail on the freeway if

$MR_k < B_k$  so the tradeoff between ramp queues and freeway level of service can be considered.

One of the potentially major problems associated with an entrance ramp metering system is the development of long queues at the metering locations. It might be desirable to introduce constraints into the model to control these queues. Since the values of the slack variables in the second set of constraints represent the (approximate) length of the queue at the end of the study period (assuming it was zero at the beginning) the queue constraints can be placed on the slack variables.

Both types of queue constraints are somewhat indirect in that neither places a restriction directly on the queue length. Both restrict quantities that are related to the queue lengths.

The two types of queue constraints which can be placed in the model are:

$$S_j \leq Q_j \quad j=1, \dots, n-1^* \quad (1)$$

and 
$$S_j = S_{j+1} \quad j=1, \dots, n-2^* \quad (2)$$

The first type assures that the number of vehicles  $S_j$  which are denied access at the  $j$ th input is less than a certain number  $Q_j$ . The second type spreads the excess demand equally over all entrance ramps in the system by denying access to the same number of vehicles at each of these ramps.

There are perhaps many other types of constraints which could be added to this linear programming model. One such type would maintain the sum of the merging volumes in the adjacent freeway lane and the entrance ramp less than or equal to the single-lane merging capacity. A constraint which would accomplish this is

$$P_a \sum_{j=a+1}^n A_{jk} X_j + X_a < L_a \quad a=1, \dots, n-1^{**}$$

where  $P_a$  is the percent of the total freeway volume upstream of input  $a$  which

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\*Assumes input  $n$  is the freeway input where no queueing is allowed.

\*\*Assumes input  $n$  is the freeway input.

is in the lane adjacent to the entrance ramp and  $L_a$  is the single lane merging capacity at the input a entrance ramp.

So far a method has been presented by which system operation is based largely on historical data (the  $A_{jk}$  and  $D_j$  are historical data and when normal operating conditons prevail the  $B_k$  are also based on historical data) which are then converted to control parameters (desired merge rates). The individual entrance ramps are controlled on the basis of present conditions not on historical data. However, if no reduced-capacity situations (accidents, disabled vehicles, etc.) occurred, the desired merge rates could be determined once and no central monitor and control computer would be required. Since reduced-capacity occurrences are fairly frequent on urban freeways, it is desirable that their effects on the operation of a freeway system be considered.

## REDUCED-CAPACITY OPERATION

### Detection

Prompt detection of a reduced-capacity occurrence is important if its adverse effect is to be minimized. In addition it is important to obtain a reliable estimate of the capacity of the freeway section at the occurrence if the system of controls can be adjusted to compensate for the capacity reduction.

Let us assume that the capacity is reduced between the Avenue D entrance ramp and the Avenue E exit in Figure 4. Assuming also that the demand there is greater than the capacity, congestion will develop and begin to propagate upstream toward entrance D. The traditional method of detecting the capacity reduction would be by an indication of congestion at the nearest detector upstream of it. Low speed, high lane occupancy or high density indicates a downstream source of congestion, in this case the event of interest.

One will note that with a detector on the avenue D entrance ramp and on the Avenue E exit ramp a closed system is defined and is enclosed by a dashed line in Figure 4. The input locations to the closed system are the freeway section downstream of the Avenue D exit ramp and the Avenue D entrance ramp while the output locations are the Avenue E exit ramp and the freeway section downstream of this ramp. Shortly after the capacity reduction, the output flow rate from the closed system will decrease while the input flow will remain about normal (unless the capacity reduction is close to the input detector stations). If  $I(t)$  and  $O(t)$  are, respectively, the number of vehicles entering and leaving the system after some time  $t_0$ , the rate at which vehicles are accumulating in the system  $[I(t + \Delta t) - I(t) - O(t + \Delta t) + O(t)] / \Delta t$  could be monitored and used to detect the reduction in capacity. Thus, when the output rate falls significantly below the input rate for some period of time a capacity reduction somewhere between the input and output location is fairly certain. Since there is a main detection station upstream of every

entrance ramp, the freeway is broken into a series of these closed systems. The central computer could monitor each of these systems for unusual behavior.

When a capacity reduction has occurred in one of these closed systems, steady-state conditions will normally exist between the point of decreased capacity and the output section of the system. If steady state does not exist in this area a more severe capacity reduction must have taken place downstream. However, in most cases the steady state exists and the output flow rate will, over a reasonable time period, equal the flow rate across the reduced-capacity section. The capacity flow rate of this section, then, can be estimated by simple volume counts downstream of it.

Once the capacity reduction has been detected and the capacity flow rate has been estimated, the modified operation of the freeway system can be determined. The revised capacity flow rate is substituted in the linear programming model and the new desired merge rates are obtained. These, as well as the expected input volume at each entrance ramp, provide a rapid estimate of the severity of the ramp controls necessary upstream of the capacity reduction. All of these operations would be conducted automatically within the central computer.

As an example of the operation of the model under normal and reduced capacity circumstances, the system shown in Figure 4 will be analyzed. Only three freeway capacity constraints will be considered in this simple example, on sections 1-3, and no queue constraints are included. An hour period is used in the analysis.

This statement of the model, then, is

$$\begin{aligned}
 &\text{maximize} && \sum_{j=1}^6 X_j \\
 &\text{subject} && \sum_{j=1}^6 A_{jk} X_j \leq B_k \quad k = 1, 2, 3 \\
 &&& \text{and} \quad X_j \leq D_j \quad j = 1, \dots, 6 \\
 &&& \text{and} \quad X_j \geq 0 \quad j = 1, \dots, 6
 \end{aligned}$$

Table 1 contains the  $A_{jk}$  used in the model under normal operating conditions. These data would be obtained from origin-destination surveys at the entrance ramps.



TABLE 1

 $A_{jk}$  USED IN MODEL FOR NORMAL OPERATION

		j					
		1	2	3	4	5	6
k	1	1.000	1.000	.949	.933	.824	.519
	2			1.000	1.000	.922	.619
	3				1.000	.969	.777

Table 2 contains the  $B_k$ , the freeway capacities under normal operation. These would be obtained from historical volume counts.

TABLE 2

 $B_k$  USED IN MODEL FOR NORMAL OPERATION

K	$B_k$
1	5900
2	6000
3	6450

The  $D_j$ , the maximum hourly demand at each input, are in Table 3. These, too, would be obtained from a series of volume counts.

TABLE 3

 $D_j$  USED IN MODEL FOR NORMAL OPERATION

j	1	2	3	4	5	6
$D_j$	600	475	450	500	825	6800

The simplex method was used to solve this problem and Table 4 contains the original simplex tableau, the  $S_1, \dots, S_9$  being the slack variables.

The optimal simplex tableau is shown in Table 5. The optimal solution is  $X_1 = 447$ ,  $X_2 = 475$ ,  $X_3 = 450$ ,  $X_4 = 367$ ,  $X_5 = 825$ ,  $X_6 = 6800$ ,  $S_2 = 213$ ,  $S_4 = 153$ , and  $S_7 = 133$ . Hence, 447 vehicles can enter the freeway via the Avenue F entrance ramp and 153 must be stored or diverted there. Only 367 vehicles can be allowed to enter at Avenue C and since the demand is 500 vehicles, 133 vehicles in the hour must be stored or diverted there. The second constraint turned out to be redundant and there is a 213 vehicle per hour excess capacity at Section 2.

Although the tableaux are not presented this same problem was solved again assuming that a capacity reduction was detected at Section 2 and that the capacity there was estimated to be 5400 vehicles per hour. In this solution  $X_1 = 600$ ,  $X_2 = 475$ ,  $X_3 = 63$ ,  $X_4 = 367$ ,  $X_5 = 825$ ,  $X_6 = 6800$ ,  $S_1 = 214$ ,  $S_6 = 387$ , and  $S_7 = 133$ . Thus, no vehicles would have to be denied access at the Avenue F entrance - in fact the capacity constraint at Section 1 is now redundant and has a 214-vehicle per hour excess capacity. The inputs  $X_2$ ,  $X_4$ ,  $X_5$ , and  $X_6$  are all unchanged. However, with the capacity at Section 2 reduced,  $X_3$  decreases to 63 vehicles (from 450). Thus, during the hour only 63 vehicles would be expected to be able to enter the Avenue D entrance ramp while 387 would have to be diverted. This ramp would probably be closed due to the capacity reduction.

The effect of the accident on the system output,  $Z_0$ , during the hour can also be seen from these analyses. Under normal operation  $Z_0 = \sum_{j=1}^n X_j = 9364$  while with the capacity reduction  $Z_0 = 9130$ . Therefore, the output of the system would be about 234 vehicles per hour less when the capacity of Section 2 is reduced to 5400 vehicles per hour.

Figure 5 shows a schematic of the central monitor and control computer operations. It receives inputs from the freeway detector and supplies outputs to the individual controller locations. The other operations and decisions are performed internally.

In summary, some of the advantages of using the total directional flow rate and capacity to operate a system of freeway ramp metering controls have been presented. The main advantages stem from the ability to make use of the continuity characteristics thereby permitting systems analysis and operation. Each local controller obtains the desired merge rate from the central computer and then determines the time headways between entering vehicles to maintain this rate of merge. The central computer monitors the freeway operation by examining the detector outputs. It determines whether normal or reduced-capacity operations prevail and establishes the proper merging rates

TABLE 4

## ORIGINAL SIMPLEX TABLEAU

$C_j$																	$b_i$
		$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	
$C_i$	$X_i$	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$	$s_1$	$s_2$	$s_3$	$s_4$	$s_5$	$s_6$	$s_7$	$s_8$	$s_9$	
0	$s_1$	1	1	.949	.933	.824	.519	1									5900
0	$s_2$			1	1	.922	.619		1								6000
0	$s_3$				1	.969	.777			1							6450
0	$s_4$	1									1						600
0	$s_5$		1									1					475
0	$s_6$			1									1				450
0	$s_7$				1									1			500
0	$s_8$					1									1		825
0	$s_9$						1									1	6800
$Z_j - C_j$		-1	-1	-1	-1	-1	-1	0	0	0	0	0	0	0	0	0	$Z_0 = 0$

TABLE 5  
OPTIMAL SIMPLEX TABLEAU

	$C_j$	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0
$C_i$	$X_i$	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$X_6$	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	$S_6$	$S_7$	$S_8$	$S_9$	
1	$X_1$	1						1		-.933		-1	-.949		.080	.206	447
0	$S_2$								1	-1			-1		.047	.158	213
1	$X_4$				1					1					-.969	-.777	367
0	$S_4$							-1		.933	1	1	.949		-.080	-.206	153
1	$X_2$		1									1					475
1	$X_3$			1									1				450
0	$S_7$									-1				1	.969	.777	133
1	$X_5$					1									1		825
1	$X_6$						1									1	6800
$Z_j - C_j$		0	0	0	0	0	0	1	0	.067	0	0	.051	0	.111	.429	$Z_0 = 9364$

at each entrance ramp in the system. It supplies these rates to the local controllers which then maintains them.

The control system suggested here is fairly elaborate and an investment in such a system should be carefully analyzed. An incremental analysis should be used. If the annual cost of the system as outlined in this report is \$200,000 and the annual benefit to the motorists is \$400,000 it might seem to be justified. However, an extremely simple, fixed-time control scheme with an annual cost of \$20,000 might result in an annual benefit to the motorists of \$300,000. In this case the incremental investment of \$180,000 for an annual benefit of \$100,000 may not be justified.

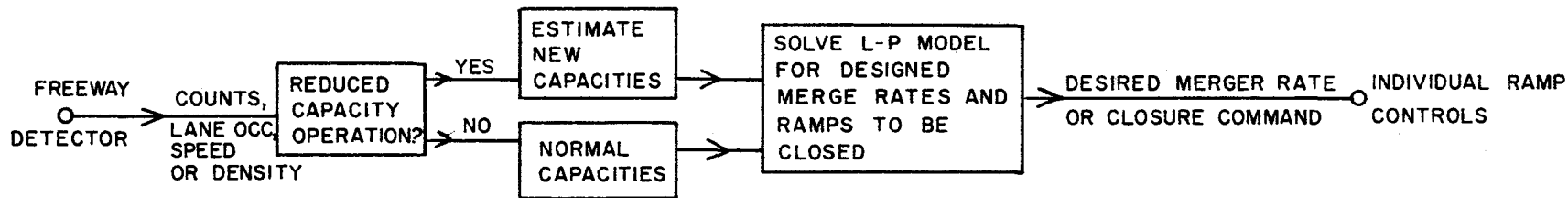


FIGURE 5- SCHEMATIC OF OPERATIONS OF CENTRAL MONITOR AND CONTROL COMPUTER

## ACKNOWLEDGEMENTS

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\*This same material constitutes Report 9 of the Chicago Area Expressway Surveillance Project.



APPENDIX



## APPENDIX

### Linear Programming

A mathematical model of the linear programming type has two essential characteristics: a linear function to be maximized or minimized and a set of linear constraints on the values of the variables. The function to be maximized or minimized is frequently called the criterion function, objective function, or figure of merit. Hereafter in order to reduce any confusion, discussion will involve only the maximizing case since there is an easy transformation from minimizing case to the maximizing case.

The variables in the model are usually designated by a subscripted  $X$  such as  $X_1, X_2, X_3, \dots, X_n$  in a problem with  $n$  variables. If a problem had three variables they would be designated  $X_1, X_2,$  and  $X_3$ .

Associated with each variable is a unit savings or unit profit if the objective function is to be maximized. The letter  $C$  is normally used for the coefficients in the objective function. There is one coefficient for each variable. So if a problem had three variables, the three coefficients would be  $C_1, C_2,$  and  $C_3$  and the objective function would be to maximize  $C_1X_1 + C_2X_2 + C_3X_3$ . For example,  $C_1$  might equal 10,  $C_2$  might equal 15 and  $C_3$  might equal 5. In this case the objective function would be

$$\text{maximize } 10X_1 + 15X_2 + 5X_3.$$

If there were no constraints placed upon this process, the objective function could be made as large as possible by increasing one or more of the variables. However, in linear programming models, constraints limit the size of the variables. For example, assume that the preceding objective function represented the daily profit of producing three products. Assume further that the three products represent three types of signs produced in a highway department sign shop. Since the sign shop is relatively small, the highway department can use all of each type of sign that the shop can produce. The department must purchase the signs which cannot be made in the sign shop. For each type 1 sign produced, the department saves \$10. It saves \$15 for each type 2 sign produced and \$5 for each type 3 sign produced.

The number of type 1 signs produced daily is  $X_1$  and the unit savings for this type of sign is \$10 per sign. Thus the total savings of producing sign 1 is  $10X_1$  and the total savings for producing all three signs is  $10X_1 + 15X_2 + 5X_3$  and the department wishes to maximize this savings.

Three machines a, b, and c, are available for production of these three types of signs and each machine operates for eight hours per day. Assume that 1 hour on machine a and a half hour on each of the other two machines is required to produce each sign of type 1. Each type 2 sign requires 1 hour on machines a and b and no time on machine c. One half hour on machines b and c is required to produce each type 3 sign.

Looking at machine a, eight hours are available to produce the three types of signs and each sign of type 1 requires 1 hour, type 2 requires 1 hour per sign and type 3 requires no time. Therefore the amount of time available on machine a limits the number of signs of each type which can be produced each day. The time constraint on machine a can be written as follows

$$1X_1 + 1X_2 + 0X_3 < 8 \text{ hours (machine a)}$$

This states that the amount of time spent on machine a in producing the three types of signs must be less than or equal to the eight hours available on this machine. Similarly the time constraints on machines b and c as follows:

$$1/2 X_1 + 1X_2 + 1/2 X_3 \leq 8 \text{ hours (machine b)}$$

and  $1/2 X_1 + 0X_2 + 1/2 X_3 \leq 8 \text{ hours (machine c)}$

This entire process can be stated in linear programming format as follows:

$$\begin{aligned} &\text{maximize } 10X_1 + 15X_2 + 5X_3 \\ &\text{subject to } \quad 1X_1 + 1X_2 + 0X_3 \leq 8 \text{ (machine a)} \\ &\quad \quad \quad 1/2 X_1 + 1X_2 + 1/2 X_3 \leq 8 \text{ (machine b)} \\ &\quad \quad \quad 1/2 X_1 + 0X_2 + 1/2 X_3 \leq 8 \text{ (machine c)} \end{aligned}$$

In this statement the constraints are presented as inequalities. Constraint inequalities are normally converted to equalities by adding a slack variable to each constraint. For example by adding the slack variable variable  $S_a$ , the first constraint above can be written as follows:

$$1X_1 + 1X_2 + 0X_3 + S_a = 8.$$

The value of the slack variable represents the amount of time each day during which machine a is not used. This entire problem can be rewritten as follows:

$$\text{maximize } 10X_1 + 15X_2 + 5X_3$$

$$\begin{array}{rcl}
\text{subject to} & 1X_1 + 1X_2 + 0X_3 + S_a & = 8 \\
& 1/2X_1 + 1X_2 + 1/2X_3 + S_b & = 8 \\
& 1/2X_1 + 0X_2 + 1/2X_3 + S_c & = 8
\end{array}$$

General linear programming problems can be solved using the simplex method (9) and special types of linear programming problems can be solved using special-purpose algorithms. This problem was solved by hand using the simplex method and two optimal solutions were obtained. In the first solution  $X_1 = 0$ ,  $X_2 = 8$ ,  $X_3 = 0$ ,  $S_a = 0$ ,  $S_b = 0$ , and  $S_c = 8$ . This means that only signs of type 2 are produced and eight signs of this type are produced. Also eight hours of capacity ( $S_c = 8$ ) remain unused on machine c, i.e., this machine is not used at all. Machines a and b are used to capacity.

In the alternate optimal solution eight signs each of type 1 and type 3 are produced and no signs of type 2 are produced. All three machines are used to capacity. Each of these optimal production schedules results in a profit of \$120 per day. The two optimal solutions can be examined from other points of view to see which one would be adopted. For example, solution 1 may be preferable because machine c is not used and, hence, is available for other work.

This small problem was used for illustrative purposes but this does not imply that linear programming is limited to small problems. Computer programs are available for the solution of problems of more than a thousand variables and several thousand constraints.

## PUBLICATIONS

### Project 2-8-61-24 Freeway Surveillance and Control

1. Research Report 24-1, "Theoretical Approaches to the Study and Control of Freeway Congestion" by Donald R. Drew.
2. Research Report 24-2, "Optimum Distribution of Traffic Over a Capacitated Street Network" by Charles Pinnell.
3. Research Report 24-3, "Freeway Level of Service as Influenced by Volume Capacity Characteristics" by Donald R. Drew and Charles J. Keese.
4. Research Report 24-4, "Deterministic Aspects of Freeway Operations and Control" by Donald R. Drew.
5. Research Report 24-5, "Stochastic Considerations in Freeway Operations and Control" by Donald R. Drew.
6. Research Report 24-6, "Some Considerations of Vehicular Density on Urban Freeways" by John J. Haynes.
7. Research Report 24-7, "Traffic Characteristics of the Freeway Interchange Traffic of the Inbound Gulf Freeway" by William R. McCasland.
8. Research Report 24-8, "System Demand-Capacity Analysis on the Inbound Gulf Freeway" by Joseph A. Wattleworth.
9. Research Report 24-9, "Capacity - Demand Analysis of the Wayside Interchange on the Gulf Freeway" by William R. McCasland.
10. Research Report 24-10, "Inbound Gulf Freeway Ramp Control Study I" by Charles Pinnell, Donald R. Drew, William R. McCasland, and Joseph A. Wattleworth.
11. Research Report 24-11, "Investigation of an Internal Energy Model for Evaluating Freeway Level of Service" by Donald R. Drew and Conrad L. Dudek.

12. Research Report 24-12, "Gap Acceptance Characteristics for Ramp-Freeway Surveillance and Control" by Donald R. Drew.
13. Research Report 24-13, "Inbound Gulf Freeway Ramp Control Study II" by Charles Pinnell, Donald R. Drew, William R. McCasland, and Joseph A. Wattleworth.
14. Research Report 24-14, "Outbound Operation of the Gulf Freeway."

