A THREE-DIMENSIONAL MATHEMATICAL MODEL OF AN AUTOMOBILE PASSENGER

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Evaluation of the Roadway Environment by
Dynamic Analysis of the Interaction Between
the Vehicle, Passenger, and Roadway

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FOREWORD

The information contained herein was developed on Research Project 2-5-69-140 entitled "Evaluation of the Roadside Environment by Dynamic Analysis of the Interaction Between the Vehicle, Passenger, and Roadway" which is a cooperative research study sponsored jointly by The Texas Highway Department and the U. S. Department of Transportation, Federal Highway Administration.

Basically, the objectives of the study are to apply mathematical simulation techniques in determining the dynamic response of vehicles and their passengers when in collision with various roadside objects or when traversing curves in the road, shoulders, or other situations. It is a three-year study with a proposed completion date of August 1971. The first year's work was presented in Research Report 140-1 entitled "Documentation of Input for Single Vehicle Accident Computer Program," July 1969.

As part of the second year's task an analytical model was developed which predicts the dynamic response of an automobile's occupant in three-dimensional space. This report presents the derivation of the passenger model, a validation study, and a description of input data to the computer program used in determining the passenger's response.

The opinions, findings, and conclusions expressed in this publication are those of the author and not necessarily those of the Federal Highway Administration.

ABSTRACT

Engineers are currently attempting to reduce the severity of vehicle accidents by designing and building a safer roadway environment. To accomplish this, consideration must be given to the expected dynamic response of the vehicle and occupant during a collision with roadside obstacles.

The reported research presents the development of an analytical model that predicts the response of an automobile passenger during vehicle motion of a general nature, i.e., a three-dimensional path including simultaneous rotations about the three directions. This model reduces the problem of predicting the accelerations and forces acting on a passenger during a collision or violent maneuver to that of specifying the path of the vehicle as a function of time plus the deformation properties of the vehicle interior.

The vehicle occupant is defined mathematically in three-dimensions as an independent system which is then placed inside the vehicle but not connected to it. The vehicle interior contains the passenger within its boundaries by applying contact forces while the vehicle moves through space. The vehicle interior is idealized with 25 planar surfaces and includes lap and torso restraint belts.

The geometry of the vehicle occupant is idealized by 12 rigid mass segments interconnected in a pattern which reflects the articulated nature of the human body. This system has 31 degrees of freedom which correspond to the set of generalized coordinates used in Lagrange's

equations to derive equations of motion. These equations are solved numerically with the aid of the IBM 360/65 computer.

"Spinal elasticity" is simulated with rotational springs in the back joints of the articulated body, and "muscle tone" is simulated with rotational viscous dampers in every body joint.

Validation of the model has been achieved for frontal collisions. The model's predicted response of a dummy on a test cart compared well with actual test data. Both restrained and unrestrained conditions were analyzed. There were two restrained conditions: (1) lap belt only and (2) lap and torso belt.

The model provides the highway engineer with a tool which will enable him to apply biomechanics data on human tolerance limits to the problem of modifying roadside structures such that occupant injuries during vehicle accidents can be reduced.

SUMMARY

During the second year's effort of this three-year study, a mathematical model was developed which describes the motion, accelerations, and forces experienced by an automobile's passenger during a collision or any violent maneuver of a vehicle in three dimensions. A computer program was then written to solve the governing equations of the model.

Features of this program include: automatic seating of the passenger inside the vehicle; seating the passenger at either the left front, right front, left rear, or right rear positions; a lap restraint belt at any of these four positions; and/or a torso restraint belt at any of these four positions.

The passenger program requires input of the position of the vehicle in space as a function of time plus the force-deformation relations of the various surfaces of the vehicle interior. The path of the vehicle is obtainable from the CAL Single Vehicle Accident computer program or by full-scale testing.

Output from the program is presented in three forms: (1) digital printout, (2) X-Y plots of selected parameters, and (3) orthographic views of the passenger with respect to the vehicle interior for any given time during an event.

It has been concluded that this passenger model provides the engineer with a valuable tool with which to study vehicle and roadway problems culminating in the saving of lives and the reduction of occupant injuries.

IMPLEMENTATION STATEMENT

The reported research describes a tool which supplements existing technology as related to the prediction of occupant injury during a given vehicle maneuver and therefore exhibits favorable implementation potential. The passenger model, described herein, can be used in conjunction with the CAL Single Vehicle Accident computer program or full-scale test results and with biomechenics data on human tolerance limits to provide a systematic approach to the evaluation of roadways and vehicle interiors for safety. Some of its suggested applications include:

1. the evaluation of roadway geometry, i.e., sideslopes, ditches, terrain involving a variation of vertical and horizontal alignment, etc.; roadside safety features such as the breakaway sign, energy absorbing impact cushions, etc.; roadside protective barriers such as guardrails, bridge rails, median barriers, etc.;
2. the design of the vehicle interior and restraint systems;
3. the study of the dynamic behavior of a pedestrian when struck by an automobile; and
4. the study of collisions involving more than one vehicle.

TABLE OF CONTENTS

TABLE OF CONTENTS (Continued)

TABLE OF CONTENTS (Continued)

LIST OF FIGURES

LIST OF FIGURES (Continued)

LIST OF FIGURES (Continued)

INTRODUCTION

Background

Engineers are currently attempting to reduce the severity of single vehicle accidents by designing and building a safer roadway environment.

To design or evaluate effectively a roadway or its immediate environment for safety, consideration must be given to the dynamic response of the vehicle and occupant during interaction with geometric features such as curves, ditches, etc., or obstacles such as guardrails, bridge rails, median barriers, sign posts, etc. Accordingly, the design of highway safety devices such as breakaway signs, energy absorbing impact cushions, earth berms (an earth embankment geometrically designed to redirect safely a vehicle which has left the roadway), etc., depends directly on the dynamic response of vehicle and passenger during collision with these objects.

The above considerations are accurately summarized in the following quotation (1)*:

> "Unless the motion time history of the vehicle can be translated into the expected kinematics of the vehicle

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*Numerals in parentheses refer to corresponding items in "APPENDIX I.-REFERENCES."

> occupant and further translated into the nature and extent of physical damage, it is not possible to establish performance requirements for roadside structure modifications that will effect a reduction in occupant injuries during single vehicle collisions."

The reported research was aimed at providing an analytical means of supplementing existing technology as related to roadside energy conversion systems. This was accomplished by developing a mathematical model to predict the response of an automobile passenger during violent vehicle motion of a general nature, i.e., a three-dimensional path including simultaneous rotations about the three directions.

Design Considerations

Usual design practice is first to determine the time history and levels of acceleration (g-levels) experienced by the vehicle during a particular maneuver or collision. These are next compared to certain tolerance limits assuming that the occupant is subjected to the same g-level. This assumption is rigorously true only if the occupant is rigidly fastened to the vehicle. In actuality the passenger is unrestrained, lap belted, or shoulder harnessed and movement is not completely restricted, such that this assumption could range anywhere from overly conservative to dangerously inadequate depending on the situation.

Another factor which influences highway safety design problems is the quantitative consideration of contact forces between vehicle occupant and vehicle interior. It is possible for an automobile passenger to suffer fatal injuries from contact forces during a vehicle maneuver which at present may appear completely tolerable from the standpoint of vehicle accelerations alone.

It is felt that an analytical model of a passenger, used in conjunction with available biomechanics data on human tolerance limits, can be of significant value in approaching highway safety design problems.

Review of Literature

A survey of the literature has shown that the mathematical modeling of a vehicle occupant has been attempted in recent years. In most cases these efforts were aimed at developing restraint systems for the occupant, but in no instance was the occupant's general dynamic response the prime consideration.

In the early 1960's, a mathematical approach to the occupant restraint problem was made by the aerospace industry (5, 16). The primary concern was the behavior of viscera for fully restrained subjects.

During 1962-63, an analytical study of occupant restraint systems was performed by Cornell Aeronautical Laboratory (CAL) (10). A seven degree-of-freedom nonlinear mathematical model of a restrained, articulated body on a test cart, for the case of a frontal collision, was formulated and programmed for an electronic computer. This study also led to the development of an eleven degree-of-freedom passenger model completed in 1966, and is the most sophisticated yet employed for the occupant restraint problem (12, 13, 14).

In 1967, Emori (2) conducted a study whose purpose was "to understand the mechanics of the automobile collision and to establish a logical background for the injury reduction of occupants." The scope of his research precluded the use of the CAL model and a single degree-of-freedom spring mass system for the occupant, and a similar representation of the automobile was used.

Renneker (17) used a two degree-of-freedom model of an occupant characterized by hip and torso restraint to study the effect of vehicle forestructure energy absorption on occupant injury.

Martinez and Garcia (8), in 1968, developed a mathematical model to represent the motion of the head and neck during rear-end collisions to study the "whiplash" phenomenon.

In 1969, Suggs, et al., (18), considered the problem of objectionable amplitudes and frequencies in the vibration of seats using a two degree-of-freedom representation of the human, for the purpose of developing more comfortable seats.

With the exception of the CAL model (12, 13, 14), the abovementioned efforts have little in common with the reported research but are acknowledged because they were mathematical simulations of the vehicle occupant.

The CAL model provided the major guidelines for performing this research since, in this writer's opinion, the results of that study reflect an adequate representation of the vehicle occupant for the two dimensional environment considered. However, the specific equations derived by CAL were not applicable to this study since this study involves a three-dimensional formulation, although the same basic geometrical configuration and concepts were applicable.

DERIVATION OF THE EQUATIONS OF MOTION FOR THE VEHICLE OCCUPANT

Mathematical Formulation

Body Geometry and Coordinate Systems

The vehicle occupant is defined separately from the vehicle, or as an independent system of articulated rigid mass segments in three-dimensional space. Consequently, the vehicle interior can be thought of as a confining environment for the occupant.

Interaction between occupant and vehicle interior is discussed in detail in a later section.

Figure 1 shows the center lines of the 12 rigid mass segments and their connection pattern, chosen as to geometrically describe the human body. A center line is defined as a straight line which connects the ends of a segment. It is assumed that the center of mass of a segment lies on its center line. For the body extremities, i.e., head-neck segments, forearm-hand segments, and leg-foot segments below the knees, the center line is defined as the straight line extending from the connecting body joint through the center of mass of the segment. Fixed at the center of mass, of each segment n, is a right-handed cartesian coordinate system denoted by axes X_n, Y_n, and Z_n. The positive directions of these axes are defined such that when the body in Figure 1 is standing upright with arms hanging vertically (downward) X_n will be positive straight ahead, Y_n will be positive to his left, and Z_n will be positive upwards. In addition, the

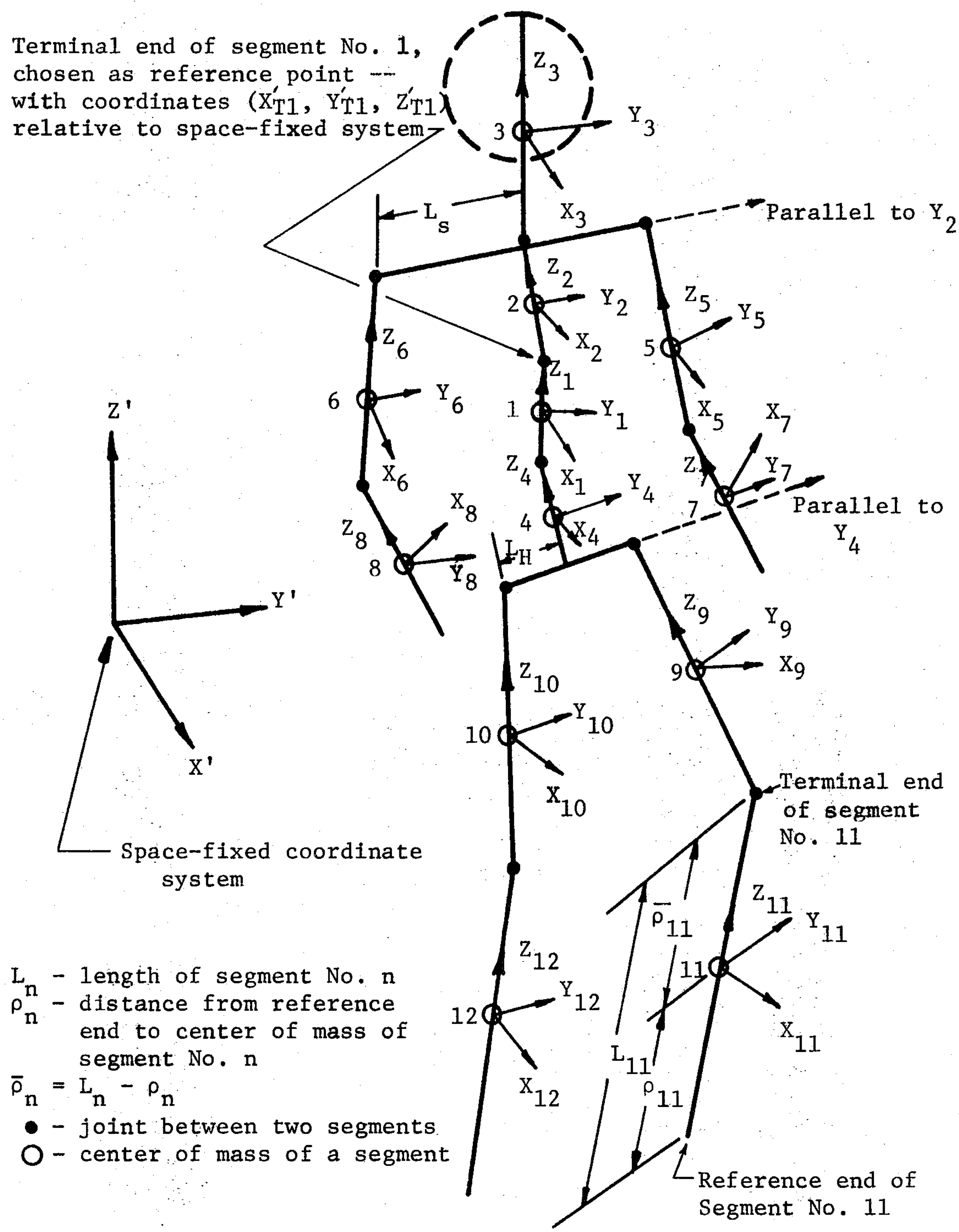

FIGURE 1.-ARTICULATED BODY WITH COORDINATE SYSTEMS

coordinate system of each segment X_n, Y_n, and Z_n will be parallel to the space-fixed coordinate system X', Y', Z'.

Euler Transformation and Generalized Coordinates

The relationship between segment-fixed coordinates (X_n, Y_n, Z_n) and space-fixed coordinates (X', Y', Z') is defined, in matrix form, as

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix} \begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} \qquad \ldots\ldots\ldots\ldots \quad (1)$$

The matrix $[T^n]$ is a transformation matrix that transforms the segment-fixed coordinates back to the space-fixed coordinates, which means that if each of the 12 segments of the articulated body assumes a different orientation in space at any particular time, it is still possible to relate each of them to a common reference frame.

$[T^n]$ could be derived in a number of ways, the most common of which utilizes direction cosines of the X_n, Y_n, Z_n axes relative to the X', Y', Z' axes. However, direction cosines would not be suitable for a Lagrangian formulation (to be discussed later) since they are not independent of each other and hence are not generalized (4). However, if the transformation from one cartesian coordinate system to another is done by means of three successive rotations performed in a specific sequence, the three angles are independent of each other and can be used as generalized coordinates. These three sequential rotations, commonly referred to as the "Eulerian angles", will be used to derive $[T^n]$.

The three sequential rotations to be used are given in Figure 2. Consider the cartesian coordinate systems X', Y', Z' and X_n, Y_n, Z_n to be initially parallel, Figure 2(a), then the rotations are as follows:

1. Rotate about the Z' axis an amount ϕ_n to first intermediate position X_{n1}, Y_{n1}, Z_{n1}, Figure 2(b).
2. Rotate about the Y_{n1} axis an amount θ_n to second intermediate position X_{n2}, Y_{n2}, Z_{n2}, Figure 2(c).
3. Rotate about the Z_{n2} axis an amount ψ_n to final position X_n, Y_n, Z_n, Figure 2(d).

The transformation from coordinates X_{n1}, Y_{n1}, Z_{n1} to coordinates X', Y', Z', Figure 2(b), is given by

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} \cos\phi_n & -\sin\phi_n & 0 \\ \sin\phi_n & \cos\phi_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_{n1} \\ Y_{n1} \\ Z_{n1} \end{Bmatrix} \quad \ldots\ldots \quad (2)$$

The transformation from the coordinates X_{n2}, Y_{n2}, Z_{n2} to coordinates X_{n1}, Y_{n1}, Z_{n1}, Figure 2(c), is given by

$$\begin{Bmatrix} X_{n1} \\ Y_{n1} \\ Z_{n1} \end{Bmatrix} = \begin{bmatrix} \cos\theta_n & 0 & \sin\theta_n \\ 0 & 1 & 0 \\ -\sin\theta_n & 0 & \cos\theta_n \end{bmatrix} \begin{Bmatrix} X_{n2} \\ Y_{n2} \\ Z_{n2} \end{Bmatrix} \quad \ldots\ldots \quad (3)$$

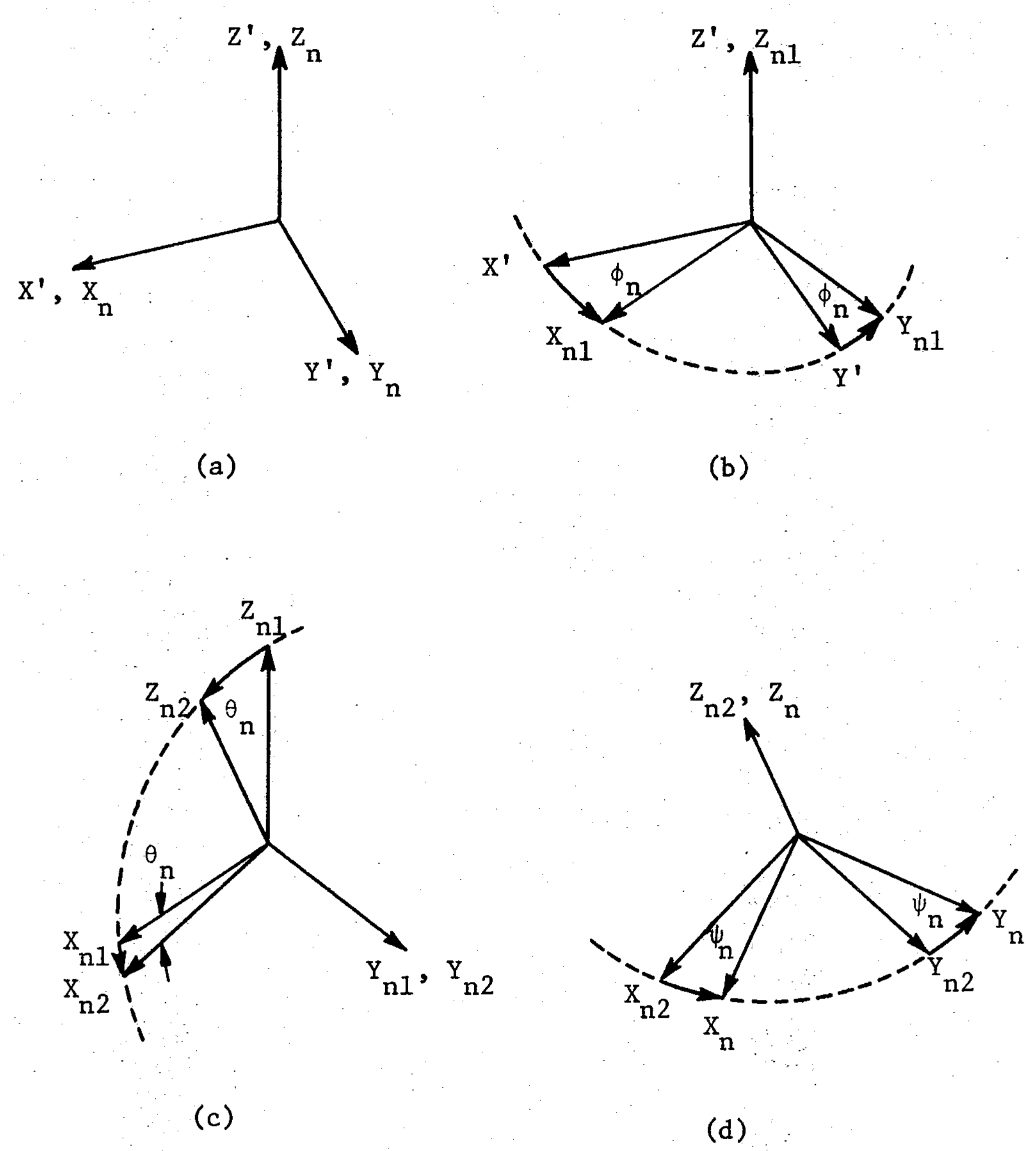

FIGURE 2.-EULERIAN ANGLES

The transformation from coordinates X_n, Y_n, Z_n to coordinates X_{n2}, Y_{n2}, Z_{n2}, Figure 2(d), is given by

$$\begin{Bmatrix} X_{n2} \\ Y_{n2} \\ Z_{n2} \end{Bmatrix} = \begin{bmatrix} \cos\psi_n & -\sin\psi_n & 0 \\ \sin\psi_n & \cos\psi_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} \quad \ldots\ldots \quad (4)$$

Substituting Equation 4 into Equation 3 yields

$$\begin{Bmatrix} X_{n1} \\ Y_{n1} \\ Z_{n1} \end{Bmatrix} = \begin{bmatrix} \cos\theta_n & 0 & \sin\theta_n \\ 0 & 1 & 0 \\ -\sin\theta_n & 0 & \cos\theta_n \end{bmatrix} \begin{bmatrix} \cos\psi_n & -\sin\psi_n & 0 \\ \sin\psi_n & \cos\psi_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} \quad . . \quad (5)$$

Substituting from Equation 5 into Equation 2 and performing the triple matrix product yield $[T^n]$ when the result is compared with Equation 1. The individual elements of matrix $[T^n]$ are

$$T^n_{11} = \cos\phi_n\cos\theta_n\cos\psi_n - \sin\phi_n\sin\psi_n \quad \ldots\ldots \quad (6a)$$

$$T^n_{12} = -\cos\phi_n\cos\theta_n\sin\psi_n - \sin\phi_n\cos\psi_n \quad \ldots\ldots \quad (6b)$$

$$T^n_{13} = \cos\phi_n\sin\theta_n \quad \ldots\ldots \quad (6c)$$

$$T^n_{21} = \sin\phi_n\cos\theta_n\cos\psi_n + \cos\phi_n\sin\psi_n \quad \ldots\ldots \quad (6d)$$

$$T^n_{22} = -\sin\phi_n\cos\theta_n\sin\psi_n + \cos\phi_n\cos\psi_n \quad \ldots\ldots \quad (6e)$$

$$T^n_{23} = \sin\phi_n\sin\theta_n \quad \ldots\ldots \quad (6f)$$

$$T^n_{31} = -\sin\theta_n\cos\psi_n \quad \ldots\ldots \quad (6g)$$

$$T^n_{32} = \sin\theta_n \sin\psi_n \quad \ldots \quad (6h)$$

$$T^n_{33} = \cos\theta_n \quad \ldots \quad (6i)$$

It can also be shown by using Figure 2 that

$$\begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix}^T \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} \quad \ldots \quad (7)$$

Furthermore, matrix multiplication shows that $[T^n][T^n]^T = [I]$, the identity matrix, such that Equation 1 is an orthogonal transformation.

At this point, the minimum number of coordinates needed to define the system is 39, namely, X'_{T1}, Y'_{T1}, and Z'_{T1} (Figure 1) plus three "Eulerian angles" for each of the 12 segments. However, realizing that the elbows and knees are really pinned joints rather than ball and socket joints, this number can be reduced.

Elbow and knee constraints. The pinned nature of the elbows and knees can be incorporated by constraining them to bend such that axes Y_7, Y_8, Y_{11}, and Y_{12} (Figure 1) are always parallel to axes Y_5, Y_6, Y_9, and Y_{10}, respectively. In the case of an arm, e.g., the left arm, this means that the orientation of segment No. 7 with respect to the space-fixed coordinate system can be uniquely specified by stating the "Eulerian angles" of segment No. 5 plus one additional angle, namely, the angular position of segment No. 7 relative to segment No. 5. The same is true for the right arm and both legs. Let m denote the segment number of an extremity (m = 7, 8, 11, or 12) and α_m represent the position angle of segment No. m relative to segment No. j, where j = m-2.

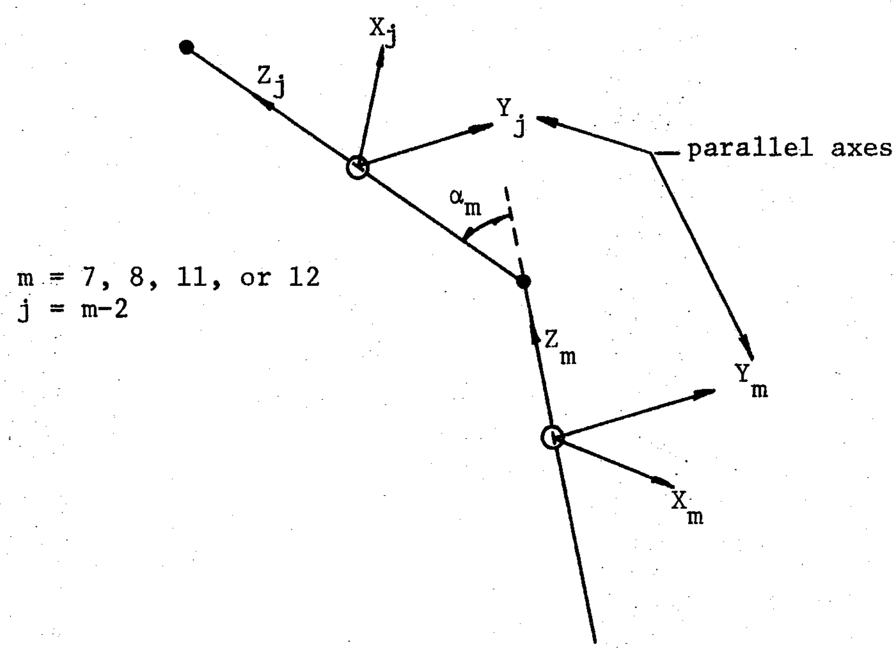

FIGURE 3.-ELBOW OR KNEE CONSTRAINT

Figure 3 shows that

$$\begin{Bmatrix} X_j \\ Y_j \\ Z_j \end{Bmatrix} = \begin{bmatrix} \cos\alpha_m & 0 & \sin\alpha_m \\ 0 & 1 & 0 \\ -\sin\alpha_m & 0 & \cos\alpha_m \end{bmatrix} \begin{Bmatrix} X_m \\ Y_m \\ Z_m \end{Bmatrix} \quad \ldots\ldots \quad (8)$$

Let j = n and substitute from Equation 8 into Equation 1, thus giving

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix} \begin{bmatrix} \cos\alpha_m & 0 & \sin\alpha_m \\ 0 & 1 & 0 \\ -\sin\alpha_m & 0 & \cos\alpha_m \end{bmatrix} \begin{Bmatrix} X_m \\ Y_m \\ Z_m \end{Bmatrix} \quad \ldots \quad (9)$$

Rewrite Equation 9 as

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} & & \\ & T^{nm} & \\ & & \end{bmatrix} \begin{Bmatrix} X_m \\ Y_m \\ Z_m \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (10)$$

Therefore the matrix $[T^{nm}]$ where m = 7, 8, 11, or 12 and n = m-2 is the transformation matrix that transforms the coordinate system fixed in segment No. m back to the space-fixed system. From Equation 9 and Equations 6a through 6i the elements of matrix $[T^{nm}]$ are

$$T^{nm}_{11} = \cos\phi_n\cos\theta_n\cos\psi_n\cos\alpha_m - \sin\phi_n\sin\psi_n\cos\alpha_m - \cos\phi_n\sin\theta_n\sin\alpha_m \quad \ldots\ldots\ldots\ldots\ldots \quad (11a)$$

$$T^{nm}_{12} = -\cos\phi_n\cos\theta_n\sin\psi_n - \sin\phi_n\cos\psi_n \quad \ldots\ldots\ldots \quad (11b)$$

$$T^{nm}_{13} = \cos\phi_n\cos\theta_n\cos\psi_n\sin\alpha_m - \sin\phi_n\sin\psi_n\sin\alpha_m + \cos\phi_n\sin\theta_n\cos\alpha_m \quad \ldots\ldots\ldots\ldots\ldots \quad (11c)$$

$$T^{nm}_{21} = \sin\phi_n\cos\theta_n\cos\psi_n\cos\alpha_m + \cos\phi_n\sin\psi_n\cos\alpha_m - \sin\phi_n\sin\theta_n\sin\alpha_m \quad \ldots\ldots\ldots\ldots\ldots \quad (11d)$$

$$T^{nm}_{22} = -\sin\phi_n\cos\theta_n\sin\psi_n + \cos\phi_n\cos\psi_n \quad \ldots\ldots\ldots \quad (11e)$$

$$T^{nm}_{23} = \sin\phi_n\cos\theta_n\cos\psi_n\sin\alpha_m + \cos\phi_n\sin\psi_n\sin\alpha_m + \sin\phi_n\sin\theta_n\cos\alpha_m \quad \ldots\ldots\ldots\ldots \quad (11f)$$

$$T^{nm}_{31} = -\sin\theta_n\cos\psi_n\cos\alpha_m - \cos\theta_n\sin\alpha_m \quad \ldots\ldots\ldots \quad (11g)$$

$$T^{nm}_{32} = \sin\theta_n\sin\psi_n \quad \ldots\ldots\ldots\ldots\ldots \quad (11h)$$

$$T^{nm}_{33} = -\sin\theta_n\cos\psi_n\sin\alpha_m + \cos\theta_n\cos\alpha_m \quad \ldots\ldots\ldots \quad (11i)$$

From Figure 3 it can be shown that

$$\begin{Bmatrix} X_m \\ Y_m \\ Z_m \end{Bmatrix} = \begin{bmatrix} & & \\ & T^{nm} & \\ & & \end{bmatrix}^T \begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} \quad \ldots\ldots\ldots\ldots \quad (12)$$

Furthermore, matrix multiplication shows that $[T^{nm}][T^{nm}]^T = [I]$, the identity matrix, such that Equation 10 is an orthogonal transformation.

Constraining the elbows and knees in the manner described has eliminated eight degrees of freedom, hence the minimum number of coordinates now needed to describe the system is 31. These generalized coordinates, denoted by q_1, q_2, ... q_{31}, are defined as follows: $q_1 = X'_{T1}$, $q_2 = Y'_{T1}$, $q_3 = Z'_{T1}$, $q_4 = \phi_1$, $q_5 = \theta_1$, $q_6 = \psi_1$, $q_7 = \phi_2$, $q_8 = \theta_2$, $q_9 = \psi_2$, $q_{10} = \phi_3$, $q_{11} = \theta_3$, $q_{12} = \psi_3$, $q_{13} = \phi_4$, $q_{14} = \theta_4$, $q_{15} = \psi_4$, $q_{16} = \phi_5$, $q_{17} = \theta_5$, $q_{18} = \psi_5$, $q_{19} = \phi_6$, $q_{20} = \theta_6$, $q_{21} = \psi_6$, $q_{22} = \alpha_7$, $q_{23} = \alpha_8$, $q_{24} = \phi_9$, $q_{25} = \theta_9$, $q_{26} = \psi_9$, $q_{27} = \phi_{10}$, $q_{28} = \theta_{10}$, $q_{29} = \psi_{10}$, $q_{30} = \alpha_{11}$, $q_{31} = \alpha_{12}$.

The Position of a Point on a Segment in Space

Using Equations 1 and 10, it is possible to express the position of any point on the articulated body with respect to the space-fixed coordinate system for any given set of values of the generalized coordinates.

To express kinetic energy of the articulated body in terms of the generalized coordinates, it is necessary to derive equations of position for the center of mass of each segment with respect to the space-fixed coordinate system. The matrix equation of position for the center of mass of segment No. 1, from Figure 1, is

$$\begin{Bmatrix} X_1' \\ Y_1' \\ Z_1' \end{Bmatrix} = \begin{Bmatrix} X_{T1}' \\ Y_{T1}' \\ Z_{T1}' \end{Bmatrix} + \begin{bmatrix} & & \\ & T^1 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\bar{\rho}_1 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots \quad (13)$$

in which (X_1', Y_1', Z_1') represents the coordinates of the center of mass of segment No. 1 and $(X_{T1}', Y_{T1}', Z_{T1}')$ represents the coordinates of the reference point* on the articulated body both with respect to the space-fixed system, whereas $(0, 0, -\bar{\rho}_1)$ represents the coordinates of the center of mass of segment No. 1 with respect to segment-fixed coordinates which are parallel to axes X_1, Y_1, Z_1 but located at the terminal end of segment No. 1.

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*Such a point is needed to account for translation of the articulated body as a unit (see Figure 1).

Likewise, the matrix equation for the center of mass of segment No. 8 is

$$\begin{Bmatrix} X'_8 \\ Y'_8 \\ Z'_8 \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ -L_s \\ 0 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^6 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -L_6 \end{Bmatrix}$$

$$+ \begin{bmatrix} & & \\ & T^{68} & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -\rho_8 \end{Bmatrix} \quad \ldots\ldots\ldots\ldots \quad (14)$$

Applying the same reasoning to all 12 segments and performing the matrix multiplication yields the following equations for the coordinates of the center of mass of each segment with respect to the space-fixed system:

$$X'_1 = X'_{T1} - \rho_1 T^1_{13} \quad \ldots\ldots\ldots\ldots \quad (15a)$$

$$Y'_1 = Y_{T1} - \rho_1 T^1_{23} \quad \ldots\ldots\ldots\ldots \quad (15b)$$

$$Z'_1 = Z'_{T1} - \rho_1 T^1_{33} \quad \ldots\ldots\ldots\ldots \quad (15c)$$

$$X'_2 = X'_{T1} + \rho_2 T^2_{13} \quad \ldots\ldots\ldots\ldots \quad (16a)$$

$$Y'_2 = Y'_{T1} + \rho_2 T^2_{23} \quad \ldots\ldots\ldots\ldots \quad (16b)$$

$$Z_2' = Z_{T1}' + \rho_2 T_{33}^2 \qquad (16c)$$

$$X_3' = X_{T1}' + L_2 T_{13}^2 + \rho_3 T_{13}^3 \qquad (17a)$$

$$Y_3' = Y_{T1}' + L_2 T_{23}^2 + \rho_3 T_{23}^3 \qquad (17b)$$

$$Z_3' = Z_{T1}' + L_2 T_{33}^2 + \rho_3 T_{33}^3 \qquad (17c)$$

$$X_4' = X_{T1}' - L_1 T_{13}^1 - \bar{\rho}_4 T_{13}^4 \qquad (18a)$$

$$Y_4' = Y_{T1}' - L_1 T_{23}^1 - \bar{\rho}_4 T_{23}^4 \qquad (18b)$$

$$Z_4' = Z_{T1}' - L_1 T_{33}^1 - \bar{\rho}_4 T_{33}^4 \qquad (18c)$$

$$X_5' = X_{T1}' + L_2 T_{13}^2 + L_s T_{12}^2 - \bar{\rho}_5 T_{13}^5 \qquad (19a)$$

$$Y_5' = Y_{T1}' + L_2 T_{23}^2 + L_s T_{22}^2 - \bar{\rho}_5 T_{23}^5 \qquad (19b)$$

$$Z_5' = Z_{T1}' + L_2 T_{33}^2 + L_s T_{32}^2 - \bar{\rho}_5 T_{33}^5 \qquad (19c)$$

$$X_6' = X_{T1}' + L_2 T_{13}^2 - L_s T_{12}^2 - \bar{\rho}_6 T_{13}^6 \qquad (20a)$$

$$Y_6' = Y_{T1}' + L_2 T_{23}^2 - L_s T_{22}^2 - \bar{\rho}_6 T_{23}^6 \qquad (20b)$$

$$Z_6' = Z_{T1}' + L_2 T_{33}^2 - L_s T_{32}^2 - \bar{\rho}_6 T_{33}^6 \qquad (20c)$$

$$X_7' = X_{T1}' + L_2 T_{13}^2 + L_s T_{12}^2 - L_5 T_{13}^5 - \bar{\rho}_7 T_{13}^{57} \qquad (21a)$$

$$Y_7' = Y_{T1}' + L_2 T_{23}^2 + L_s T_{22}^2 - L_5 T_{23}^5 - \bar{\rho}_7 T_{23}^{57} \qquad (21b)$$

$$Z_7' = Z_{T1}' + L_2 T_{33}^2 + L_s T_{32}^2 - L_5 T_{33}^5 - \bar{\rho}_7 T_{33}^{57} \qquad (21c)$$

$$X_8' = X_{T1}' + L_2T_{13}^2 - L_sT_{12}^2 - L_6T_{13}^6 - \bar{\rho}_8T_{13}^{68} \quad \ldots . \quad (22a)$$

$$Y_8' = Y_{T1}' + L_2T_{23}^2 - L_sT_{22}^2 - L_6T_{23}^6 - \bar{\rho}_8T_{23}^{68} \quad \ldots . \quad (22b)$$

$$Z_8' = Z_{T1}' + L_2T_{33}^2 - L_sT_{32}^2 - L_6T_{33}^6 - \bar{\rho}_8T_{33}^{68} \quad \ldots . \quad (22c)$$

$$X_9' = X_{T1}' - L_1T_{13}^1 - L_4T_{13}^4 + L_HT_{12}^4 - \bar{\rho}_9T_{13}^9 \quad \ldots . \quad (23a)$$

$$Y_9' = Y_{T1}' - L_1T_{23}^1 - L_4T_{23}^4 + L_HT_{22}^4 - \bar{\rho}_9T_{23}^9 \quad \ldots . \quad (23b)$$

$$Z_9' = Z_{T1}' - L_1T_{33}^1 - L_4T_{33}^4 + L_HT_{32}^4 - \bar{\rho}_9T_{33}^9 \quad \ldots . \quad (23c)$$

$$X_{10}' = X_{T1}' - L_1T_{13}^1 - L_4T_{13}^4 - L_HT_{12}^4 - \bar{\rho}_{10}T_{13}^{10} \quad \ldots \quad (24a)$$

$$Y_{10}' = Y_{T1}' - L_1T_{23}^1 - L_4T_{23}^4 - L_HT_{22}^4 - \bar{\rho}_{10}T_{23}^{10} \quad \ldots \quad (24b)$$

$$Z_{10}' = Z_{T1}' - L_1T_{33}^1 - L_4T_{33}^4 - L_HT_{32}^4 - \bar{\rho}_{10}T_{33}^{10} \quad \ldots \quad (24c)$$

$$X_{11}' = X_{T1}' - L_1T_{13}^1 - L_4T_{13}^4 + L_HT_{12}^4 - L_9T_{13}^9 - \bar{\rho}_{11}T_{13}^{911} \quad . \quad (25a)$$

$$Y_{11}' = Y_{T1}' - L_1T_{23}^1 - L_4T_{23}^4 + L_HT_{22}^4 - L_9T_{23}^9 - \bar{\rho}_{11}T_{23}^{911} \quad . \quad (25b)$$

$$Z_{11}' = Z_{T1}' - L_1T_{33}^1 - L_4T_{33}^4 + L_HT_{32}^4 - L_9T_{33}^9 - \bar{\rho}_{11}T_{33}^{911} \quad . \quad (25c)$$

$$X_{12}' = X_{T1}' - L_1T_{13}^1 - L_4T_{13}^4 - L_HT_{12}^4 - L_{10}T_{13}^{10} - \bar{\rho}_{12}T_{13}^{1012} \quad (26a)$$

$$Y_{12}' = Y_{T1}' - L_1T_{23}^1 - L_4T_{23}^4 - L_HT_{22}^4 - L_{10}T_{23}^{10} - \bar{\rho}_{12}T_{23}^{1012} \quad (26b)$$

$$Z_{12}' = Z_{T1}' - L_1T_{33}^1 - L_4T_{33}^4 - L_HT_{32}^4 - L_{10}T_{33}^{10} - \bar{\rho}_{12}T_{33}^{1012} \quad (26c)$$

Velocity squared. Taking the first derivative with respect to time* of X_1', Y_1', Z_1' through X_{12}', Y_{12}', Z_{12}' (Equations 15a through 26c) and squaring the results yields the following equations for components of velocity squared of the center of mass of each segment with respect to the space-fixed system:

$$(\dot{X}_1')^2 = (\dot{X}_{T1}')^2 - 2\bar{\rho}_1\dot{X}_{T1}'\dot{T}_{13}^1 + \bar{\rho}_1^2(\dot{T}_{13}^1)^2 \quad \ldots\ldots \quad (27a)$$

$$(\dot{Y}_1')^2 = (\dot{Y}_{T1}')^2 - 2\bar{\rho}_1\dot{X}_{T1}'\dot{T}_{23}^1 + \bar{\rho}_1^2(\dot{T}_{23}^1)^2 \quad \ldots\ldots \quad (27b)$$

$$(\dot{Z}_1')^2 = (\dot{Z}_{T1}')^2 - 2\bar{\rho}_1\dot{Z}_{T1}'\dot{T}_{33}^1 + \bar{\rho}_1^2(\dot{T}_{33}^1)^2 \quad \ldots\ldots \quad (27c)$$

.
.
.

$$\begin{aligned}(\dot{Z}_{12}')^2 = {} & (\dot{Z}_{T1}')^2 + (L_1\dot{T}_{33}^1)^2 + (L_4\dot{T}_{33}^4)^2 + (L_H\dot{T}_{32}^4)^2 \\ & + (L_{10}\dot{T}_{33}^{10})^2 + (\bar{\rho}_{12}\dot{T}_{33}^{1012})^2 - 2L_1\dot{Z}_{T1}'\dot{T}_{33}^1 - 2L_4\dot{Z}_{T1}'\dot{T}_{33}^4 \\ & - 2L_H\dot{Z}_{T1}'\dot{T}_{32}^4 - 2L_{10}\dot{Z}_{T1}'\dot{T}_{33}^{10} - 2\bar{\rho}_{12}\dot{Z}_{T1}'\dot{T}_{33}^{1012} \\ & + 2L_1L_4\dot{T}_{33}^1\dot{T}_{33}^4 + 2L_1L_H\dot{T}_{33}^1\dot{T}_{32}^4 + 2L_1L_{10}\dot{T}_{33}^1\dot{T}_{33}^{10} \\ & + 2L_1\bar{\rho}_{12}\dot{T}_{33}^1\dot{T}_{33}^{1012} + 2L_4L_H\dot{T}_{33}^4\dot{T}_{32}^4 + 2L_4L_{10}\dot{T}_{33}^4\dot{T}_{33}^{10} \\ & + 2L_4\bar{\rho}_{12}\dot{T}_{33}^4\dot{T}_{33}^{1012} + 2L_HL_{10}\dot{T}_{32}^4\dot{T}_{33}^{10} + 2L_H\bar{\rho}_{12}\dot{T}_{32}^4\dot{T}_{33}^{1012} \\ & + 2L_{10}\bar{\rho}_{12}\dot{T}_{33}^{10}\dot{T}_{33}^{1012} \quad \ldots\ldots\ldots\ldots\ldots \quad (38c)\end{aligned}$$

*A dot above a variable name will be taken to mean the first derivative of that variable with respect to time.

Angular Velocities

To express the rotational portion of the articulated body's kinetic energy in terms of the generalized coordinates, it was necessary to relate the angular velocities of a segment to the generalized velocities of the system. This was accomplished by realizing that angular velocities are vectors which obey the transformation relationships that were derived for Figures 2 and 3 (Equations 1 through 12).

The generalized angular velocities $\dot{\phi}_n$, $\dot{\theta}_n$, $\dot{\psi}_n$ are vectors parallel to axes Z', Y_{n1}, and Z_{n2}, respectively, Figure 2. Let ω_{xn}, ω_{yn}, and ω_{zn} be angular velocities about segment coordinate axes X_n, Y_n, and Z_n, respectively.* Similarly, define $\dot{\phi}_{xn}$, $\dot{\phi}_{yn}$, and $\dot{\phi}_{zn}$ as components of $\dot{\phi}_n$; $\dot{\theta}_{xn}$, $\dot{\theta}_{yn}$, and $\dot{\theta}_{zn}$ as components of $\dot{\theta}_n$; and $\dot{\psi}_{xn}$, $\dot{\psi}_{yn}$, and $\dot{\psi}_{zn}$ as components of $\dot{\psi}_n$ where all components are parallel to axes X_n, Y_n, and Z_n, respectively, such that

$$\begin{Bmatrix} \omega_{xn} \\ \omega_{yn} \\ \omega_{zn} \end{Bmatrix} = \begin{Bmatrix} \dot{\phi}_{xn} \\ \dot{\phi}_{yn} \\ \dot{\phi}_{zn} \end{Bmatrix} + \begin{Bmatrix} \dot{\theta}_{xn} \\ \dot{\theta}_{yn} \\ \dot{\theta}_{zn} \end{Bmatrix} + \begin{Bmatrix} \dot{\psi}_{xn} \\ \dot{\psi}_{yn} \\ \dot{\psi}_{zn} \end{Bmatrix} \quad . . . \quad (39)$$

*This can be interpreted to mean that ω_{xn}, ω_{yn}, and ω_{zn} are components of the angular velocity vector of segment No. n parallel to directions X_n, Y_n, and Z_n, respectively.

From Equation 7 it is evident that

$$\begin{Bmatrix} \dot{\phi}_{xn} \\ \dot{\phi}_{yn} \\ \dot{\phi}_{zn} \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ \dot{\phi}_n \end{Bmatrix} \quad \ldots\ldots\ldots \quad (40)$$

From Figure 2 it can be shown that

$$\begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} = \begin{bmatrix} \cos\psi_n\cos\theta_n & \sin\psi_n & -\cos\psi_n\sin\theta_n \\ -\sin\psi_n\cos\theta_n & \cos\psi_n & \sin\psi_n\sin\theta_n \\ \sin\theta_n & 0 & \cos\theta_n \end{bmatrix} \begin{Bmatrix} X_{n1} \\ Y_{n1} \\ Z_{n1} \end{Bmatrix} \quad \ldots \quad (41)$$

and it immediately follows that

$$\begin{Bmatrix} \dot{\theta}_{xn} \\ \dot{\theta}_{yn} \\ \dot{\theta}_{zn} \end{Bmatrix} = \begin{bmatrix} \cos\psi_n\cos\theta_n & \sin\psi_n & -\cos\psi_n\sin\theta_n \\ -\sin\psi_n\cos\theta_n & \cos\psi_n & \sin\psi_n\sin\theta_n \\ \sin\theta_n & 0 & \cos\theta_n \end{bmatrix} \begin{Bmatrix} 0 \\ \dot{\theta}_n \\ 0 \end{Bmatrix} \quad \ldots \quad (42)$$

Again, using Figure 2 along with the definitions for $\dot{\psi}_n$, $\dot{\psi}_{xn}$, $\dot{\psi}_{yn}$, and $\dot{\psi}_{zn}$ it can be shown that

$$\begin{Bmatrix} \dot{\psi}_{xn} \\ \dot{\psi}_{yn} \\ \dot{\psi}_{zn} \end{Bmatrix} = \begin{bmatrix} \cos\psi_n & \sin\psi_n & 0 \\ -\sin\psi_n & \cos\psi_n & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ \dot{\psi}_n \end{Bmatrix} \quad \ldots\ldots\ldots \quad (43)$$

Substituting Equations 40, 42, and 43 into Equation 39 yields

$$\omega_{xn} = -\dot{\phi}_n\sin\theta_n\cos\psi_n + \dot{\theta}_n\sin\psi_n \quad \ldots\ldots\ldots \quad (44a)$$

$$\omega_{yn} = \dot{\phi}_n\sin\theta_n\sin\psi_n + \dot{\theta}_n\cos\psi_n \quad \ldots\ldots\ldots \quad (44b)$$

$$\omega_{zn} = \dot{\phi}_n\cos\theta_n + \dot{\psi}_n \quad \ldots\ldots\ldots \quad (44c)$$

For segment No. m (m = 7, 8, 11, or 12), i.e., an arm or leg extremity, define $\dot{\phi}_n$, $\dot{\theta}_n$, and $\dot{\psi}_n$ (n = m-2) as before and introduce $\dot{\alpha}_m$ as the generalized velocity vector that is parallel to both Y_n and Y_m (see Figure 3). In addition, let $\bar{\omega}_{xm}$, $\bar{\omega}_{ym}$, and $\bar{\omega}_{zm}$ be angular velocities about segment coordinate axes X_m, Y_m, and Z_m, respectively.

Using Figure 3 it can be shown that

$$\begin{Bmatrix} \bar{\omega}_{xm} \\ \bar{\omega}_{ym} \\ \bar{\omega}_{zm} \end{Bmatrix} = \begin{bmatrix} \cos\alpha_m & 0 & -\sin\alpha_m \\ 0 & 1 & 0 \\ \sin\alpha_m & 0 & \cos\alpha_m \end{bmatrix} \begin{Bmatrix} \omega_{xn} \\ \omega_{yn} + \dot{\alpha}_m \\ \omega_{zn} \end{Bmatrix} \qquad (45)$$

Substituting from Equations 44a, 44b, and 44c into Equation 45 and performing the matrix multiplication yields

$$\bar{\omega}_{xm} = \dot{\phi}_n(\sin\theta_n\cos\psi_n\cos\alpha_m + \cos\theta_n\sin\alpha_m) + \dot{\theta}_n\sin\psi_n\cos\alpha_m - \dot{\psi}_n\sin\alpha_m \qquad (46a)$$

$$\bar{\omega}_{ym} = \dot{\phi}_n\sin\theta_n\sin\psi_n + \dot{\theta}_n\cos\psi_n + \dot{\alpha}_m \qquad (46b)$$

$$\bar{\omega}_{zm} = \dot{\phi}_n(\cos\theta_n\cos\alpha_m - \sin\theta_n\cos\psi_n\sin\alpha_m) + \dot{\theta}_n\sin\psi_n\sin\alpha_m + \dot{\psi}_n\cos\alpha_m \qquad (46c)$$

Lagrange's Equations

The equations of motion were derived according to the set of differential equations known as Lagrange's equations for nonconservative systems (4, 6, 19). These may be written as

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j} + \frac{\partial V}{\partial q_j} = Q_j \quad \text{. . . . . (47)}$$

in which t = time, U = kinetic energy of the system, V = potential energy of the system, q_j = generalized coordinates, $\dot{q}_j$ = generalized velocities, Q_j = generalized forces acting on the system which are not necessarily derivable from a potential function, and j = 1, 2, ... 31, for this particular problem.

It was convenient to define

$$Q_j = (Q_f)_j + R_j \quad \text{. . . . . (48)}$$

in which $(Q_f)_j$ = generalized forces resulting from externally applied loads (contact forces)* and R_j = generalized forces due to frictional resistance in the joints (viscous damping) to simulate muscle tone.*

In addition, the potential energy V is of two types, namely V_p, potential energy of position, and V_s, potential energy due to restoring springs* located in each of the two back joints (spinal elasticity), such that

$$V = V_p + V_s \quad \text{. (49)}$$

therefore

$$\frac{\partial V}{\partial q_j} = \frac{\partial V_p}{\partial q_j} + \frac{\partial V_s}{\partial q_j} \quad \text{. . . . . (50)}$$

Define $(F_p)_j$ and $(F_s)_j$ as the generalized forces due to potential energy of position and strain energy potential, respectively,

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*To be discussed in detail later under separate heading.

hence

$$(F_p)_j = -\frac{\partial V_p}{\partial q_j} \quad \text{. . . . . (51a)}$$

$$(F_s)_j = -\frac{\partial V_s}{\partial q_j} \quad \text{. . . . . (51b)}$$

Substituting from Equations 48, 50, 51a and 51b into Equation 47 gives

$$\frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) - \frac{\partial U}{\partial q_j} = (F_p)_j + (F_s)_j + (Q_f)_j + R_j \quad \text{. . . (52)}$$

For the articulated body under consideration, Equation 52 represents a set of 31 differential equations which can be categorized as being ordinary, of second order, simultaneous, and nonlinear.

The fact that the differential equations are nonlinear immediately dictates a solution by numerical integration, and the particular approach used was the "Runge-Kutta" method because it is inherently stable. As will be discussed in a later section, this method lends itself well to matrix manipulation, therefore, it was for convenience of solution that the 31 differential equations represented by Equation 52 were rewritten in matrix form as

$$[D]\{\ddot{q}\} = \{E\} + \{F_p\} + \{F_s\} + \{Q_f\} + \{R\} \quad \text{. . . . . (53)}$$

in which matrices [D] and {E} are simply a rearrangement of the terms stemming from the left side of Equation 52, i.e., [D] is the matrix of coefficients for the generalized accelerations, $\{\ddot{q}\}$; and {E} is a column vector containing all remaining terms with signs reversed.

The mathematical formulation of the vehicle occupant is complete and the remainder of this section will be devoted to evaluating the matrix components of Equation 53.

Contribution of Kinetic Energy to Equations of Motion

The kinetic energy of the articulated body is given by

$$U = \frac{1}{2} \sum_{i=1}^{12} M_i[(\dot{X}_i')^2 + (\dot{Y}_i')^2 + (\dot{Z}_i')^2]$$

$$+ \frac{1}{2} \sum_{l} [I_{xl}(\omega_{xl})^2 + I_{yl}(\omega_{yl})^2 + I_{zl}(\omega_{zl})^2]$$

$$+ \frac{1}{2} \sum_{k} [I_{xk}(\bar{\omega}_{xk})^2 + I_{yk}(\bar{\omega}_{yk})^2 + I_{zk}(\bar{\omega}_{zk})^2] \quad . . . (54)$$

in which l = 1, 2, 3, 4, 5, 6, 9, and 10; k = 7, 8, 11, and 12; M_i is the mass of segment No. i; and I_{xn}, I_{yn}, and I_{zn} ($n = l$ or k) are the mass moments of inertia about axes X_n, Y_n, and Z_n, respectively* of segment No. n. Equations 27a through 38c define $(\dot{X}_i')^2$, $(\dot{Y}_i')^2$, and $(\dot{Z}_i')^2$; whereas ω_{xl}, ω_{yl}, and ω_{zl} are given by Equations 44a through 44c and $\bar{\omega}_{xk}$, $\bar{\omega}_{yk}$; and $\bar{\omega}_{zk}$ are expressed by Equations 46a through 46c.

The kinetic energy contributes matrices [D] and {E} to the matrix equation of motion, Equation 53, in the manner prescribed by the left

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*Note that no products of inertia (I_{xy}, etc.) appear in Equation 54 which means that axes X_n, Y_n, and Z_n are assumed to be a principal set.

side of Equation 52. Computing the elements of these matrices was a very tedious operation and thus is omitted from this presentation; however, the final result is given in "APPENDIX III.-EQUATIONS OF MOTION."

Contribution of Potential Energy to Equations of Motion

As previously stated, the potential energy is of two types, namely V_p, potential energy of position, and V_s, potential energy resulting from "spinal elasticity." For clarity, these are discussed separately.

Potential Energy of Position

For the case where the vehicle occupant is in a gravitational field, the direction of the space-fixed axis, Z', was defined as being parallel to the direction of gravitational attraction with opposite sense, e.g., pointing vertically upward when on the surface of the earth. Furthermore, Z'_o was defined as the arbitrary position of a datum plane (parallel to X'-Y' plane), such that

$$V_p = \sum_{i=1}^{12} M_i g(Z'_i - Z'_o) \quad \ldots\ldots\ldots\ldots\ldots \quad (55)$$

in which M_i is the mass of segment No. i, g is acceleration due to gravity, and Z'_i is given by Equations 15c through 26c.

The potential energy of position provided the column vector $\{F_p\}$ to the matrix equation of motion, Equation 53, in the manner dictated by Equation 51a such that the j^{th} element of $\{F_p\}$ is

$$(F_p)_j = -g \sum_{i=1}^{12} M_i \frac{\partial Z_i}{\partial q_j} \qquad \ldots\ldots\ldots\ldots\ldots \quad (56)$$

The result of Equation 56 is given in "APPENDIX III.-EQUATIONS OF MOTION."

Potential Energy due to Spinal Elasticity

To simulate in the simplest manner, the ability of the human spine to recover its initial configuration after bending, it was decided to use rotational elastic springs in the two back joints shown in Figure 1. These springs, shown in Figure 4, are defined as lying in the plane formed by the center lines of the two adjoining back segments, regardless of their spatial orientation. Also, the springs are adjustable such that their undeformed positions are assumed to be compatible with the initial configuration of the spine, i.e., the spinal configuration oscillates (in space) about its initial position. Defining the undeformed positions of the two springs as $(\pi - \beta_{10})$ and $(\pi - \beta_{20})$, respectively, the potential energy stored in rotating to positions $(\pi - \beta_1)$ and $(\pi - \beta_2)$, respectively, can be expressed as

$$V_s = \frac{1}{2} K_1(\beta_1 - \beta_{10})^2 + \frac{1}{2} K_2(\beta_2 - \beta_{20})^2 \qquad \ldots\ldots \quad (57)$$

in which K_1 and K_2 are spring stiffnesses having units of torque per angular measure.

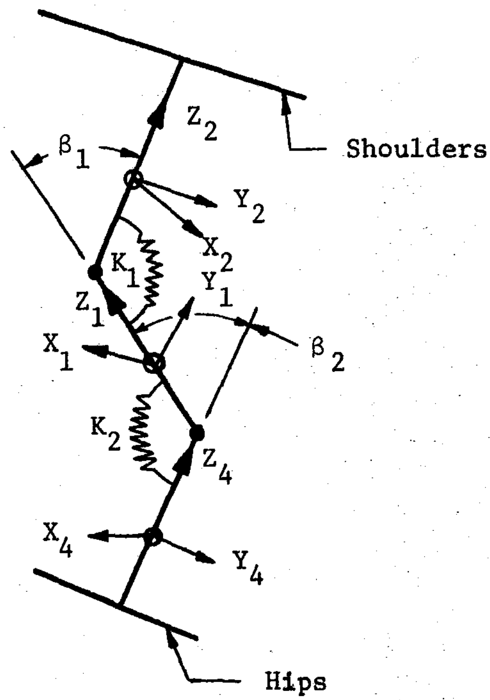

FIGURE 4.-SPRINGS IN BACK JOINTS

To express β_1 and β_2 in terms of the generalized coordinates (q_j's), define $\bar{i}_n$, $\bar{j}_n$, and $\bar{k}_n$ as a triad of unit vectors which coincides with the segment-fixed coordinate axes, X_n, Y_n, and Z_n, of segment No. n. Therefore, from Figure 4 and the definition of the scalar product (dot product) of two vectors, it follows that

$$\cos\beta_1 = \bar{k}_1 \cdot \bar{k}_2 \quad \text{. . . . . (58a)}$$

$$\cos\beta_2 = \bar{k}_1 \cdot \bar{k}_4 \quad \text{. . . . . (58b)}$$

since $\bar{k}_n$ coincides with the segment's center line.

Before the "dot products" in Equations 58a and 58b can be performed, it is necessary that $\bar{k}_1$, $\bar{k}_2$, and $\bar{k}_4$ be resolved to a common coordinate system. Choosing the space-fixed system, shown in Figure 2, as that common reference, and defining $\bar{I}$, $\bar{J}$, and $\bar{K}$ as a triad of unit vectors coinciding with space-fixed axes X', Y', and Z', respectively, it follows from Equation 7 that

$$\begin{Bmatrix} \bar{i}_n \\ \bar{j}_n \\ \bar{k}_n \end{Bmatrix} = \begin{bmatrix} & & \\ & T^n & \\ & & \end{bmatrix}^T \begin{Bmatrix} \bar{I} \\ \bar{J} \\ \bar{K} \end{Bmatrix} \qquad \text{. . . . . (59)}$$

from which

$$\bar{k}_n = T^n_{13}\bar{I} + T^n_{23}\bar{J} + T^n_{33}\bar{K} \qquad \text{. . . . . (60)}$$

Substituting this result into Equations 58a and 58b yields

$$\beta_1 = \cos^{-1}(T^1_{13}T^2_{13} + T^1_{23}T^2_{23} + T^1_{33}T^2_{33}) \qquad \text{. . . . . . . (61a)}$$

$$\beta_2 = \cos^{-1}(T^1_{13}T^4_{13} + T^1_{23}T^4_{23} + T^1_{33}T^4_{33}) \qquad \text{. . . . . . . (61b)}$$

where T^n_{ij} is a function of the generalized coordinates as given by Equations 6a through 6i.

Equation 57, the potential energy due to "spinal elasticity", contributes column vector, $\{F_s\}$, to the matrix equation of motion, Equation 53, in the manner dictated by Equation 51b such that the j^{th} element of $\{F_s\}$ is

$$(F_s)_j = -K_1(\beta_1 - \beta_{10}) \frac{\partial\beta_1}{\partial q_j} - K_2(\beta_2 - \beta_{20}) \frac{\partial\beta_2}{\partial q_j} \qquad \text{. . . (62)}$$

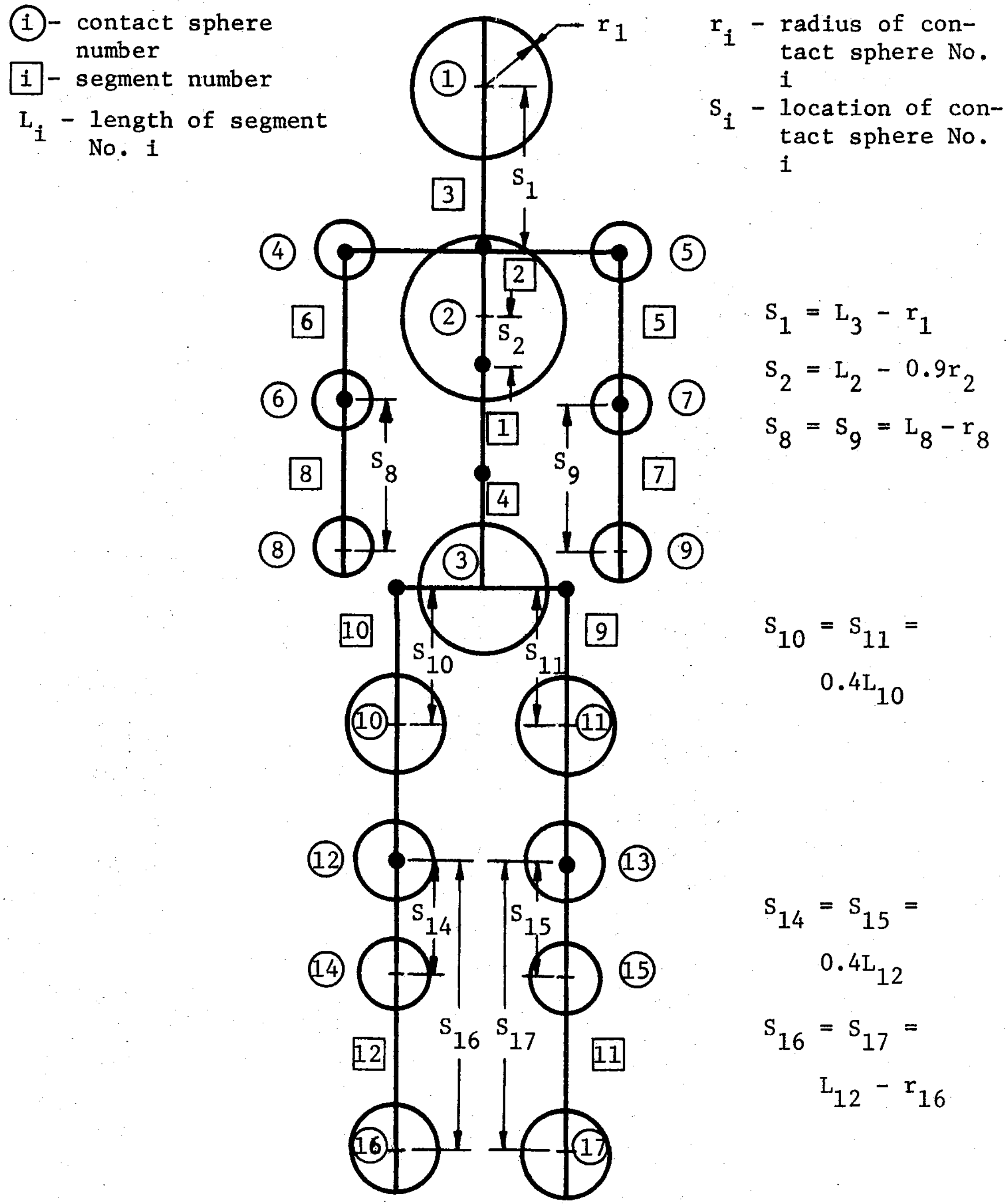

FIGURE 5.-LOCATIONS OF CONTACT SPHERES

contact spheres chosen for the vehicle occupant and their respective locations on the center lines of the body segments. The location of contact sphere No. i is denoted by S_i and its radius by r_i.

Force on a Contact Sphere

Whenever a contact sphere encounters a boundary of the vehicle interior, a contact force results. Assuming that the magnitude* of this force can be computed, there still remains the task of resolving it into components which correspond to the generalized coordinates (q_j's) for use in the equations of motion. These contact force components are the generalized forces which compose the column vector, $\{Q_f\}$, in Equation 53, and are evaluated using the concept of virtual work (4,6,19).

Consider, as in Figure 6, a resultant contact force, F_i, acting on each contact sphere No. i (i = 1,2,...17) at point A defined by (X'_{Ai}, Y'_{Ai}, Z'_{Ai}) in the space-fixed coordinate system, and let each force move through a virtual displacement having components $\delta X'_{Ai}$, $\delta Y'_{Ai}$, and $\delta Z'_{Ai}$; then the total virtual work done is

$$\delta W = \sum_{i=1}^{17} [F_{xi}\, \delta X'_{Ai} + F_{yi} \delta Y'_{Ai} + F_{zi} \delta Z'_{Ai}] \qquad \ldots\ldots \quad (63)$$

in which F_{xi}, F_{yi}, and F_{zi} are components of force F_i with respect to the space-fixed coordinate system.

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*To be discussed in the section entitled "PASSENGER-VEHICLE INTERACTION."

Assuming that F_i can be resolved into equivalent generalized forces, $(Q_f)_j$, then the virtual work is also

$$\delta W = \sum_{j=1}^{31} (Q_f)_j \delta q_j \quad \text{. . . . . (64)}$$

in which δq_j is a generalized virtual displacement.

Define $\bar{r}_i$ as the radius vector extending from the center of contact sphere No. i to the point on the sphere at which F_i acts, i.e., point A in Figure 6.

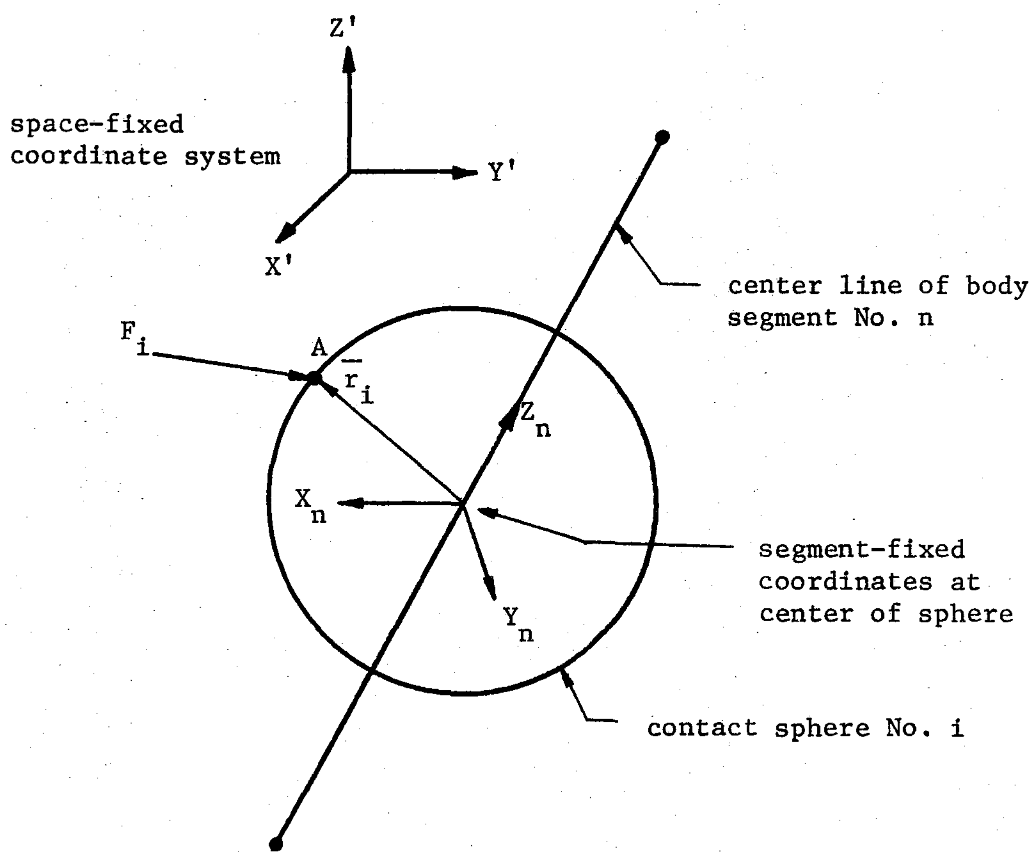


FIGURE 6.-FORCE ON A CONTACT SPHERE

The components of $\bar{r}_i$ are defined as r_{xi}, r_{yi}, and r_{zi} with respect to X_n, Y_n, and Z_n, respectively; i.e., the segment-fixed coordinates in Figure 6. With this definition of F_i, it is now possible to derive expressions for X'_{Ai}, Y'_{Ai}, Z'_{Ai} in terms of the generalized coordinates for use in Equation 63. From Figures 1, 5, and 6 and Equations 1 and 10,

$$\begin{Bmatrix} X'_{A1} \\ Y'_{A1} \\ Z'_{A1} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ L_2 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^3 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x1} \\ r_{y1} \\ S_1+r_{z1} \end{Bmatrix} \quad . \; . \; (65a)$$

$$\begin{Bmatrix} X'_{A2} \\ Y'_{A2} \\ Z'_{A2} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x2} \\ r_{y2} \\ S_2+r_{z2} \end{Bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; (65b)$$

$$\begin{Bmatrix} X'_{A3} \\ Y'_{A3} \\ Z'_{A3} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^1 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -L_1 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^4 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x3} \\ r_{y3} \\ r_{z3}-L_4 \end{Bmatrix} \quad . \; . \; (65c)$$

$$\begin{Bmatrix} X'_{A4} \\ Y'_{A4} \\ Z'_{A4} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x4} \\ -L_s+r_{y4} \\ L_2+r_{z4} \end{Bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; (65d)$$

$$\begin{Bmatrix} X'_{A5} \\ Y'_{A5} \\ Z'_{A5} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x5} \\ L_s+r_{y5} \\ L_2+r_{z5} \end{Bmatrix} \quad . \; . \; . \; . \; . \; . \; . \; . \; (65e)$$

$$\begin{Bmatrix} X'_{A6} \\ Y'_{A6} \\ Z'_{A6} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ -L_s \\ L_2 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^6 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x6} \\ r_{y6} \\ r_{z6}-L_6 \end{Bmatrix} \quad . \ . \ (65f)$$

$$\begin{Bmatrix} X'_{A7} \\ Y'_{A7} \\ Z'_{A7} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ L_s \\ L_2 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^5 & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x7} \\ r_{y7} \\ r_{z7}-L_5 \end{Bmatrix} \quad . \ (65g)$$

$$\vdots$$

$$\begin{Bmatrix} X'_{A17} \\ Y'_{A17} \\ Z'_{A17} \end{Bmatrix} = \begin{Bmatrix} X'_{T1} \\ Y'_{T1} \\ Z'_{T1} \end{Bmatrix} + \begin{bmatrix} & & \\ & T^2 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -L_1 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^4 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ L_H \\ -L_4 \end{Bmatrix}$$

$$+ \begin{bmatrix} & & \\ & T^9 & \\ & & \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -L_9 \end{Bmatrix} + \begin{bmatrix} & & \\ & T^{911} & \\ & & \end{bmatrix} \begin{Bmatrix} r_{x17} \\ r_{y17} \\ r_{z17}-S_{17} \end{Bmatrix} \quad . \ . \ . \ (65q)$$

where the elements of $[T^n]$ and $[T^{nm}]$ are given by Equations 6a through 6i and Equations 11a through 11i, respectively.

By performing the matrix multiplication indicated by Equations 65a through 65q*, expressions for X'_{Ai}, Y'_{Ai}, and Z'_{Ai} $(i = 1,2...17)$ are obtained as a function of the generalized coordinates, q_j $(j = 1,2...31)$, such that $\delta X'_{Ai}$, $\delta Y'_{Ai}$, and $\delta Z'_{Ai}$ now become

---

*Equations 65h through 65p are not given, to avoid unnecessary illustration.

$$\delta X'_{Ai} = \sum_{j=1}^{31} \frac{\partial X'_{Ai}}{\partial q_j} \delta q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots (66a)$$

$$\delta Y'_{Ai} = \sum_{j=1}^{31} \frac{\partial Y'_{Ai}}{\partial q_j} \delta q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots (66b)$$

$$\delta Z'_{Ai} = \sum_{j=1}^{31} \frac{\partial Z'_{Ai}}{\partial q_j} \delta q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots (66c)$$

Substituting from Equations 66a through 66c into Equation 63 and rearranging the order of summation yields

$$\delta W = \sum_{j=1}^{31} \sum_{i=1}^{17} \left(F_{xi} \frac{\partial X'_{Ai}}{\partial q_j} + F_{yi} \frac{\partial Y'_{Ai}}{\partial q_j} + F_{zi} \frac{\partial Z'_{Ai}}{\partial q_j}\right) \delta q_j \quad \ldots (67)$$

By comparison of Equation 64 with Equation 67, it is obvious that

$$(Q_f)_j = \sum_{i=1}^{17} \left(F_{xi} \frac{\partial X'_{Ai}}{\partial q_j} + F_{yi} \frac{\partial Y'_{Ai}}{\partial q_j} + F_{zi} \frac{\partial Z'_{Ai}}{\partial q_j}\right) \quad \ldots (68)$$

which defines the terms of column vector $\{Q_f\}$, the result of which is given in "APPENDIX III.-EQUATIONS OF MOTION."

Generalized Forces due to Viscous Damping in All Joints

The human body's muscular network is principally a source of kinetic energy for body motion, however, it can also act as a dissipator of rotational kinetic energy that is derived from an external source. The latter property of the muscular system could have a significant effect on body kinematics in a panic situation such as a vehicle collision, since the tensing of muscles is a rather natural reaction. In an attempt to account for this phenomenon in a simple manner, rotational resistance (in the form of viscous damping) was introduced in every joint of the articulated body.

The Viscous Damper

Viscous damping is, by definition (6), a velocity dependent resistance (force or torque) to motion. Figure 7 shows the viscous damper for body joint No. i (i = 1,2...11) which joins body segment No. k to segment No. l. Each body joint contains a similar damper in that it may differ from all others only in the damping constant, J_i, which is a measure of the mechanism's resistance capability.

Note: The plane formed by the center lines of segments k and l contains the viscous damper and the angle β_i.

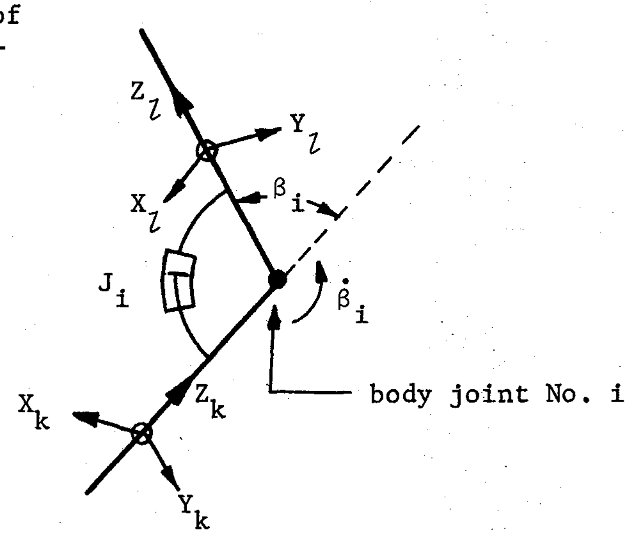


FIGURE 7.-TYPICAL VISCOUS DAMPER

The resisting torque, $(R_T)_i$, in joint No. i, Figure 7, resulting from the damping constant, J_i, and the angular velocity, $\dot{\beta}_i$, is defined as

$$(R_T)_i = J_i\dot{\beta}_i \qquad (69)$$

Virtual Work of the Damping Torque

The total virtual work done by the instantaneous damping torques, $(R_T)_i$, acting during the virtual angular displacement, $\delta\beta_i$, of each joint No. i, can be expressed as

$$\delta W = -\sum_{i=1}^{11} J_i\dot{\beta}_i\delta\beta_i \qquad (70)$$

where the numbering of the 11 joints is given in Figure 8.

Employing the same reasoning as was used for the two back joints under the heading of "Potential Energy due to Spinal Elasticity," it can be shown that

$$\beta_i = \cos^{-1}(T^k_{13}T^l_{13} + T^k_{23}T^l_{23} + T^k_{33}T^l_{33}) \qquad (71)$$

in which i = 1,2,3,4,7,8, and 9; and k and l represent adjoining segment numbers as shown in Figure 8.

For the arm and leg extremities

$$\beta_i = \alpha_m \qquad (72)$$

in which i = 5,6,10, and 11; and m = 7,8,12, and 11, respectively, as indicated by Figure 3.

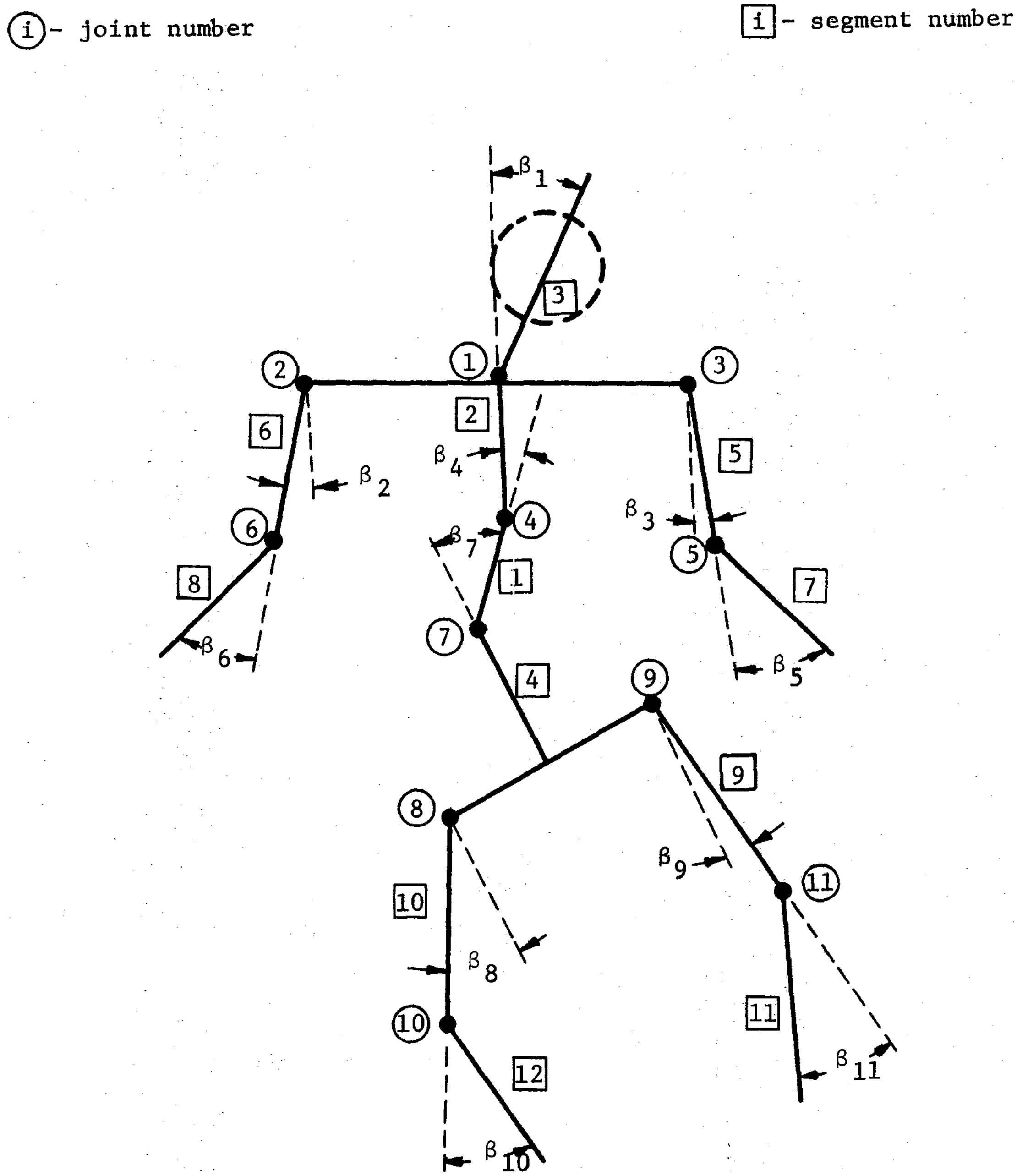


FIGURE 8.-NUMBERING OF JOINTS FOR DAMPING

Equations 71 and 72 show that β_i is a function of the generalized coordinates, q_j, therefore

$$\delta\beta_i = \sum_{j=1}^{31} \frac{\partial\beta_i}{\partial q_j} \delta q_j \qquad (73)$$

Substituting from Equation 73 into Equation 70 and interchanging the order of summation yield

$$\delta W = -\sum_{j=1}^{31} \sum_{i=1}^{11} J_i \dot{\beta}_i \frac{\partial\beta_i}{\partial q_j} \delta q_j \qquad (74)$$

Define R_j as the generalized damping force (torque) which corresponds to generalized coordinate, q_j, therefore the total virtual work done by all R_j's, each acting through a virtual displacement δq_j, is

$$\delta W = \sum_{j=1}^{31} R_j \, \delta q_j \qquad (75)$$

By comparison of Equations 74 and 75, it is obvious that

$$R_j = -\sum_{i=1}^{11} J_i \dot{\beta}_i \frac{\partial\beta_i}{\partial q_j} \qquad (76)$$

Incorporating the knee and elbow constraints described in Figure 3, Equation 76 reduces to

$$R_j = -\sum_i J_i \dot{\beta}_i \frac{\partial \beta_i}{\partial q_j} - J_5 \dot{\alpha}_7 \frac{\partial \alpha_7}{\partial q_j} - J_6 \dot{\alpha}_8 \frac{\partial \alpha_8}{\partial q_j}$$

$$- J_{10} \dot{\alpha}_{12} \frac{\partial \alpha_{12}}{\partial q_j} - J_{11} \dot{\alpha}_{11} \frac{\partial \alpha_{11}}{\partial q_j} \quad \ldots\ldots\ldots\ldots \quad (77)$$

in which i = 1,2,3,4,7,8, and 9 and $\dot{\beta}_i$ is obtained by differentiating Equation 71 with respect to time.

Equation 77 defines the elements of column vector, {R}, in the matrix equation of motion, Equation 53, and the final result is in "APPENDIX III.-EQUATIONS OF MOTION."

Joint Stops

To prevent unrealistic angular travel of a body segment relative to its adjoining segment, stops were introduced in the joints.

These "joint stops" are achieved by simply increasing the damping coefficient of joint No. i, J_i, to some large number when β_i approaches an estimated limiting value depending upon whether $\dot{\beta}_i$ is positive or negative.

Figure 9 shows the J_i versus β_i relationship for a typical joint (all body joints except knees and elbows) in which J_i increases from a "normal operation value", $(J_N)_i$, at $(\beta_N)_i$ to a "stop value" of $(J_s)_i$ at $(\beta_s)_i$ if $\dot{\beta}_i$ is positive; where $[(\beta_s)_i - (\beta_N)_i]$ is simply a transition region (approximately five degrees) to prevent abrupt motion (14).

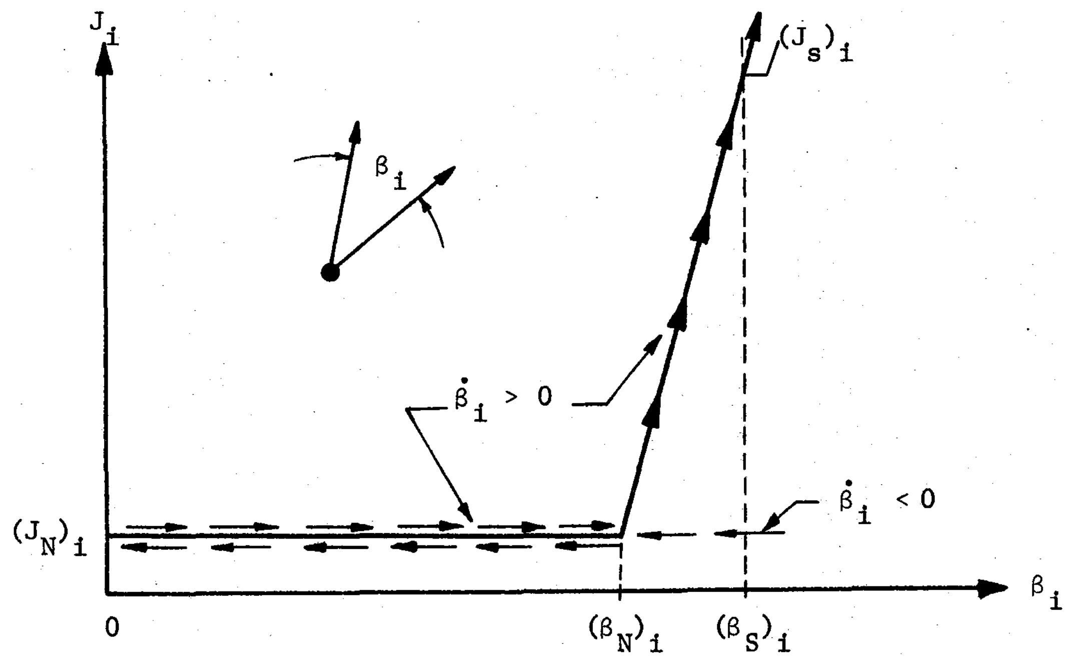


FIGURE 9-RELATIONSHIP FOR BODY JOINT STOPS

Figures 10 and 11 show the J_m versus α_m relationship for the elbows and knees, respectively. It should be noted that two "joint stops" are needed for these since α_m is a generalized coordinate capable of assuming both negative and positive values as opposed to β_i being always positive (dot product of two vectors), hence the subscripts 1 and 2.

This concludes the derivation of the equations of motion for the vehicle occupant.

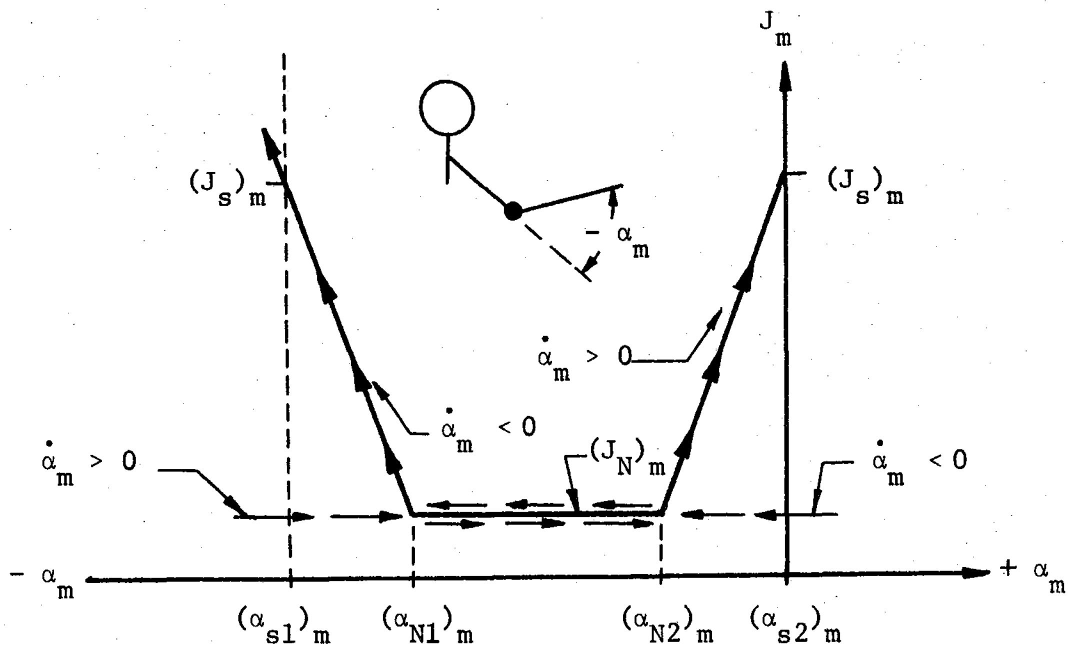


FIGURE 10.-RELATIONSHIP FOR ELBOW STOPS

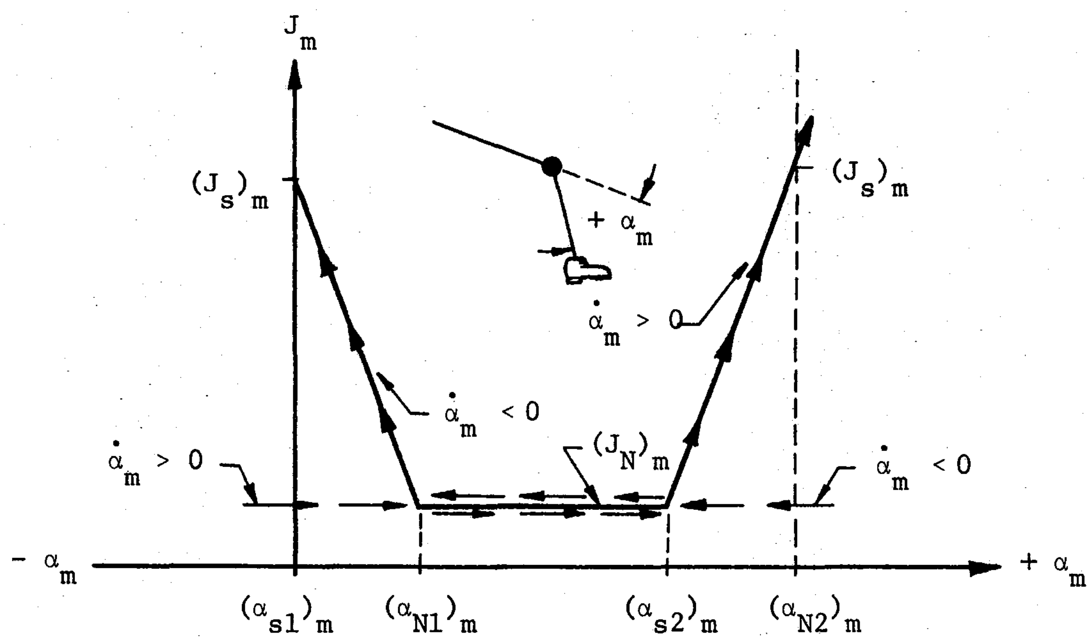


FIGURE 11.-RELATIONSHIP FOR KNEE STOPS

PASSENGER-VEHICLE INTERACTION

The idea of passenger-vehicle interaction is analogous to that of placing an object in a glass box, fastening the lid, then observing the motion of the object while the box is shaken. One could conclude from such an experiment that the motion of the object is totally dependent upon the forces afforded to it by the walls of the box (with the exception of gravity) and that these forces are dependent upon the path of the box in space as a function of time. Likewise, before contact forces on the passenger can be computed, it is necessary to define the path of the vehicle.

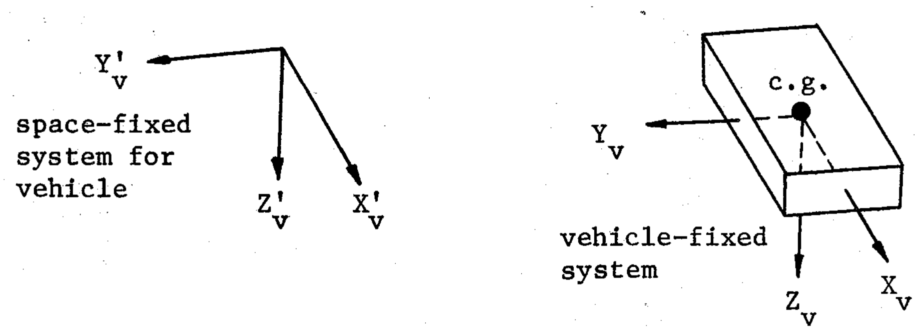
The Path of the Vehicle in Space

A tabular record of the vehicle's path in space as a function of time is sufficient for purposes of computing contact forces. This record can be fed to the computer program and, if necessary, interpolation between time stations can be performed. However, it is necessary that this tabular information conform to a predetermined standard for specifying the orientation of a body in space. There are many such standards available, one of which has already been discussed, i.e., Equation 1 and Figure 2; hence it would seem to be the logical choice. Nevertheless, a different standard was chosen for a very good reason.

There is a computer program available (9,11,15,20) for the solution of a mathematical model of an automobile capable of three-dimensional motion while interacting with the roadway or certain roadside features, obstacles, etc. Since the overall aim for the project, of which this passenger model is only a part, is to study the interaction between vehicle, passenger, and roadway, it is only fitting that input to the passenger program be made compatible with output from the vehicle program. Achieving this compatibility involves the introduction of two additional coordinate systems, namely, the vehicle-fixed system and the space-fixed system for the vehicle; plus a different system of "Eulerian angles," namely, the "aeronautical standard."

Coordinate Systems for the Vehicle

Let X_v', Y_v', and Z_v' represent a space-fixed cartesian coordinate system to which is referenced a vehicle-fixed coordinate system, denoted by X_v, Y_v, and Z_v, as shown in Figure 12.



$(X_{vc}', Y_{vc}', Z_{vc}')$ - coordinates of c.g. of vehicle with respect to space-fixed system

FIGURE 12.-VEHICLE COORDINATE SYSTEMS

The vehicle-fixed system transforms back to its space-fixed system as

$$\begin{Bmatrix} X_v' \\ Y_v' \\ Z_v' \end{Bmatrix} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{Bmatrix} X_v \\ Y_v \\ Z_v \end{Bmatrix} \qquad (78)$$

The elements of [A] are derived using the same procedure that was employed in deriving [T] (see "Euler Transformation and Generalized Coordinates") except that the "Eulerian angles" for this case are slightly different. For the "aeronautical standard" the three successive rotations are as follows:

1. Rotate about the Z_v' axis an amount ψ_c to first intermediate position X_{v1}, Y_{v1}, Z_{v1}. This angle is called yaw.
2. Rotate about the Y_{v1} axis an amount θ_c to second intermediate position X_{v2}, Y_{v2}, Z_{v2}. This angle is called pitch.
3. Rotate about the X_{v2} axis an amount ϕ_c to final position X_v, Y_v, Z_v. This angle is called roll.

Using the above definition, the elements of [A] are found to be

$$A_{11} = \cos\psi_c \cos\theta_c \qquad (79a)$$

$$A_{12} = -\sin\psi_c \cos\phi_c + \cos\psi_c \sin\theta_c \sin\phi_c \qquad (79b)$$

$$A_{13} = \sin\psi_c \sin\phi_c + \cos\psi_c \sin\theta_c \cos\phi_c \qquad (79c)$$

$$A_{21} = \sin\psi_c \cos\theta_c \qquad (79d)$$

$$A_{22} = \cos\psi_c \cos\phi_c + \sin\psi_c \sin\theta_c \sin\phi_c \qquad (79e)$$

$$A_{23} = -\cos\psi_c \sin\phi_c + \sin\psi_c \sin\theta_c \cos\phi_c \quad \ldots\ldots\ldots \quad (79f)$$

$$A_{31} = -\sin\theta_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (79g)$$

$$A_{32} = \cos\theta_c \sin\phi_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (79h)$$

$$A_{33} = \cos\theta_c \cos\phi_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (79i)$$

Consider a point i on or in the vehicle whose coordinates are (X_{vi}, Y_{vi}, Z_{vi}) in the vehicle-fixed system. With respect to the space-fixed system for the vehicle, the coordinates of point i become

$$\begin{Bmatrix} X'_{vi} \\ Y'_{vi} \\ Z'_{vi} \end{Bmatrix} = \begin{Bmatrix} X'_{vc} \\ Y'_{vc} \\ Z'_{vc} \end{Bmatrix} + \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{Bmatrix} X_{vi} \\ Y_{vi} \\ Z_{vi} \end{Bmatrix} \quad \ldots\ldots \quad (80)$$

in which $(X'_{vc}, Y'_{vc}, Z'_{vc})$ represents the coordinates of the center of gravity (c.g.) of the vehicle with respect to the space-fixed system for the vehicle, as shown in Figure 12.

To simplify the inconvenience of having two independent space-fixed coordinate systems, i.e., one for the articulated body, Figure 1, and another for the vehicle, Figure 12, the following conditions are specified:

1. Axes X' and X'_v are parallel with the same sense which simplifies the most common initial condition of both the vehicle and passenger heading in the same direction.
2. Axes Z' and Z'_v are parallel but of opposite sense which satisfies the fact that acceleration due to gravity should

have the same direction for both passenger and vehicle.

3. From 1 and 2 above it follows that Y' and Y'_v are parallel but of opposite sense if both systems are to be right handed.

4. For additional simplification, the origins of both systems are located at the same point in space.

From the above four statements it follows that

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} X'_v \\ Y'_v \\ Z'_v \end{Bmatrix} \quad \ldots\ldots\ldots\ldots \quad (81)$$

From Equations 80 and 81 it follows that

$$\begin{Bmatrix} X'_i \\ Y'_i \\ Z'_i \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} X'_{vc} \\ Y'_{vc} \\ Z'_{vc} \end{Bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \\ A \\ \\ \end{bmatrix} \begin{Bmatrix} X_{vi} \\ Y_{vi} \\ Z_{vi} \end{Bmatrix} \quad (82)$$

in which (X'_i, Y'_i, Z'_i) represents the coordinates of a point i, located in or on the vehicle, with respect to the space-fixed coordinate system for the passenger.

Using the fact* that $[A]^T = [A]^{-1}$, Equation 82 may be rewritten as

$$\begin{Bmatrix} X_{vi} \\ Y_{vi} \\ Z_{vi} \end{Bmatrix} = \begin{bmatrix} \\ A \\ \\ \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} X'_i \\ Y'_i \\ Z'_i \end{Bmatrix} - \begin{bmatrix} \\ A \\ \\ \end{bmatrix}^T \begin{Bmatrix} X'_{vc} \\ Y'_{vc} \\ Z'_{vc} \end{Bmatrix} \quad (83)$$

*Matrix multiplication shows that $[A]^T[A] = [I]$, the identity matrix.

Equation 83 may be interpreted as the transformation of a point i, whose coordinates are (X_i', Y_i', Z_i'), from the space-fixed system for the passenger to the vehicle-fixed system; assuming that the position of the vehicle with respect to the space-fixed system for the vehicle is known, i.e., X_{vc}', Y_{vc}', Z_{vc}', ψ_c, θ_c, and ϕ_c.

The Computation of Contact Forces

The Idealized Passenger Compartment

To facilitate the computation of contact forces the vehicle interior or passenger compartment is idealized by a series of planar surfaces. This greatly simplifies the geometry considerations for predicting contact between the articulated body and its confining environment.

Figure 13 shows the numbering of the points where coordinates are necessary for defining the geometry of the idealized passenger compartment. These points are used to express the equations of the planar surfaces and their inward normal vectors, shown in Figure 14.

The Prediction of Contact using Lines and Planes

The computer program is written such that each of the 17 "contact spheres," Figure 5, is checked for contact with each of the 25 planar surfaces of the vehicle interior, Figure 14. The technique used to predict contact is illustrated in Figure 15.

(X_{vi}, Z_{vi}) - coordinate of point i in X_v - Z_v plane

Note: $Z_{v5} = Z_{v18}$; $Z_{v20} = Z_{v21}$; $Z_{v6} = Z_{v11} = Z_{v19}$.

$Z_{v2} = Z_{v3}$; $Z_{v17} = Z_{v8} = Z_{v13} = Z_{v14}$; $X_{v11} = X_{v12} = X_{v15}$;

$X_{v6} = X_{v7} = X_{v22}$.

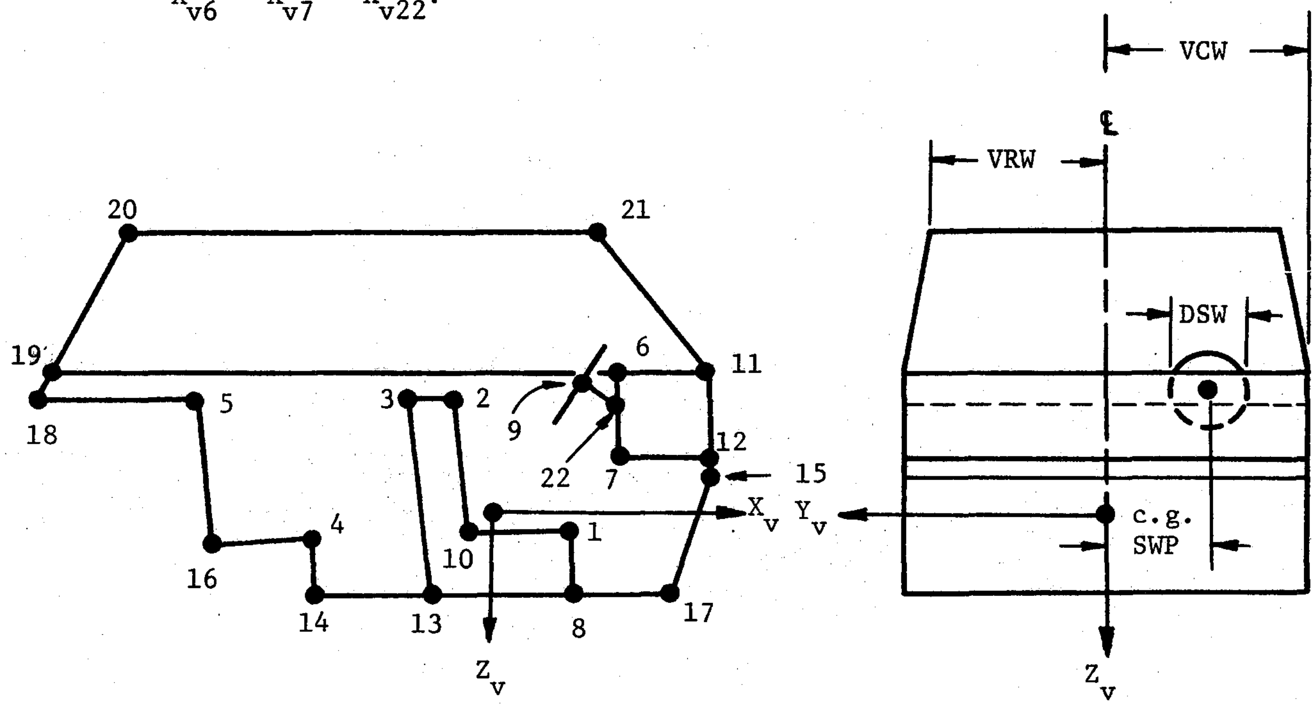


FIGURE 13.-COORDINATES AND DIMENSIONS OF THE IDEALIZED PASSENGER COMPARTMENT

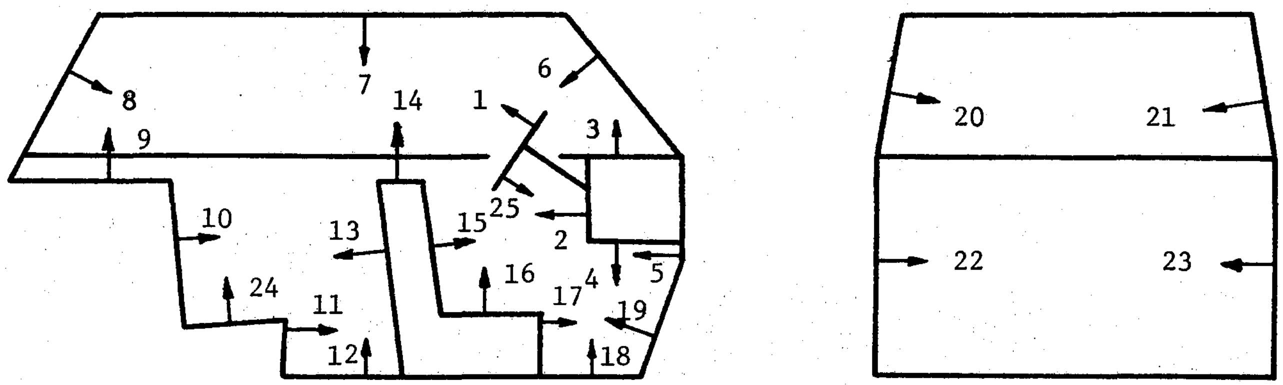


FIGURE 14.-PLANAR SURFACES AND NORMALS OF THE IDEALIZED PASSENGER COMPARTMENT

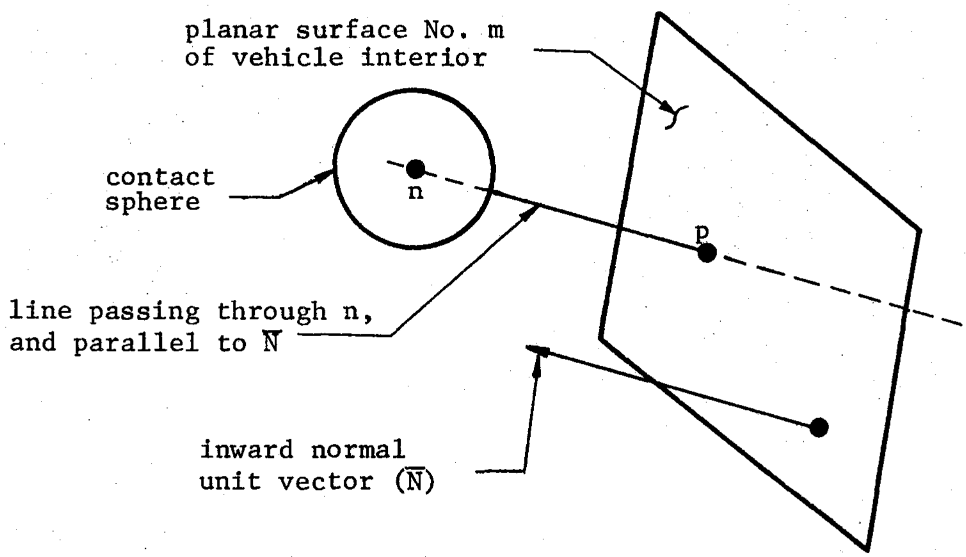


FIGURE 15.-CONTACT SPHERE AND PLANAR SURFACE

Consider "contact sphere" No. n whose center has coordinates (X'_n, Y'_n, Z'_n) with respect to the space-fixed coordinate system for the passenger which becomes (X_{vn}, Y_{vn}, Z_{vn}) with respect to the vehicle-fixed coordinate system through the use of Equation 83. The distance of this point from the planar surface No. m is found by passing a line through the point, in a direction normal to the plane, then finding the point P where the line and plane intersect, as shown in Figure 15.

The inward normal unit vector $\bar{N}$ is given by

$$\bar{N} = A_m\bar{i} + B_m\bar{j} + C_m\bar{k} \quad \dots\dots\dots\dots\dots\dots \quad (84)$$

in which $\bar{i}$, $\bar{j}$, and $\bar{k}$ form a triad of unit vectors parallel to vehicle-fixed axes, X_v, Y_v, and Z_v, respectively; and A_m, B_m, and C_m are calculated from the coordinates of points given in Figure 13.

The equation of the line parallel to $\bar{N}$ and passing through point n, of Figure 15, is

$$\frac{X_v - X_{vn}}{A_m} = \frac{Y_v - Y_{vn}}{B_m} = \frac{Z_v - Z_{vn}}{C_m} \quad . . . . \quad (85)$$

The equation of planar surface No. m, having $\bar{N}$ as its unit normal is

$$A_m X_v + B_m Y_v + C_m Z_v = G_m \quad . . . . \quad (86)$$

The constant G_m in Equation 86 can be evaluated as

$$G_m = A_m X_{vi} + B_m Y_{vi} + C_m Z_{vi} \quad . . . . \quad (87)$$

in which (X_{vi}, Y_{vi}, Z_{vi}) are the coordinates of any known point on planar surface No. m.

Solving Equations 85 and 86, simultaneously, produces the coordinates of intersection point P (X_{vp}, Y_{vp}, Z_{vp}), shown in Figure 15. These are

$$X_{vp} = X_{vn} + A_m L_{xm} \quad . . . . \quad (88a)$$

$$Y_{vp} = Y_{vn} + B_m L_{xm} \quad . . . . \quad (88b)$$

$$Z_{vp} = Z_{vn} + C_m L_{xm} \quad . . . . \quad (88c)$$

in which

$$L_{xm} = G_m - (A_m X_{vn} + B_m Y_{vn} + C_m Z_{vn}) \quad \text{. . . . .} \quad (89)$$

Define $\bar{V}_{pn}$ as the vector extending from point n to point P in Figure 15. From Equations 88a, 88b, and 88c

$$\bar{V}_{pn} = L_{xm}(A_m \bar{i} + B_m \bar{j} + C_m \bar{k}) \quad \text{. . . . .} \quad (90)$$

in which the parenthetical expression is the unit vector, $\bar{N}$, therefore the distance from the "contact sphere" center to the planar surface is simply

$$\left|\bar{V}_{pn}\right| = \left|L_{xm}\right| \quad \text{. . . . .} \quad (91)$$

Contact is predicted by L_{xm}, Equations 89 and 91, considering its sign as well as its absolute value. If L_{xm} is a positive quantity, then $\bar{V}_{pn}$ and $\bar{N}$ have the same sense meaning that the center of the contact sphere has traveled through the planar surface; hence contact has certainly occurred. Likewise, contact has occurred when L_{xm} is zero, since points n and P in Figure 15 are concurrent. When L_{xm} is negative, contact has occurred only if $\left|L_{xm}\right|$ is less than r_n, the radius of "contact sphere" No. n. The above considerations can be summarized mathematically as

$$\Delta = r_n + L_{xm} \quad \text{. . . . .} \quad (92)$$

from which contact is assured if Δ is positive. In fact, when Δ is positive not only is contact defined but Δ is also the amount of deformation present.

Magnitude of the Contact Force

With deformation defined by Equation 92, the final step towards computing a force is that of introducing a force-deformation relation for the materials involved; and testing is obviously the only way to obtain such information.

Testing was outside the scope of this particular project, nevertheless a limited source of information (12) was available from which evolved an idealized force-deformation relation. In particular, these tests which consisted of the dynamic loading of various parts of an antropomorphic dummy against cardboard honeycombed material exposed the following pertinent characteristics:

1. During the loading process, force increased at a low rate in the lower range of deformation as compared to that of the upper range of deformation.
2. Forces were considerably less during the unloading process, thus pointing to a certain amount of energy dissipation.

These observations are responsible for the proposed idealized force-deformation curve, shown in Figure 16.

In Figure 16, K_{c1} and K_{c2} are the two slopes of the bilinear curve during loading. The factors μ_1 and μ_2 represent the fraction of strain energy to be conserved during the unloading process in ranges of deformations 1 and 2, respectively; e.g., $\mu_1 = \mu_2 = 1.0$ defines a perfectly elastic collision whereas $\mu_1 = \mu_2 = 0$ defines a completely plastic collision between "contact sphere" and planar

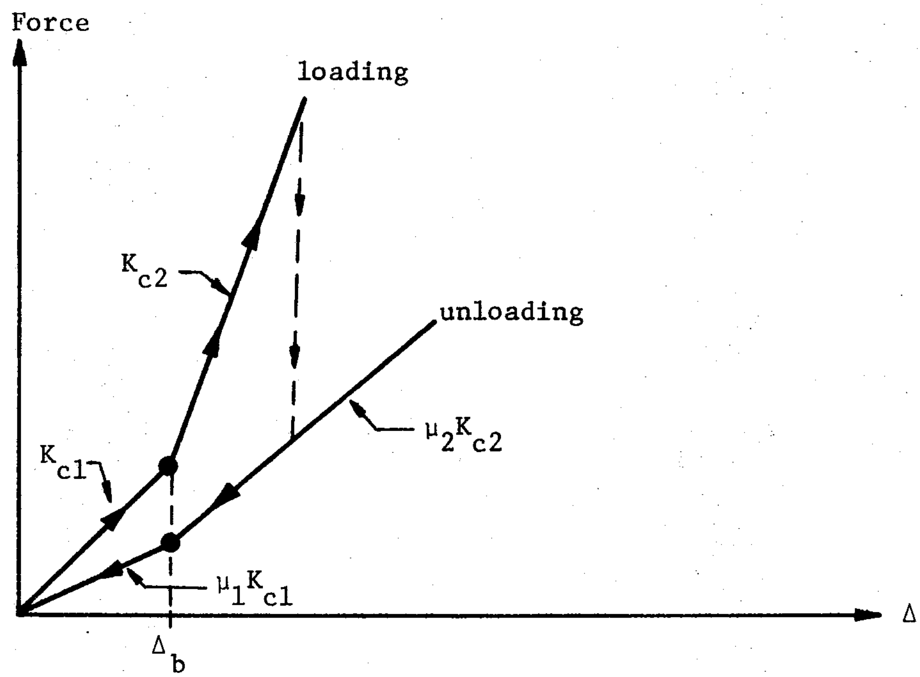


FIGURE 16.-THE BILINEAR DISSIPATIVE SPRING

surface. It should be noted that no attempt is being made to separate body deformation from vehicle deformation, hence the stiffness coefficients (K_{c1}, K_{c2}) for a given surface in the vehicle interior are actually "lumped" parameters which include body stiffness properties.

With the use of Equation 92 and Figure 16 a contact force ($\bar{F}_N$) is computed when contact occurs. The direction of this force is chosen as normal to the planar surface being contacted since the deformation (Δ) is computed in the normal direction. In addition, a friction force ($\bar{F}_f$) is calculated as some fraction of the normal force. The direction of $\bar{F}_f$ is tangent to the planar surface being contacted such that it opposes the motion of the "contact sphere." The forces $\bar{F}_N$ and $\bar{F}_f$ are added vectorially to produce a resultant force ($\bar{F}$) as

$$\bar{F} = \bar{F}_N + \bar{F}_f \qquad (93)$$

whose components are F_{vx}, F_{vy}, and F_{vz} with respect to vehicle-fixed axes X_v, Y_v, and Z_v, respectively.

From Equations 78 and 81

$$\begin{Bmatrix} F_{xi} \\ F_{yi} \\ F_{zi} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{Bmatrix} F_{vx} \\ F_{vy} \\ F_{vz} \end{Bmatrix} \qquad (94)$$

in which F_{xi}, F_{yi}, and F_{zi} are the components of $\bar{F}$, with respect to the space-fixed coordinate system for the passenger, acting on "contact sphere" No. i. These three components appear in column vector, $\{Q_f\}$, of the matrix equation of motion, Equation 53.

Lap and Torso Restraint Belts

Other sources of contact forces to which the vehicle occupant may be subjected are the safety belts. The lap belt has its ends anchored at arbitrary points and loops around the pelvic area ("contact sphere" No. 3). Likewise, the torso belt has its ends anchored at arbitrary points and loops around the upper torso area ("contact sphere" No. 2).

It is assumed that the center line of a belt defines a plane which contains the center of its respective "contact sphere" at all times. This facilitates the definition of the restraining force vector which, by definition, also lies in this plane, as shown in Figure 17.

The angles shown in Figure 17 are expressed as

$$\bar{\theta}_1 = \cos^{-1}\left(\frac{r_i}{|\bar{P}_{c1}|}\right) \quad \ldots \ldots \quad (95)$$

$$\bar{\theta}_2 = \cos^{-1}\left(\frac{r_i}{|\bar{P}_{c2}|}\right) \quad \ldots \ldots \quad (96)$$

$$\bar{\theta}_3 = \cos^{-1}\left(\frac{\bar{P}_{c1} \cdot \bar{P}_{c2}}{|\bar{P}_{c1}|\,|\bar{P}_{c2}|}\right) \quad \ldots \ldots \quad (97)$$

$$\beta = \pi - \frac{1}{2}(\bar{\theta}_1 + \bar{\theta}_2 + \bar{\theta}_3) \quad \ldots \ldots \quad (98)$$

$$\alpha = \frac{1}{2}\left|\bar{\theta}_1 + \bar{\theta}_2 + \bar{\theta}_3 - \pi\right| \quad \ldots \ldots \quad (99)$$

$$\gamma = \frac{1}{2}(\bar{\theta}_1 + \bar{\theta}_3 - \bar{\theta}_2) \quad \ldots \ldots \quad (100)$$

Equations 95 through 100 hold for all special cases, e.g., the sides of the belt being parallel or even pointing away from each other, etc.

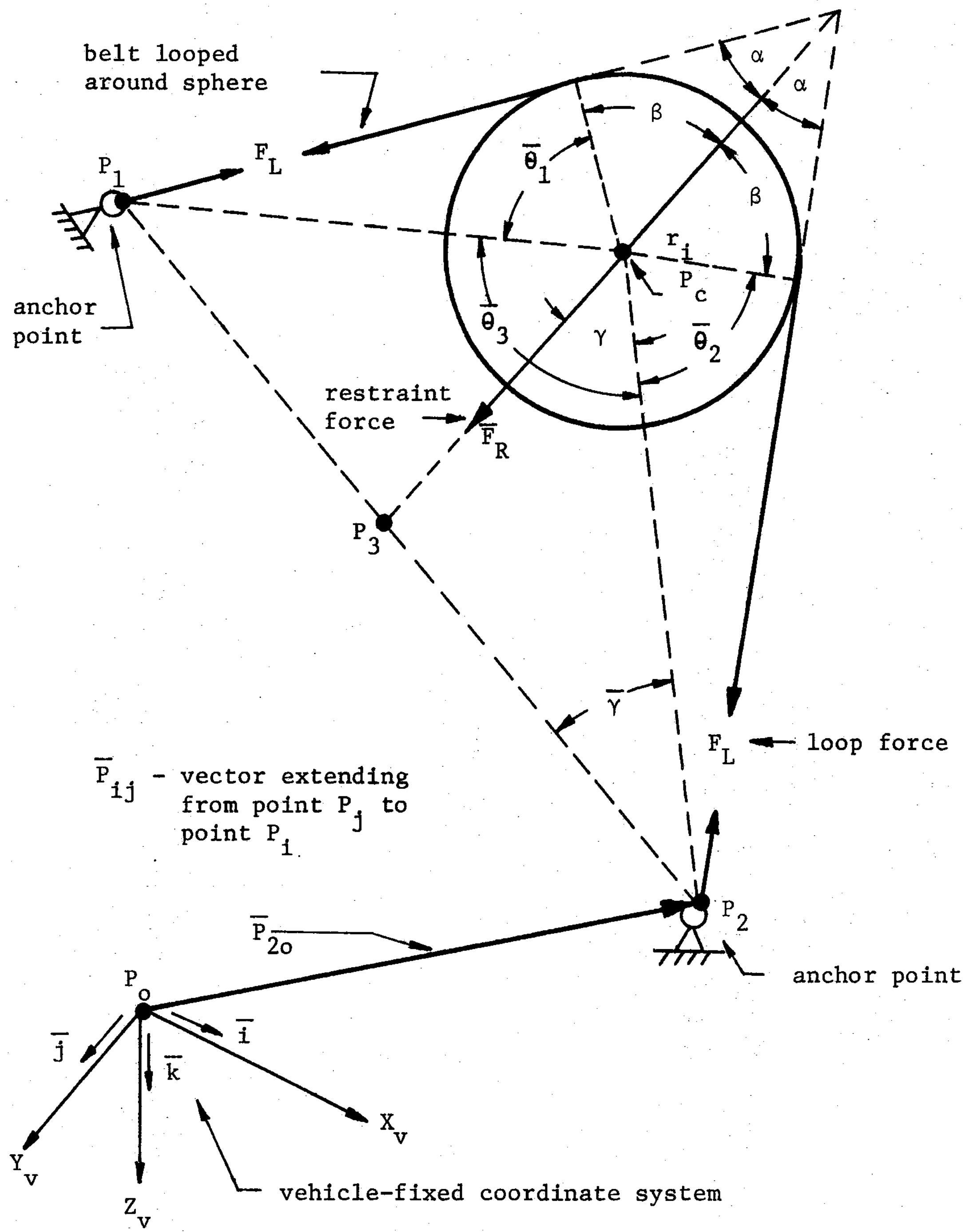


FIGURE 17.-VIEW NORMAL TO PLANE CONTAINING
BELT AND CENTER OF CONTACT SPHERE

The length of the belt at any time, t, is

$$L_{Bt} = |\bar{P}_{c2}| \sin\bar{\theta}_2 + |\bar{P}_{c1}| \sin\bar{\theta}_1 + 2r_i\beta \qquad \text{(101)}$$

in which r_i is the radius of "contact sphere" No. i (i = 2 or 3).

The belt elongation at any time, t, is

$$\Delta L_B = L_{Bt} - L_{Bo} \qquad \text{(102)}$$

in which the initial length (L_{Bo}) contains "slack" for loose-fitting belts. If $\Delta L_B \leq 0$, the loop force (F_L) is zero. For $\Delta L_B \geq 0$, F_L is computed from the idealized curve shown in Figure 18.

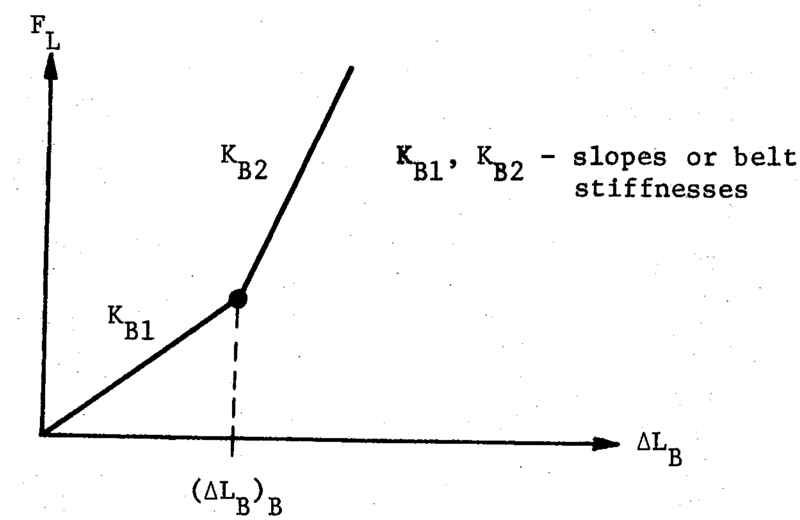


FIGURE 18.-IDEALIZED FORCE-ELONGATION CURVE FOR A RESTRAINT BELT

The magnitude of the restraining force $|\bar{F}_R|$ is defined as

$$|\bar{F}_R| = 2\, F_L \cos\alpha \qquad \text{(103)}$$

The restraining force vector is expressed as

$$\bar{F}_R = \frac{|\bar{F}_R|}{|\bar{P}_{3c}|}\,\bar{P}_{3c} \qquad \text{(104)}$$

in which the coordinates of point P_3 (needed to find $\bar{P}_{3c}$) correspond to the components of $\bar{P}_{30}$, which is found by

$$\bar{P}_{30} = \bar{P}_{20} + \bar{P}_{32} \qquad \text{(105)}$$

while the vector $\bar{P}_{32}$ is

$$\bar{P}_{32} = \frac{|\bar{P}_{32}|}{|\bar{P}_{12}|}\,\bar{P}_{12} \qquad \text{(106)}$$

It can be shown that

$$|\bar{P}_{32}| = \frac{|\bar{P}_{c2}|\sin\gamma}{\sin(\gamma+\bar{\gamma})} \qquad \text{(107)}$$

in which, as shown in Figure 17

$$\bar{\gamma} = \cos^{-1}\left(\frac{\bar{P}_{12}\cdot\bar{P}_{c2}}{|\bar{P}_{12}||\bar{P}_{c2}|}\right) \qquad \text{(108)}$$

The components of $\bar{F}_R$, Equation 104, with respect to the vehicle-fixed axes are added to F_{vx}, F_{vy}, and F_{vz} of Equation 94, for the case of $i = 2$ or 3; i.e., "contact spheres" for upper torso and pelvic area, respectively. Therefore the restraint force simply becomes part of the resultant force, on a "contact" sphere, which is transformed to the space-fixed system of the passenger for use in column vector {Q} of Equation 53.

Initial Position of Passenger

The initial position of the vehicle occupant is completely arbitrary and is specified by initializing all of the 31 generalized coordinates or degrees of freedom ($q_j, j = 1,\ldots 31$). This could prove to be a difficult task in the case where the vehicle assumes a skewed initial position. To alleviate this situation, a routine was devised by which the passenger is automatically seated properly in the vehicle, regardless of the vehicle's initial position.

Seating the Passenger

To simplify matters it is assumed that, initially, the passenger is heading in the same direction as the vehicle; i.e., the Y_n axis for each of the 12 body segments ($n = 1,\ldots 12$) is parallel (opposite sense) to Y_v, the vehicle-fixed Y direction. However, as shown in Figure 19, the vehicle occupant's posture remains arbitrary and is defined by parameters γ_1, γ_2, γ_3, γ_4, γ_5, α_7 ($\alpha_7 = \alpha_8$), X_{VH}, and Y_{VH}. The angle γ_n specifies the position of body segment No. n with respect to the vehicle-fixed coordinate system as shown in Figure 20.

From Figure 20 it is apparent that

$$\begin{Bmatrix} X_v \\ Y_v \\ Z_v \end{Bmatrix} = \begin{bmatrix} \cos\gamma_n & 0 & -\sin\gamma_n \\ 0 & -1 & 0 \\ -\sin\gamma_n & 0 & -\cos\gamma_n \end{bmatrix} \begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} \quad \ldots\ldots\ldots\ldots (109)$$

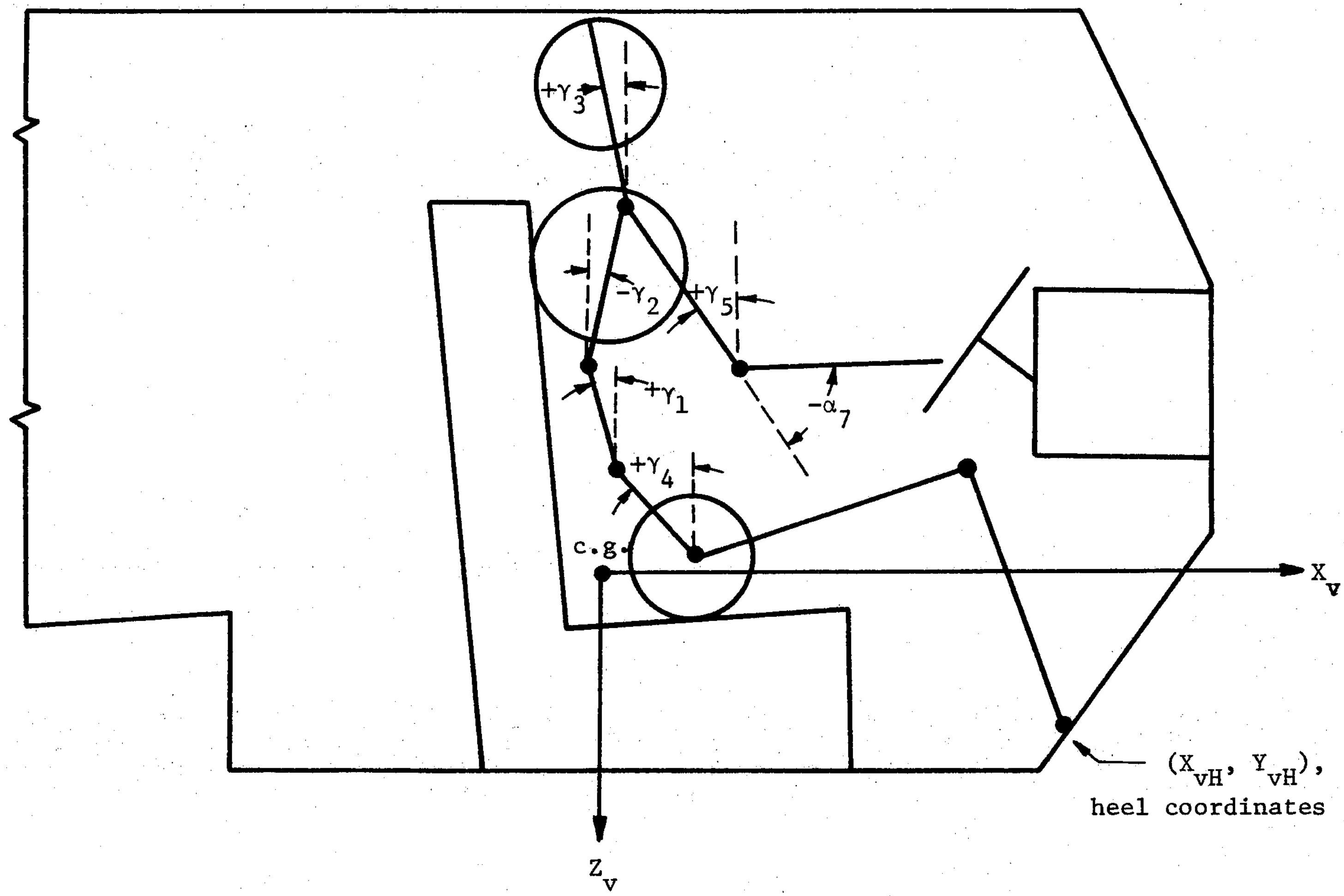


FIGURE 19.-INPUT PARAMETERS FOR PASSENGER SEATING OPTION

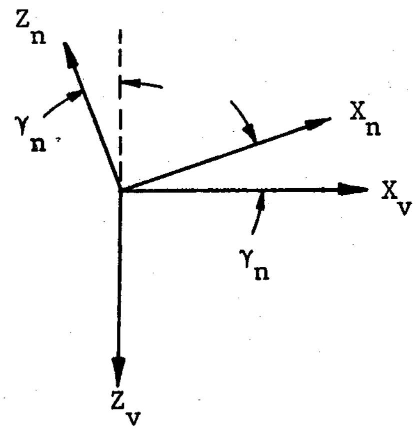


FIGURE 20.-RELATION BETWEEN SEGMENT-FIXED AND VEHICLE-FIXED COORDINATE SYSTEMS

Manipulation of Equations 78, 81, and 109 produces

$$\begin{Bmatrix} X' \\ Y' \\ Z' \end{Bmatrix} = \begin{bmatrix} & & \\ & B^n & \\ & & \end{bmatrix} \begin{Bmatrix} X_n \\ Y_n \\ Z_n \end{Bmatrix} \quad \ldots\ldots\ldots\ldots\ldots \quad (110)$$

in which

$$[B^n] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} \cos\gamma_n & 0 & -\sin\gamma_n \\ 0 & -1 & 0 \\ -\sin\gamma_n & 0 & -\cos\gamma_n \end{bmatrix} \quad . \; . \quad (111)$$

"Seating the passenger" or rather defining the generalized coordinates, such that the occupant is initially sitting as shown in Figure 19, is accomplished as follows:

1. From the information specified for Figures 13 and 19 along with the force-deformation properties of the seat and the initial position of the vehicle, it is possible, with the use of Equation 82, to determine the proper coordinates of the reference point on the articulated body as shown in Figure 1. This initially establishes X'_{T1}, Y'_{T1}, and Z'_{T1} or generalized coordinates q_1, q_2, and q_3, respectively.
2. Since $[T^n] = [B^n]$ at initial time*, initial values of generalized coordinates q_4 through q_{23} can be computed by equating terms of the matrices for $n = 1,...6$.
3. From the coordinates of the heels, X_{VH} and Z_{VH}, in Figure 20, it is possible to compute the required angular positions of the leg segments. Knowing these, initial values of generalized coordinates q_{24} through q_{31} can be computed by setting $[T^n] = [B^n]$ for $n = 9$ and 10.

The operations described in items 1, 2, and 3 above are included in the computer program as a convenience to the user. However, the user of the program still has the option of specifying arbitrary values for the generalized coordinates if such is desired. This and other options are given in "APPENDIX IV.-DESCRIPTION OF INPUT TO THE COMPUTER PROGRAM."

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*This is realized by comparison of Equations 1 and 110.

SOLUTION OF EQUATIONS

The Runge-Kutta Method

Equation 53, the matrix equation of motion for the vehicle occupant, was solved using the Runge-Kutta method for ordinary differential equations (3). This method was applicable since Equation 53 is of the form where the highest derivative, $\ddot{q}_j$ (j=1, ... 31), can be expressed as a function of the lower derivative, $\dot{q}_j$, the dependent variable, q_j, and the independent variable, t (time).

Define the right-hand side of Equation 53 as

$$\{\overline{E}\} = \{E\} + \{F_p\} + \{F_s\} + \{Q_f\} + \{R\} \quad \text{. . . . .} \quad (112)$$

Substituting from Equation 112 into Equation 53 and solving for $\{\ddot{q}\}$, such that

$$\{\ddot{q}\} = [D]^{-1}\{\overline{E}\} \quad \text{. . . . .} \quad (113)$$

Using the notation, $[D(\{q\})]$, to indicate that [D] is a function of the elements of {q}, Equation 113 may be rewritten as

$$\{\ddot{q}\} = [D(\{q\})]^{-1}\{\overline{E}(\{\dot{q}\}, \{q\})\} \quad \text{. . . . .} \quad (114)$$

in which $\{\overline{E}\}$ is shown to be a function of the elements of $\{\dot{q}\}$ and {q}.

Solution of Equation 114 was achieved through the following successive computations, for the increment of integration, Δt:

$$\{\bar{K}_1\} = \Delta t\{\dot{q}\} \quad \ldots \quad (115)$$

$$\{\bar{L}_1\} = \Delta t[D(\{q\})]^{-1}\{\bar{E}(\{\dot{q}\}, \{q\})\} \quad \ldots \quad (116)$$

$$\{\bar{K}_2\} = \Delta t\{\dot{q}\} + \frac{1}{2}\Delta t\{\bar{L}_1\} \quad \ldots \quad (117)$$

$$\{\bar{L}_2\} = \Delta t[D(\{q\} + \frac{1}{2}\{\bar{K}_1\})]^{-1}\{\bar{E}(\{\dot{q}\} + \frac{1}{2}\{\bar{L}_1\}, \{q\} + \frac{1}{2}\{\bar{K}_1\})\} \quad \ldots \quad (118)$$

$$\{\bar{K}_3\} = \Delta t\{\dot{q}\} + \frac{1}{2}\Delta t\{\bar{L}_2\} \quad \ldots \quad (119)$$

$$\{\bar{L}_3\} = \Delta t[D(\{q\} + \frac{1}{2}\{\bar{K}_2\})]^{-1}\{\bar{E}(\{\dot{q}\} + \frac{1}{2}\{\bar{L}_2\}, \{q\} + \frac{1}{2}\{\bar{K}_2\})\} \quad \ldots \quad (120)$$

$$\{\bar{K}_4\} = \Delta t\{\dot{q}\} + \Delta t\{\bar{L}_3\} \quad \ldots \quad (121)$$

$$\{\bar{L}_4\} = \Delta t[D(\{q\} + \{\bar{K}_3\})]^{-1}\{\bar{E}(\{\dot{q}\} + \{\bar{L}_3\}, \{q\} + \{\bar{K}_3\})\} \quad \ldots \quad (122)$$

$$\{\bar{K}\} = \frac{1}{6}\{\bar{K}_1\} + \frac{1}{3}\{\bar{K}_2\} + \frac{1}{3}\{\bar{K}_3\} + \frac{1}{6}\{\bar{K}_4\} \quad \ldots \quad (123)$$

$$\{\bar{L}\} = \frac{1}{6}\{\bar{L}_1\} + \frac{1}{3}\{\bar{L}_2\} + \frac{1}{3}\{\bar{L}_3\} + \frac{1}{6}\{\bar{L}_4\} \quad \ldots \quad (124)$$

The new values at t_i are

$$t_i = t_{i-1} + \Delta t \quad \ldots \quad (125)$$

$$\{\dot{q}\}_i = \{\dot{q}\}_{i-1} + \{\bar{L}\} \quad \ldots \quad (126)$$

$$\{q\}_i = \{q\}_{i-1} + \{\bar{K}\} \quad \ldots \quad (127)$$

The solution technique is stepwise and repetitive with each step incorporating Equations 115 through 127 while utilizing the previous values of $\{\dot{q}\}$ and $\{q\}$.

Discussion of the Computer Program

The computer program for the passenger model was written in Fortran IV for the IBM 360/65 computer. The program listing of about 4,000 lines was considered too lengthy for presentation in this report. Documentation of the "input data" for the program is given in "APPENDIX IV. DESCRIPTION OF INPUT TO THE COMPUTER PROGRAM."

The Subroutines

"MAIN" is the control center for the program from which subroutines "INPUT", "CONST", "SEATIN", "PRNTIN", RNGKTA", "POSVEH", and "OUTPUT" are called.

Subroutine "INPUT" reads all data required by the program.

Subroutine "CONST" computes terms which remain constant throughout the program.

Subroutine "SEATIN" computes the passenger's initial position such that he is seated properly inside the vehicle.

Subroutine "PRNTIN" prints the information which is read in "INPUT" and computer in "SEATIN".

Subroutine "RNGKTA" integrates the equations of motion according to the Runge-Kutta method, Equations 115 through 127, and calls subroutine "SETUP" in the process.

Subroutine "SETUP" calculates the elements of [D] and $\{\bar{E}\}$ according to the functional changes indicated in Equations 115 through 127 then solves for $\{\ddot{q}\}$ as shown in Equation 114. This is accomplished by calling subroutines "DMATX", "EMATX", "QFORCE", "JOINTS", "POTEGY", and "DGELS".

Subroutine "DMATX" computes the elements of [D] according to the equations given in "APPENDIX III. - EQUATIONS OF MOTION".

Subroutine "EMATX" computes the elements of {E} according to the equations given in "APPENDIX III. - EQUATIONS OF MOTION".

Subroutine "QFORCE" computes the elements of $\{Q_f\}$ according to the equations given in "APPENDIX III. - EQUATIONS OF MOTION". This is accomplished by first calling subroutine "CONTAC".

Subroutine "JOINTS" computes the elements of $\{F_s\}$ and {R} according to the equations given in "APPENDIX III. - EQUATIONS OF MOTION".

Subroutine "POTEGY" computes the elements of $\{F_p\}$ according to the equations given in "APPENDIX III. - EQUATIONS OF MOTION".

Subroutine "DGELS" solves a set of simultaneous equations utilizing the "Gaussian elimination" technique on the upper triangular portion of a symmetric matrix of coefficients.

Subroutine "CONTAC" (called by "QFORCE") computes contact forces due to passenger-vehicle interaction. "CONTAC" calls subroutine "CENTER".

Subroutine "CENTER" calculates the position of each "contact sphere" center with respect to the vehicle-fixed coordinate system.

Subroutine "POSVEH" updates the position of the vehicle in space by linearly interpolating between time stations of the tabular input for vehicle position versus time; it is called by "MAIN".

Subroutine "OUTPUT" prepares the solution for printing; it is called by "MAIN".

Subroutine "ANSWER" prints all pertinent information obtained from the solution and is called by "OUTPUT".

Output

The solution of the equations of motion consists of a time history of the following quantities:

1. the coordinates of the end points of each body segment with respect to the vehicle-fixed coordinate system;
2. acceleration components of the center of mass of each body segment with respect to the segment-fixed coordinate system for that segment (this is total acceleration);
3. angular acceleration components of each body segment with respect to its segment-fixed coordinate system;
4. angular velocity components of each body segment with respect to its segment fixed coordinate system;

5. the force on each body "contact sphere" plus the identification of whatever vehicle interior surface is being hit;
6. the coordinates of the point of application of the contact force with respect to the center of the contact sphere in segment-fixed coordinates (only for the head, chest and pelvic area);
7. the restraining force applied to the body by the lap and torso restraint belts.

VALIDATION STUDY

Scope

An original objective of this project was to validate the passenger model's three-dimensional response capabilities by comparison with existing test data of this nature. To produce conclusive results, any such data should provide the following information:

1. a time history of the vehicle's path in three-dimensional space (preferably numerical instead of photographic*);
2. a corresponding time history of the occupant's dynamic behavior, e.g., accelerations, forces, or a photographic record of motion;
3. a quantitative description of the occupant, i.e., dimensions, weight, etc.;
4. force-deformation properties of the pertinent vehicle suffaces (could be measured).

Unfortunately, test results possessing all these qualities were not to be located and funds for full scale testing were not available thus precluding a validation of the general case at this time.

However, suitable test results were available (12) for a partial validation, i.e., the case of a frontal automobile collision.

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*Photographic records can be used for application of the passenger model but only after validation is achieved.

Test Data

The test data used for comparison were generated at the Biomechanics Research Center of Wayne State University, Detroit, Michigan under the direction of Cornell Aeronautical Laboratory (CAL), Inc., Buffalo, New York for the United States Public Health Service, March, 1967. All experimental results shown in this report were copied directly from CAL's documentation of these tests (12).

The tests consisted of a dummy seated on a cart capable of controlled deceleration. Mounted on the cart were target assemblies for head, chest and knee impact. Accelerations of various parts of the dummy and impact forces were measured by instrumentation while the motion of the dummy was recorded on high speed film. Several cases were run consisting of lap restraint, lap and torso restraint, and no restraint for initial velocities of 10 and 20 miles per hour.

Also measured and documented (12) were the force-deformation characteristics of the targets, seat, and restraint belts plus the dummy's initial position and the amount of friction in each of its joints.

Discussion of Results

Response Comparison for No Restraint at 20 MPH Cart Velocity

Figures 21 through 28 show the comparison of dummy kinematics, head and chest forces, plus head and chest accelerations for the case of no restraint with 20 MPH cart velocity. Agreement between simulated motion and the high speed film record is excellent. Quantitative comparisons (forces and accelerations) are only fair as a result of the idealized vehicle interior (simulation) being geometrically different from the target assemblies used in the test. The simulation utilized a full instrument panel and steering wheel as opposed to isolated targets of about six to eight inches in diameter for the test. This resulted in hand contact in the simulation which was absent during the test. Also the knee targets were inclined for the test producing a downward force component whereas the knees in the simulation contacted a vertical surface (instrument panel) with friction as the only downward force.

Response Comparison for Lap Restraint at 20 MPH Cart Velocity

Figures 29 through 36 show the comparison of dummy kinematics, head and chest forces, plus head and chest accelerations for the case of lap restraint with 20 MPH cart velocity. Agreement between simulated motion and the high speed film record is good, however it seems that the lap belt in the simulation is stiffer than that of the test causing a higher contribution of energy to the rotational mode. This observation is consistent with the higher predicted contact forces and accelerations as shown in Figures 31 through 36. This comparison was also subject to discrepancies resulting from the geometrical differences between idealized vehicle interior and target assemblies as discussed for the case of no restraint.

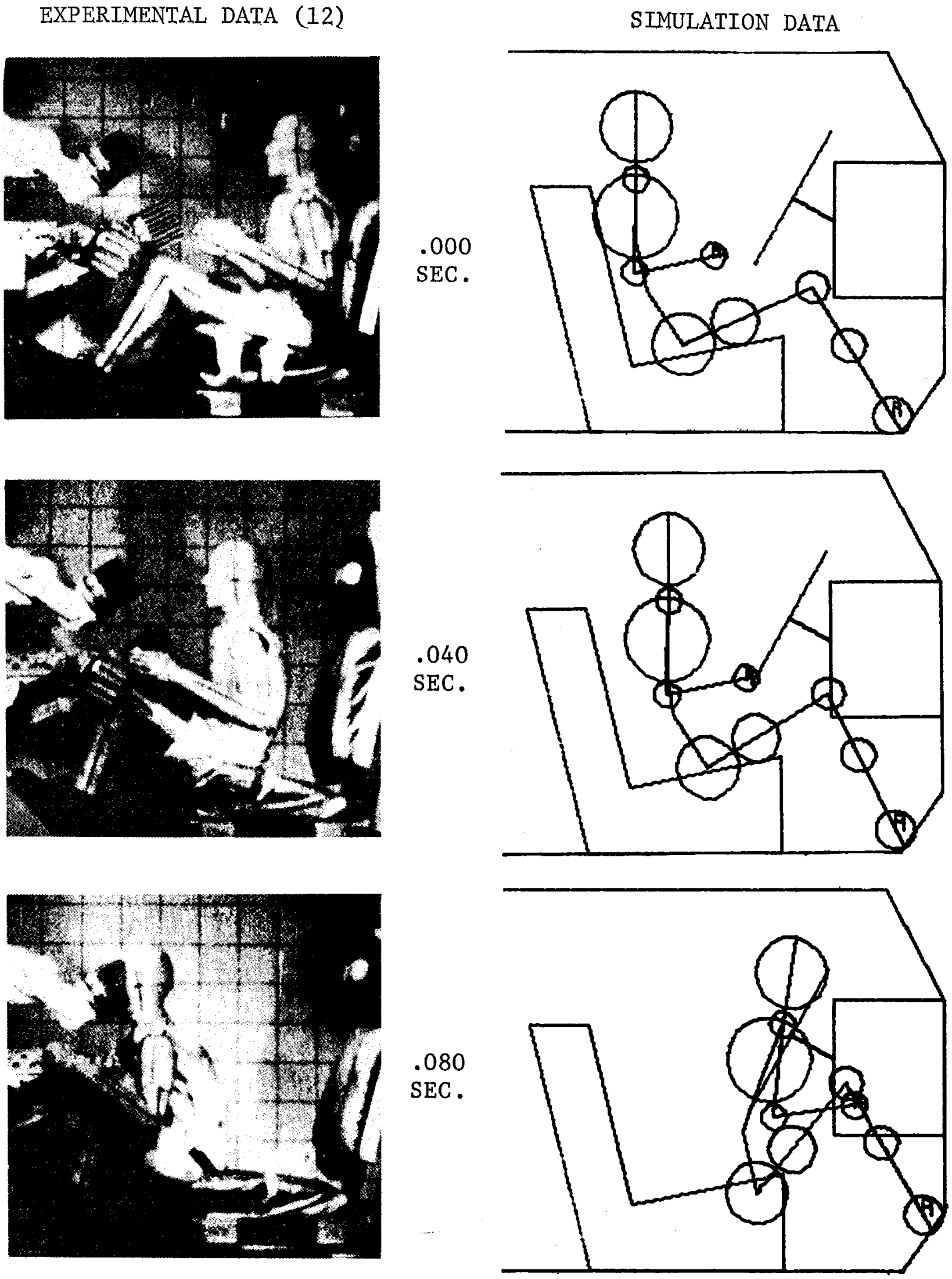

FIGURE 21.-DUMMY KINEMATIC COMPARISON NO RESTRAINT-CART VELOCITY 20 MPH, .000-.080 SEC.

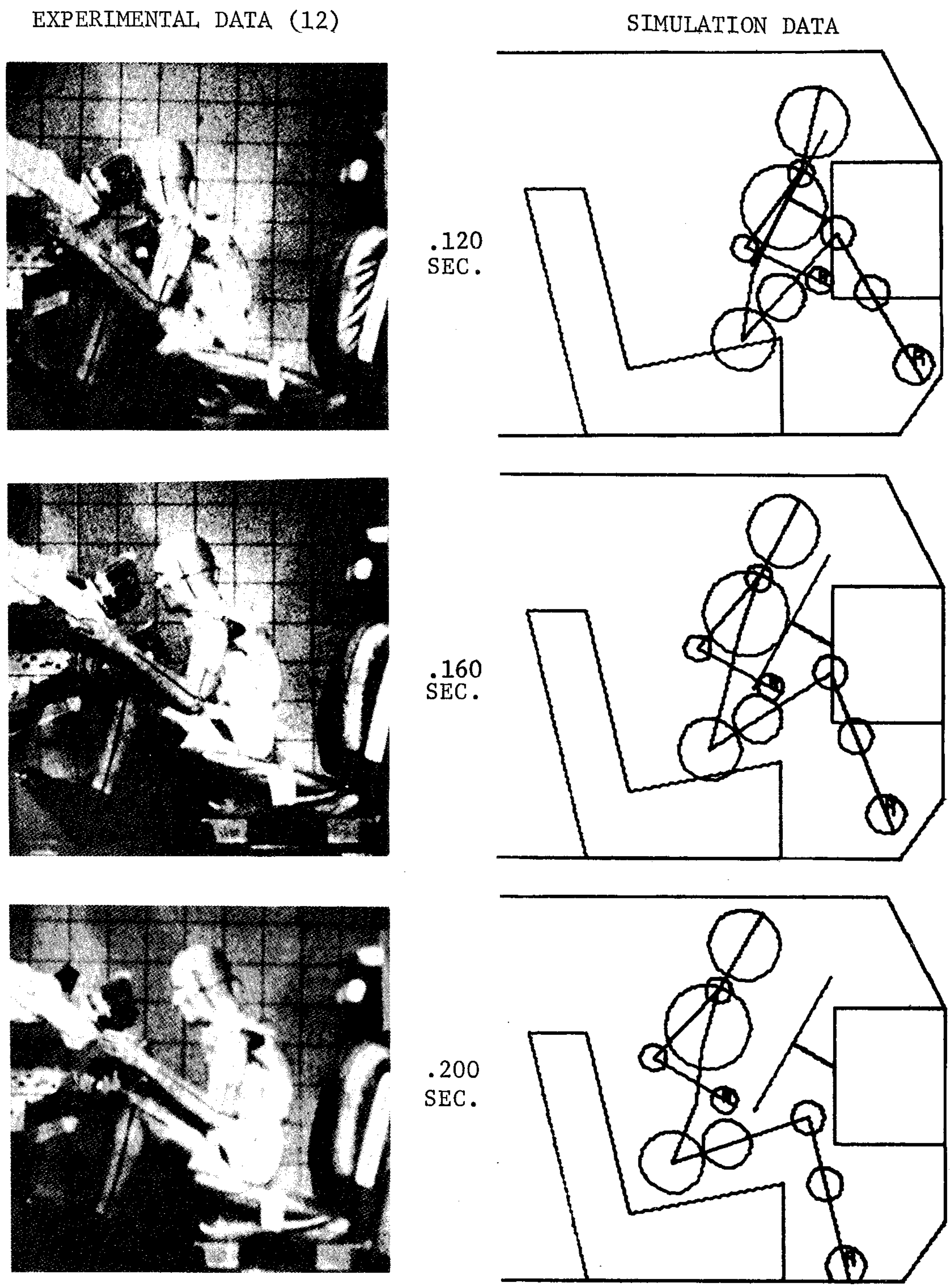

FIGURE 22.-DUMMY KINEMATIC COMPARISON NO RESTRAINT-CART VELOCITY 20 MPH, .120-.200 SEC.

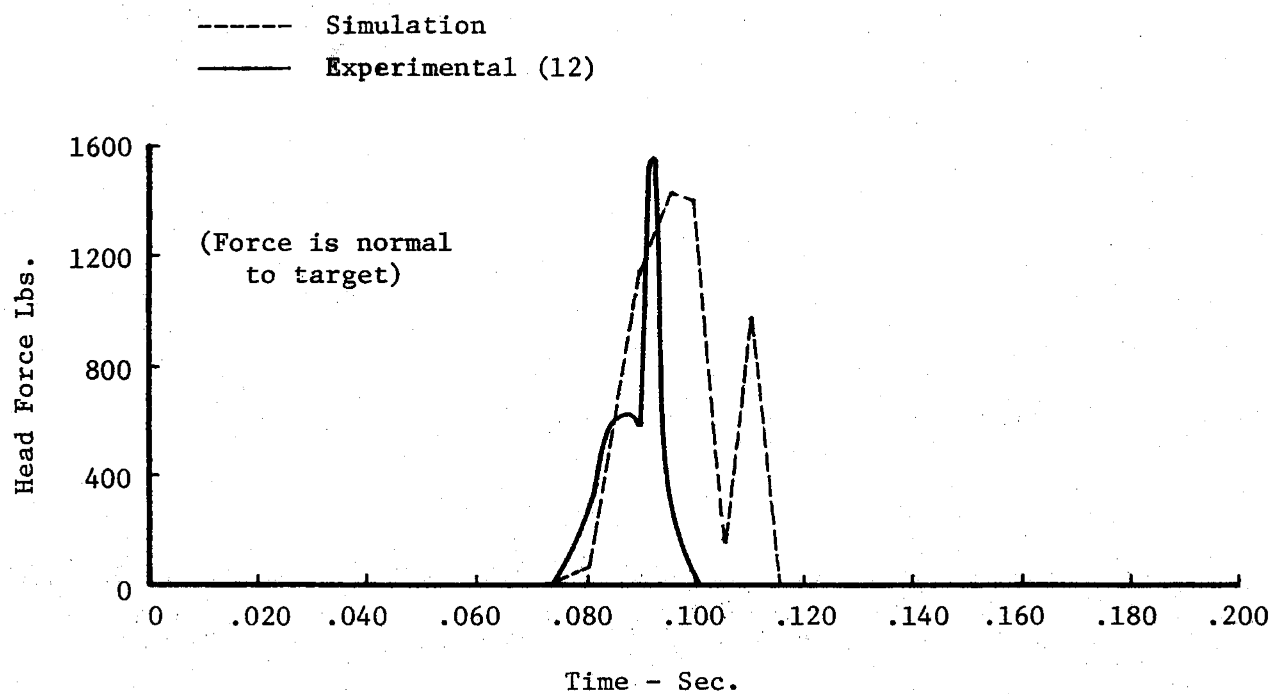

FIGURE 23.-HEAD FORCE-NO RESTRAINT-CART VELOCITY 20 MPH

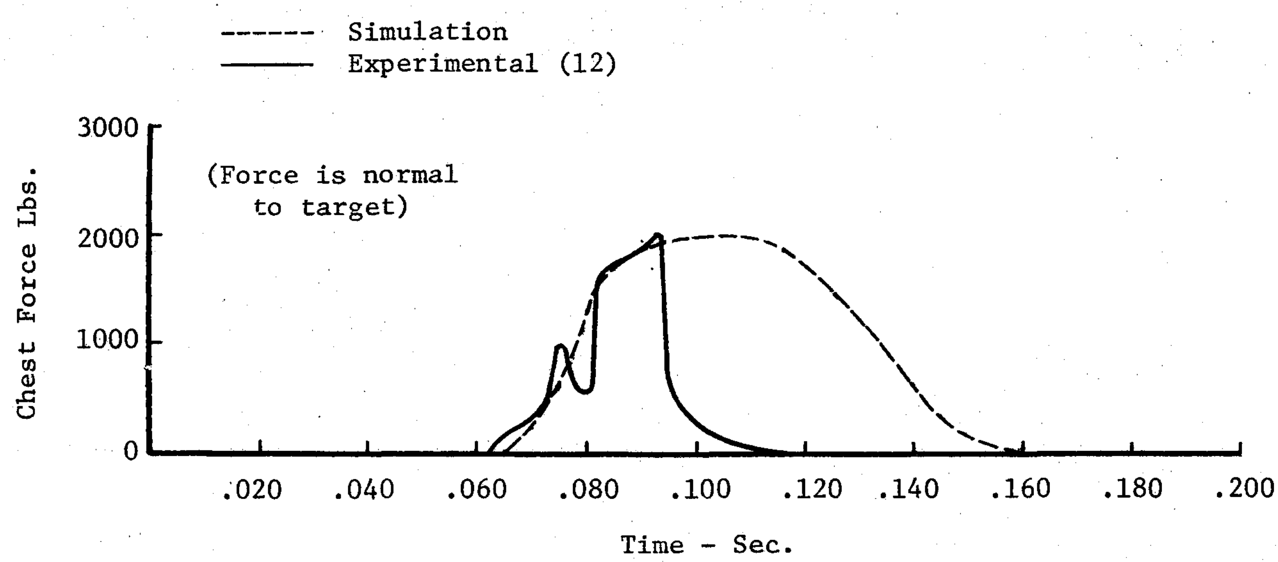

FIGURE 24.-CHEST FORCE-NO RESTRAINT-CART VELOCITY 20 MPH

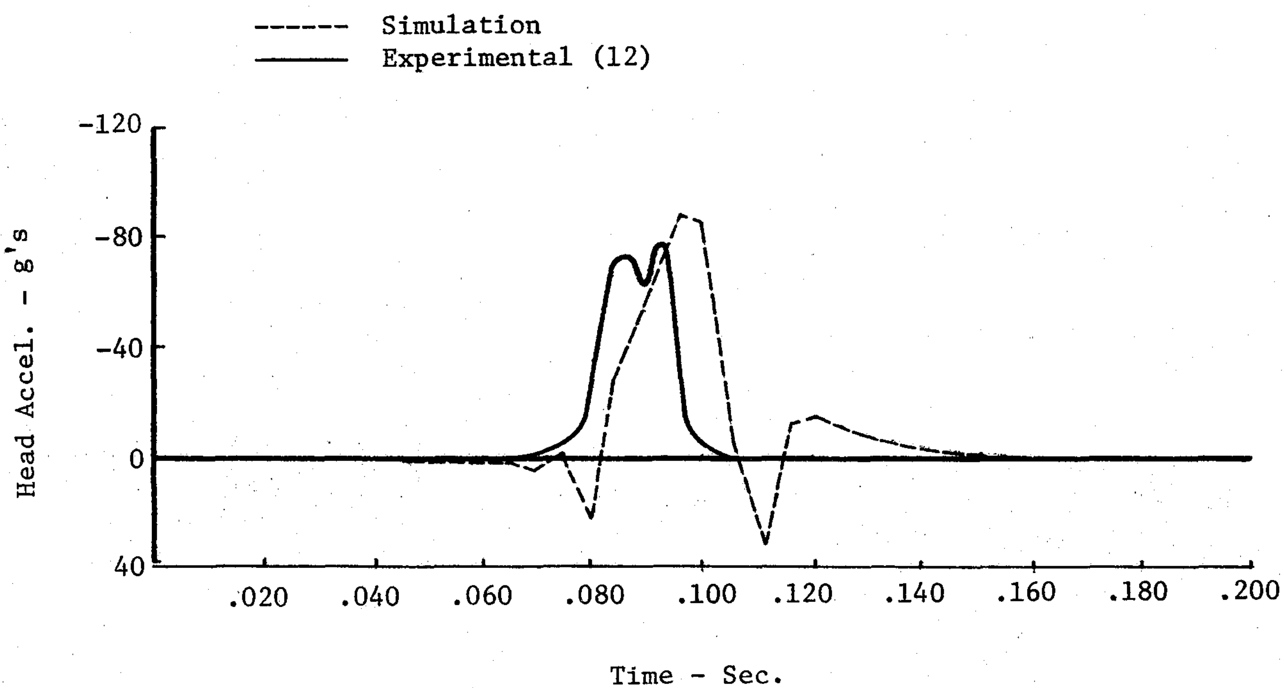

FIGURE 25.-HEAD ACCELERATION IN SEGMENT X DIRECTION-
NO RESTRAINT-CART VELOCITY 20 MPH

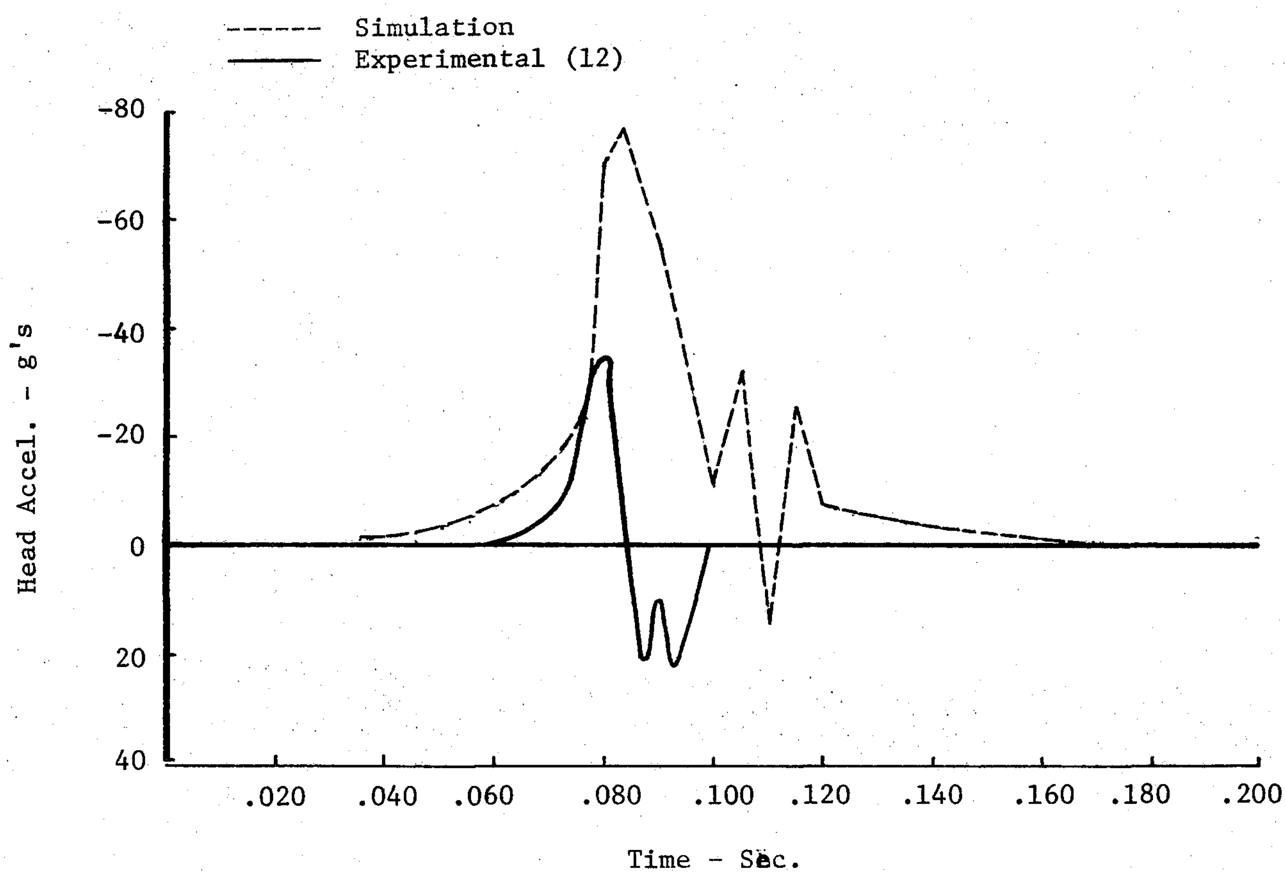

FIGURE 26.-HEAD ACCELERATION IN SEGMENT Z DIRECTION-
NO RESTRAINT-CART VELOCITY 20 MPH

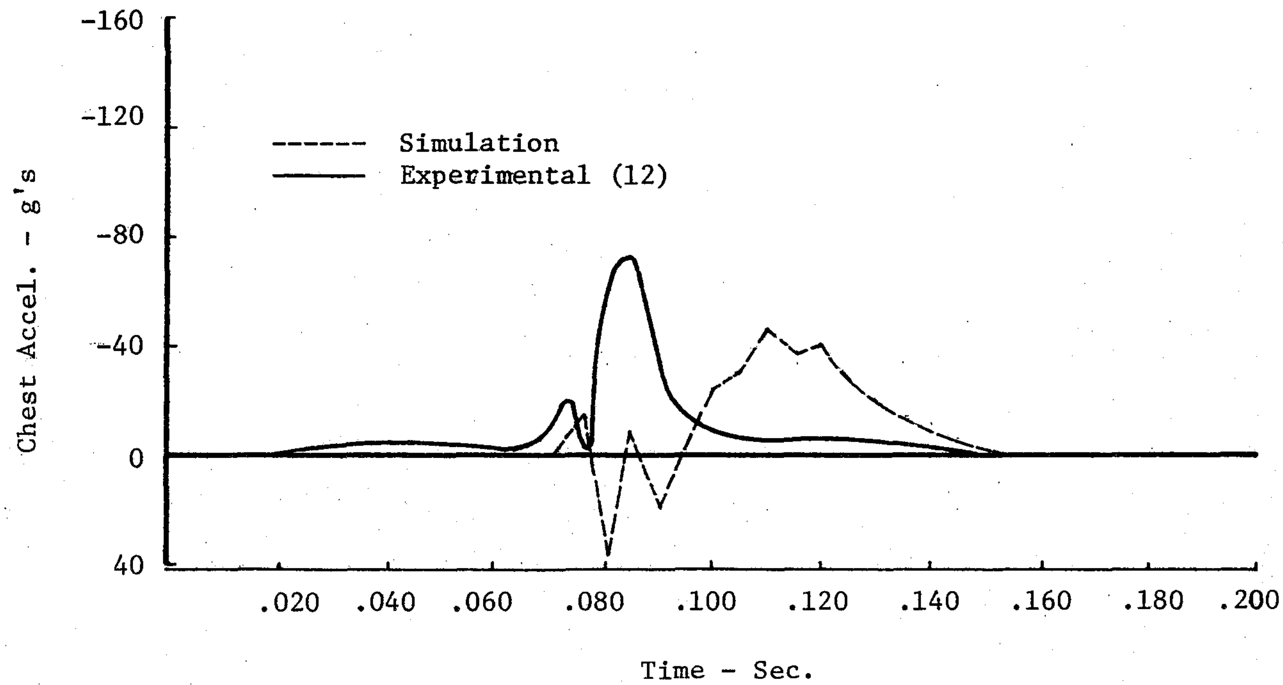

FIGURE 27.-CHEST ACCELERATION IN SEGMENT X DIRECTION-
NO RESTRAINT-CART VELOCITY 20 MPH

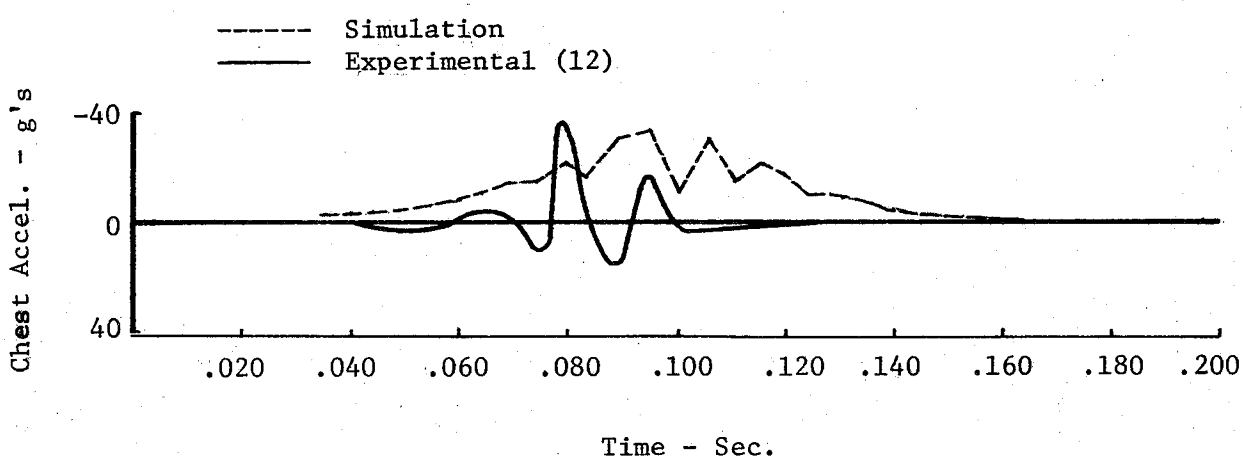

FIGURE 28.-CHEST ACCELERATION IN SEGMENT Z DIRECTION-
NO RESTRAINT-CART VELOCITY 20 MPH

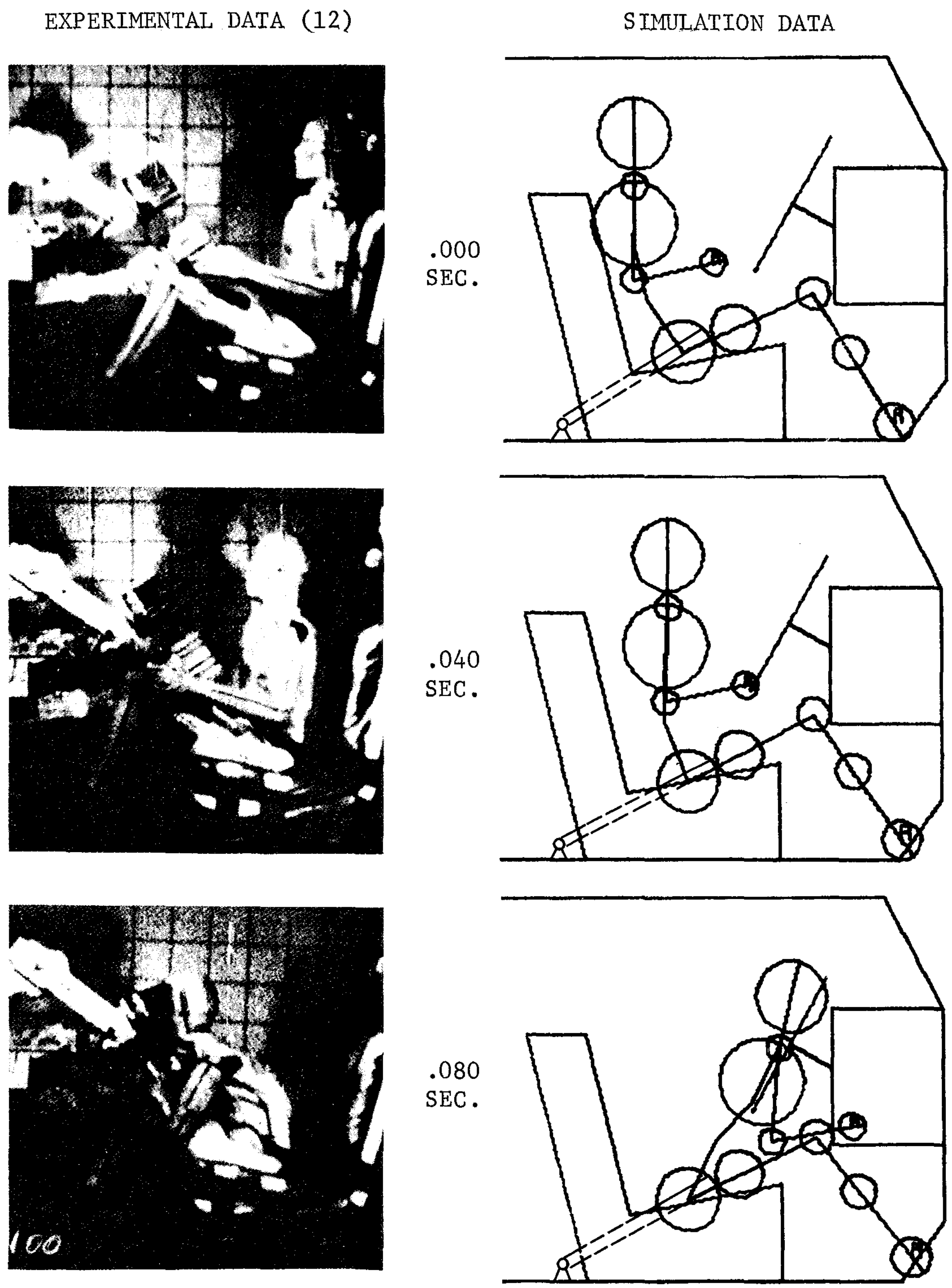

FIGURE 29.-DUMMY KINEMATIC COMPARISON LAP RESTRAINT-CART VELOCITY 20 MPH, .000-.080 SEC.

EXPERIMENTAL DATA (12) SIMULATION DATA

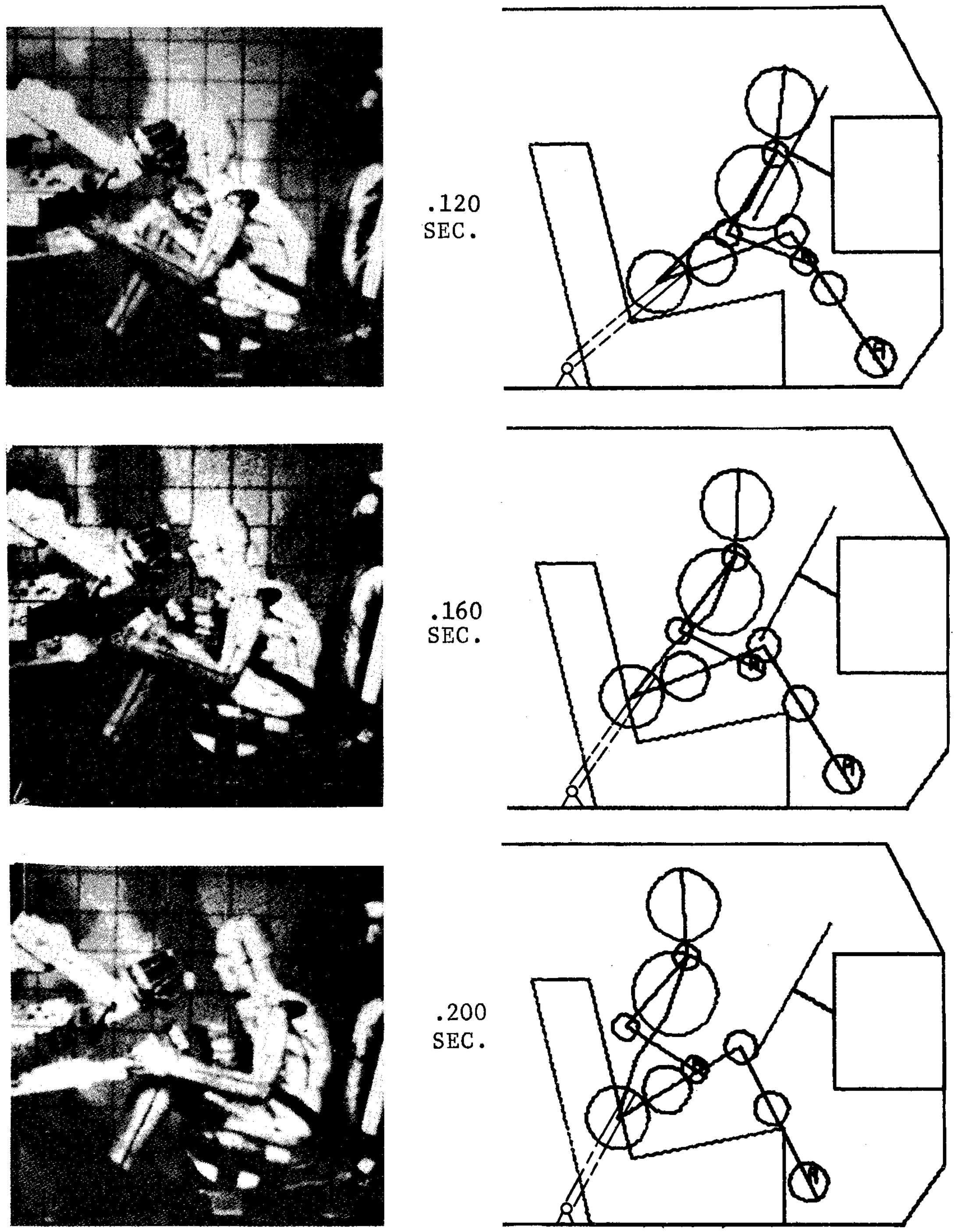

FIGURE 30.-DUMMY KINEMATIC COMPARISON LAP RESTRAINT-CART VELOCITY 20 MPH, .120-.200 SEC.

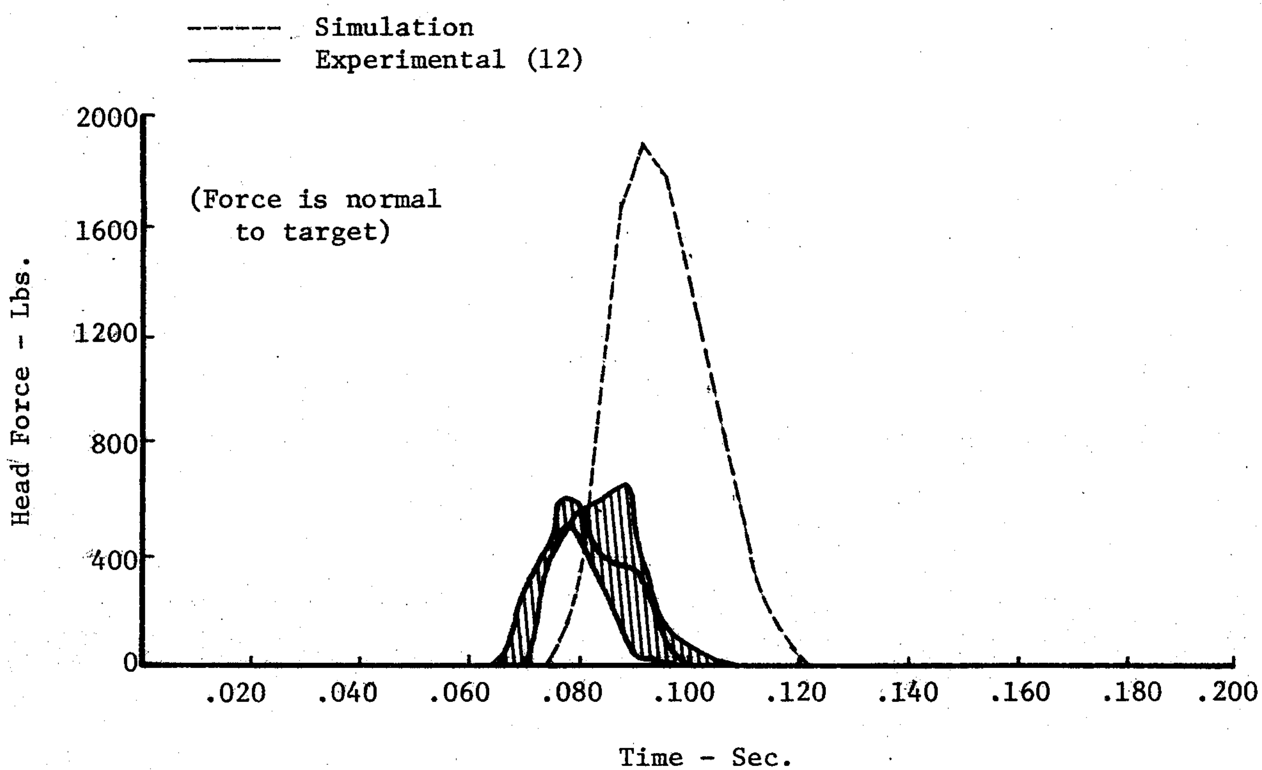

FIGURE 31.-HEAD FORCE-LAP RESTRAINT-CART VELOCITY 20 MPH

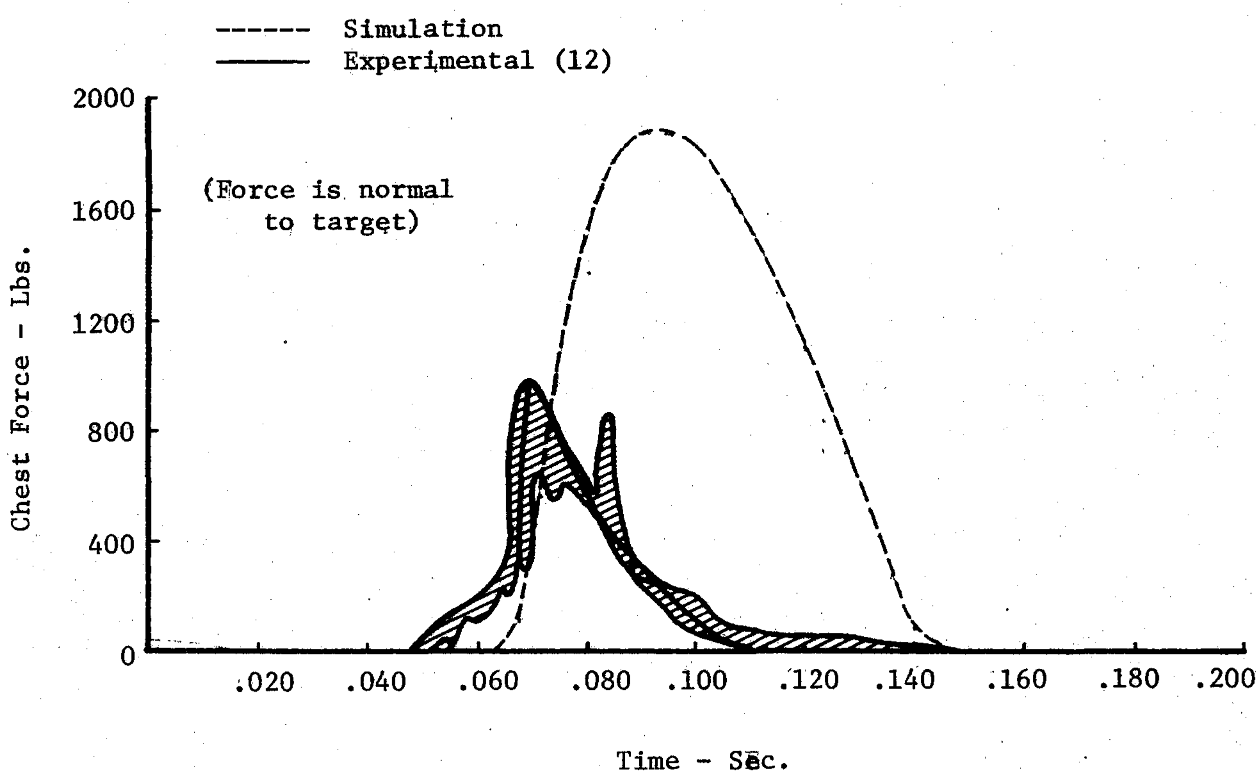

FIGURE 32.-CHEST FORCE-LAP RESTRAINT-CART VELOCITY 20 MPH

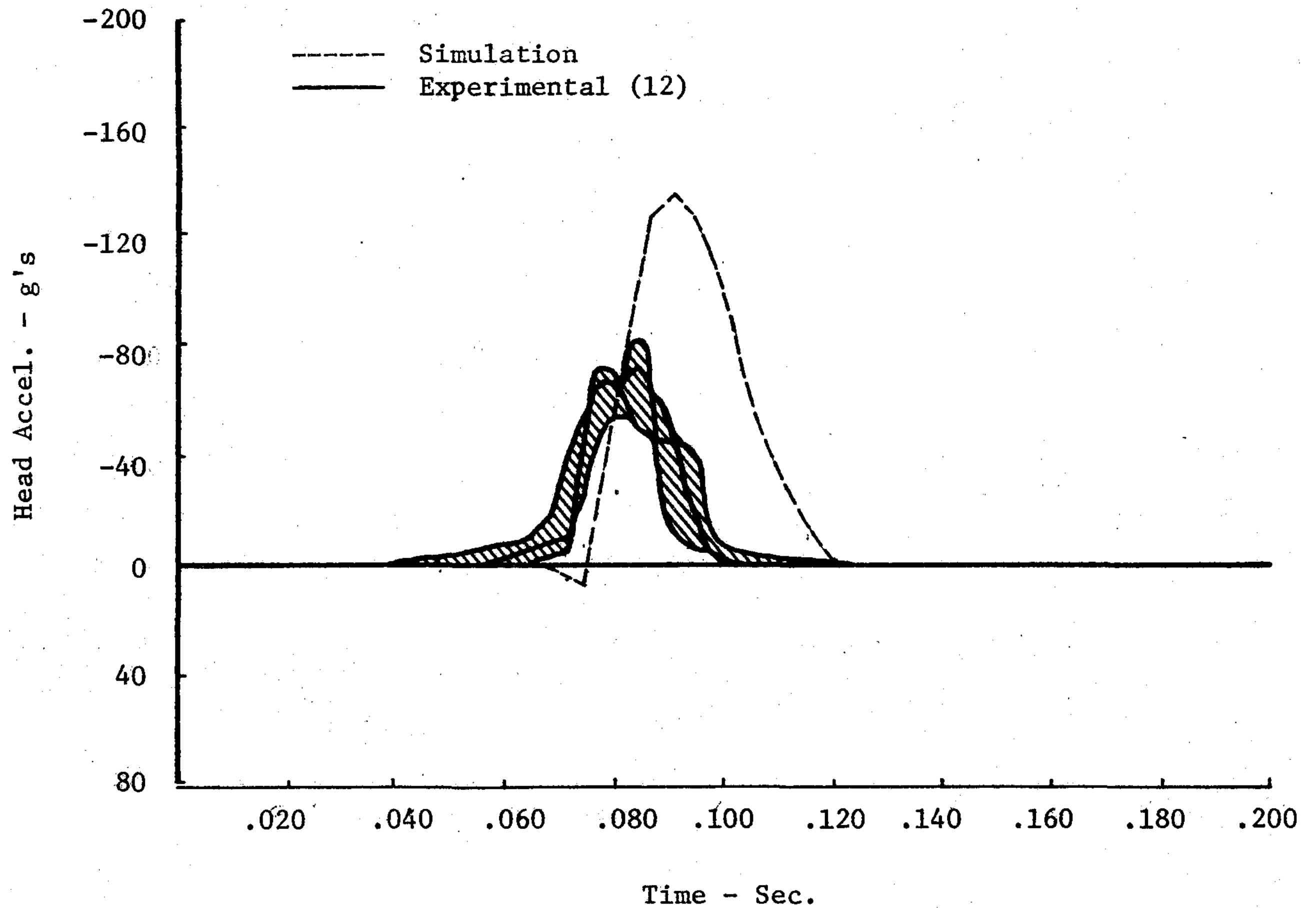

FIGURE 33.-HEAD ACCELERATION IN SEGMENT X DIRECTION-
LAP RESTRAINT-CART VELOCITY 20 MPH

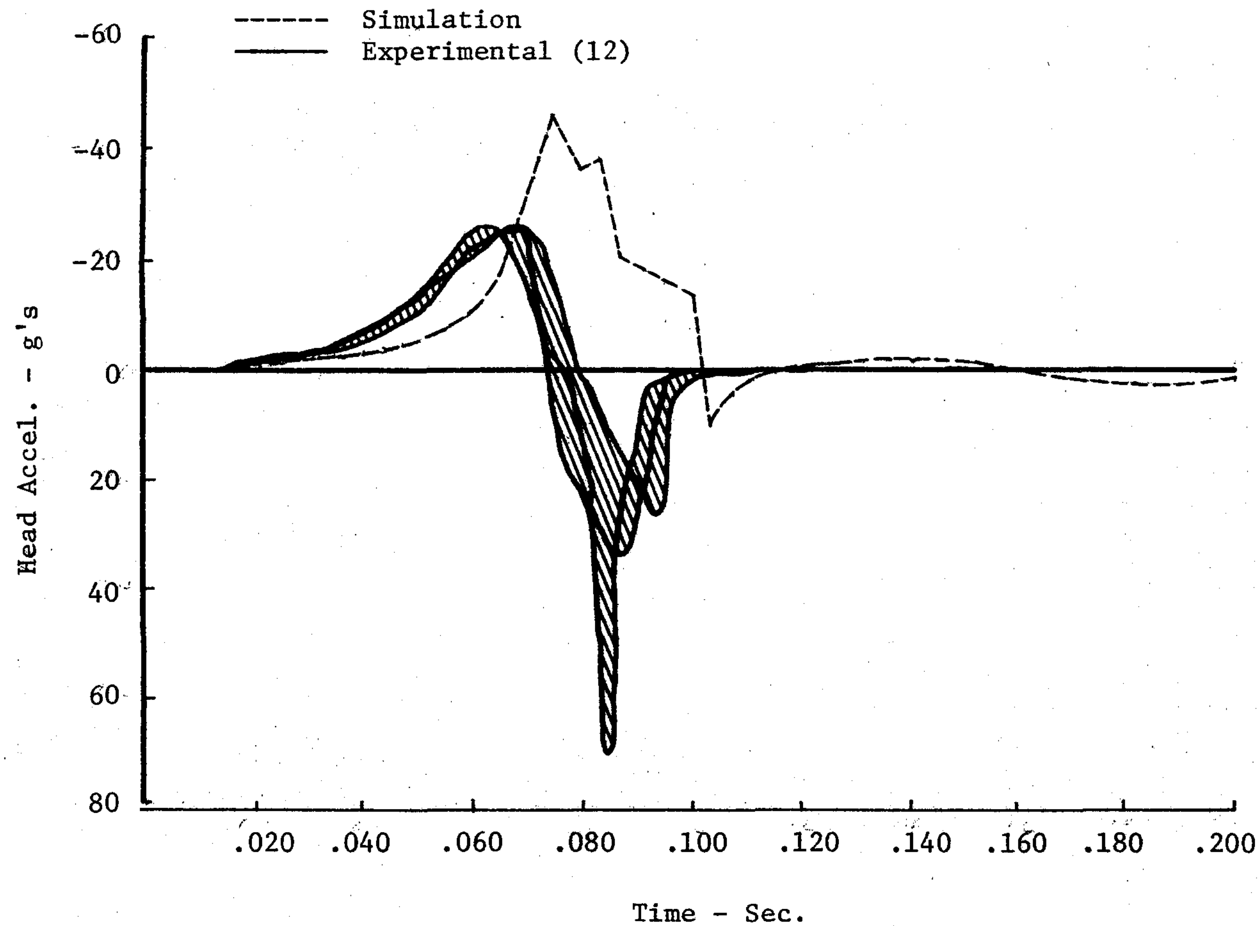

FIGURE 34.-HEAD ACCELERATION IN SEGMENT Z DIRECTION-
LAP RESTRAINT-CART VELOCITY 20 MPH

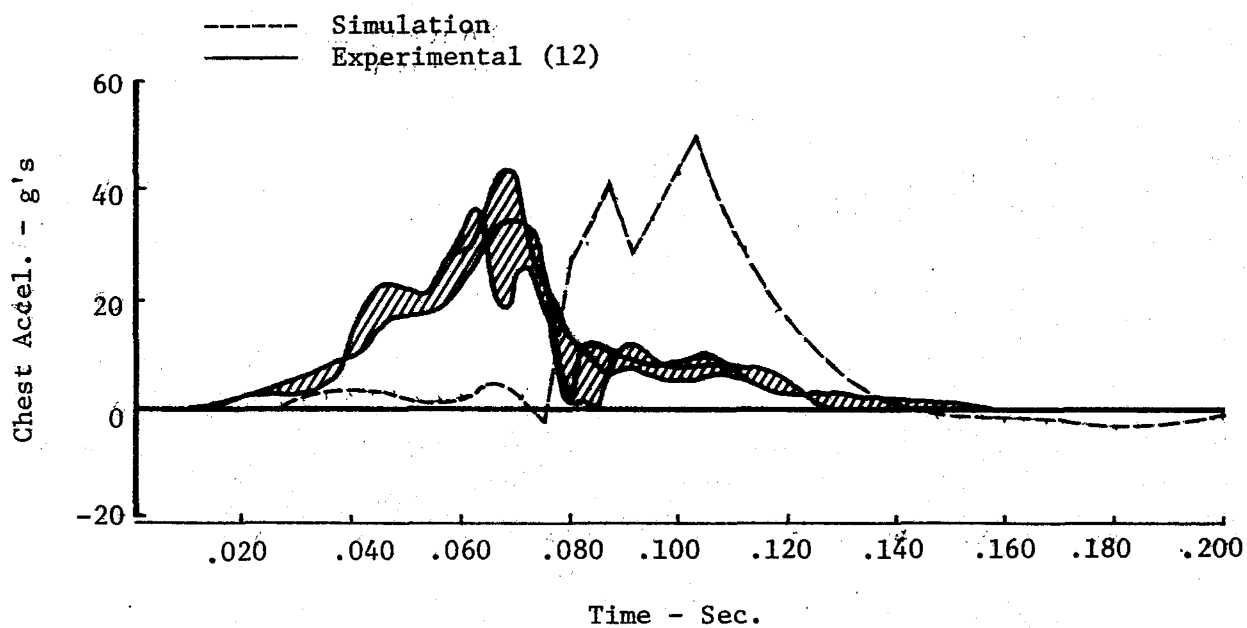

FIGURE 35.-CHEST ACCELERATION IN SEGMENT X DIRECTION-
LAP RESTRAINT-CART VELOCITY 20 MPH

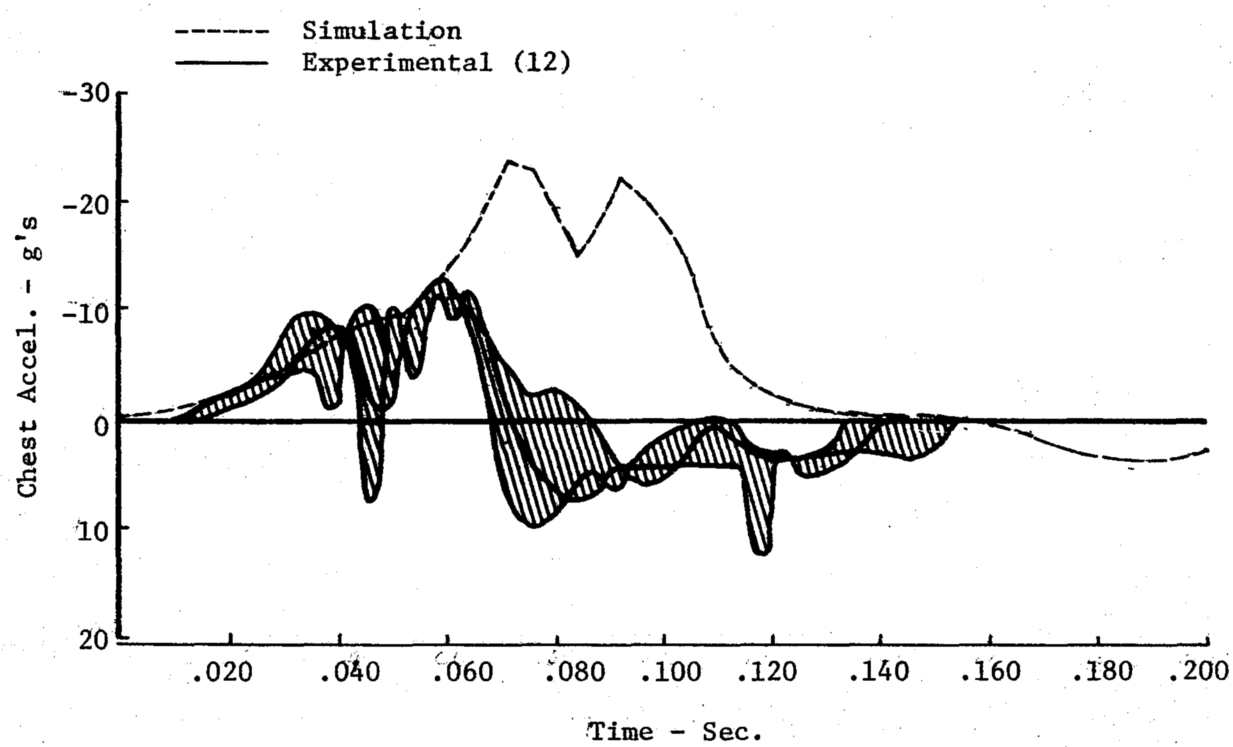

FIGURE 36.-CHEST ACCELERATION IN SEGMENT Z DIRECTION-
LAP RESTRAINT-CART VELOCITY 20 MPH

Response Comparison for Lap and Torso Restraint at 20 MPH Cart Velocity

Figures 37 through 42 show the comparison of dummy kinematics plus head and chest accelerations for the case of lap and torso restraint with 20 MPH cart velocity. Agreement between simulated motion and the high speed film record is good including unsymmetrical body movements as a result of the unsymmetrical torso restraint belt. However the spring action of the torso belt seems to be excessive in the simulation since the arms are whipped back into the seat as shown in the last frame of Figure 38. This test was subject to the same sources of possible discrepancy, as the two cases previously discussed, plus an additional one. The anchor points for the ends of the belt were unknown and therefore estimated for the simulation. This could account for some of the difference in arm kinematics.

The comparison of accelerations, Figures 39 through 42 is also fair but better than the other two cases in that peak values are closer and the shapes of the curves are more compatible with test results.

Closure

The deceleration pattern used in the test (12) approached a 20 g square wave for a duration of about .08 second. It is interesting to note that the passenger experienced levels of acceleration on the order of 80 g's with durations of approximately .03 second for the cases of no restraint and lap restraint; and levels of approximately 40 g's with durations of about .03 second for the case of lap and torso restraint. This points to the fact that in some instances, the response of the vehicle is no indication of what the passenger actually feels, as was stated in the "INTRODUCTION."

EXPERIMENTAL DATA (12) SIMULATION DATA

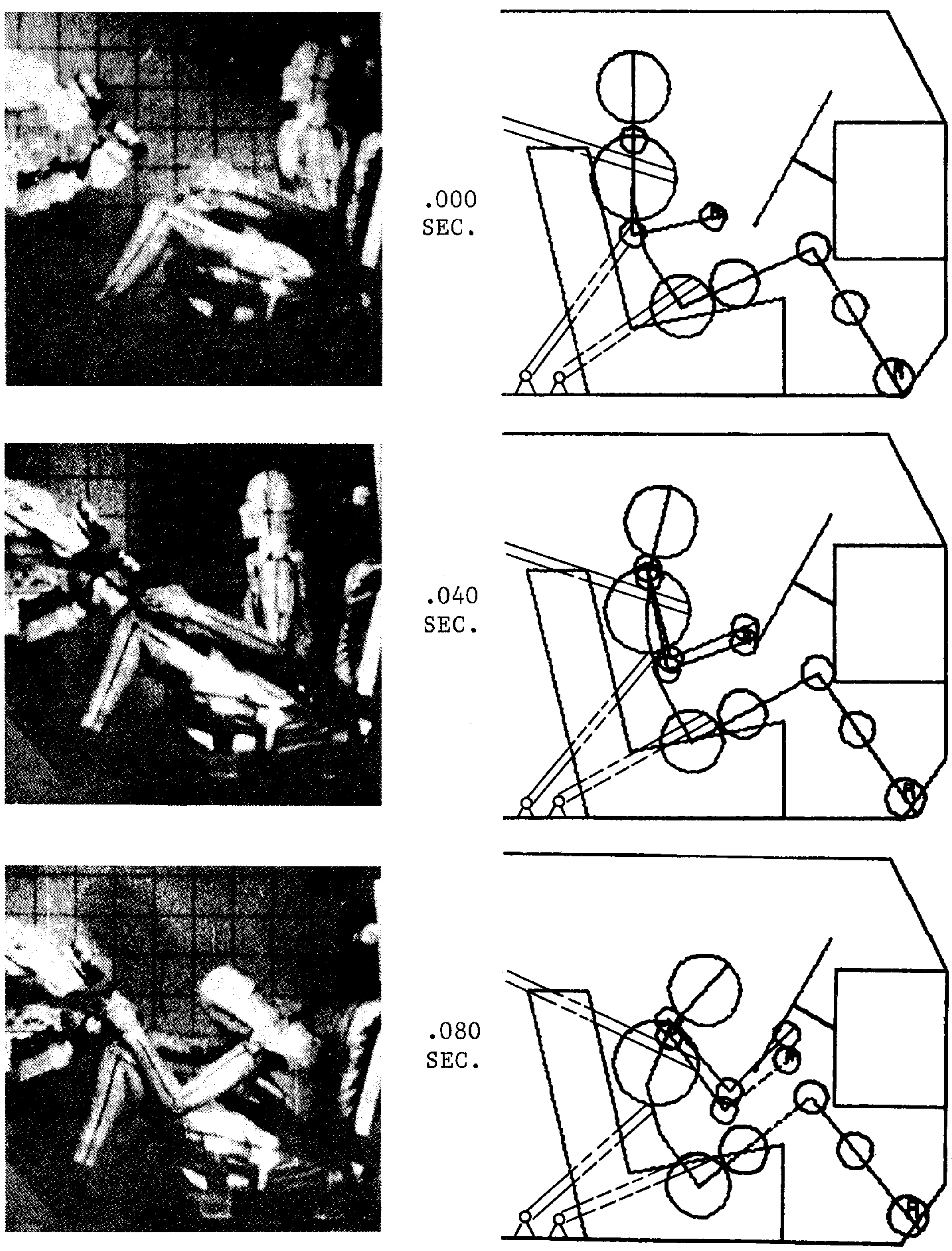

FIGURE 37.-DUMMY KINEMATIC COMPARISON LAP AND TORSO RESTRAINT-CART VELOCITY 20 MPH, .000-.080 SEC.

EXPERIMENTAL DATA (12) SIMULATION DATA

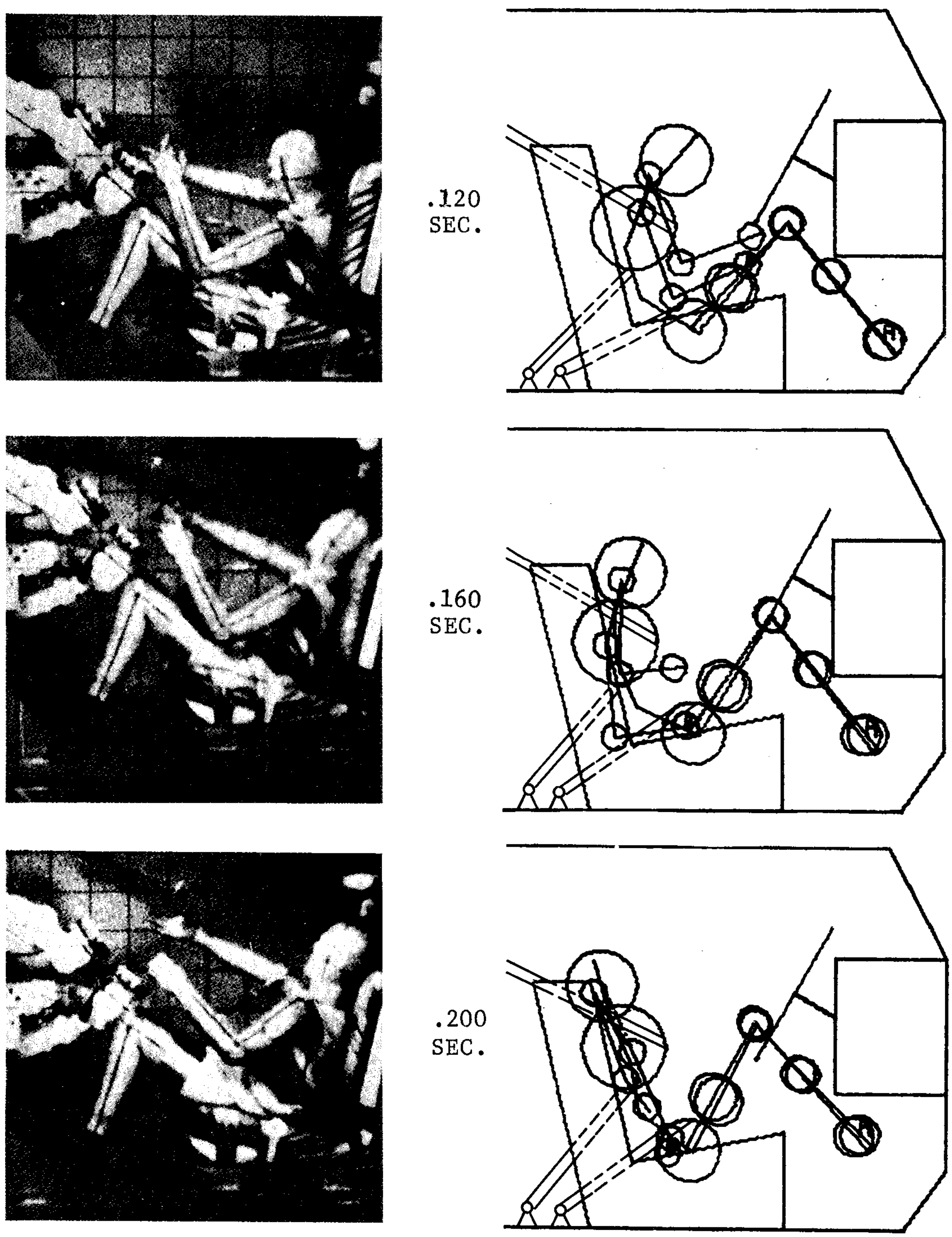

FIGURE 38.-DUMMY KINEMATIC COMPARISON LAP AND TORSO RESTRAINT-CART VELOCITY 20 MPH, .120-.200 SEC.

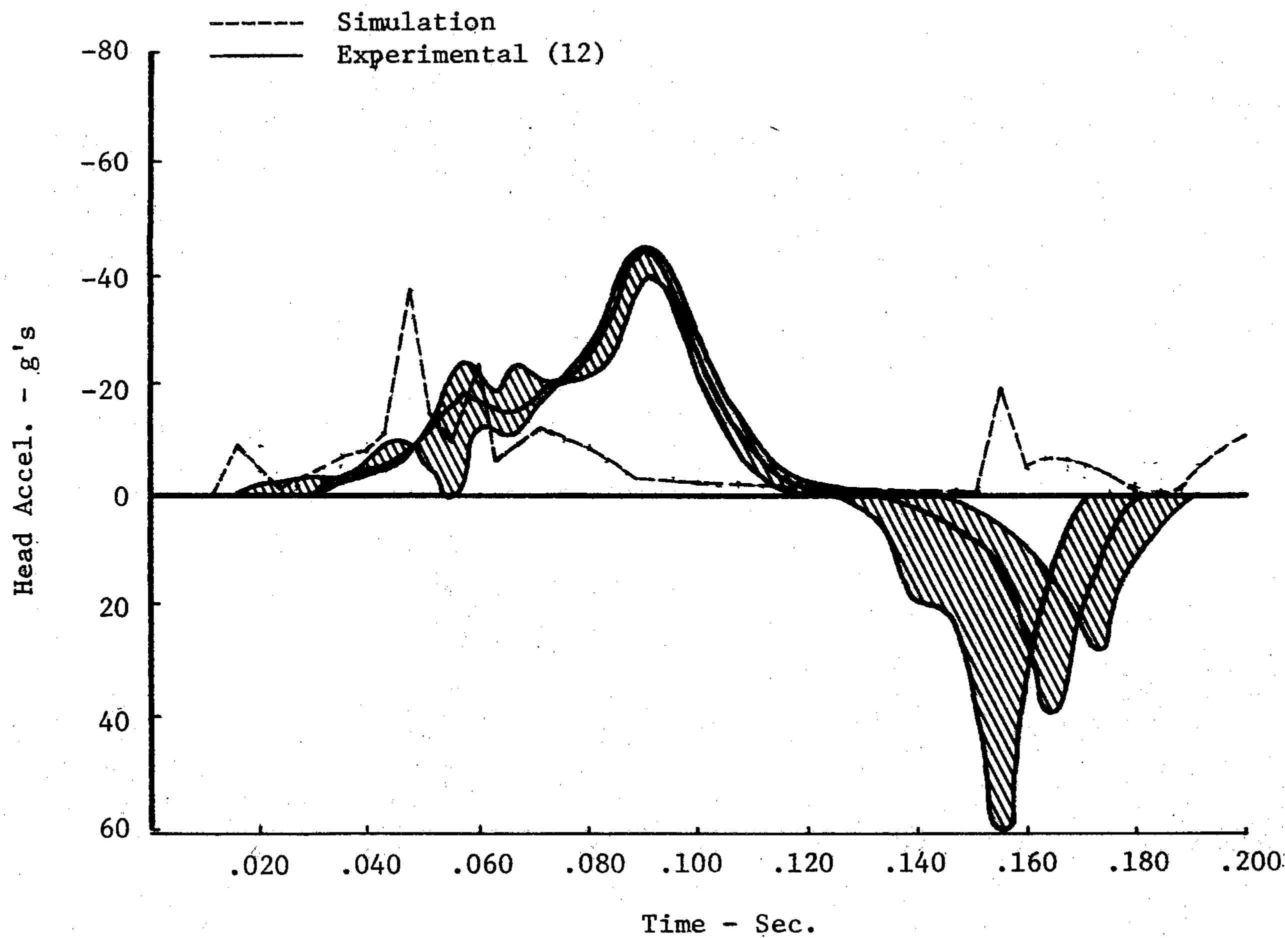

FIGURE 39.-HEAD ACCELERATION IN SEGMENT X DIRECTION-
LAP AND TORSO RESTRAINT-CART VELOCITY 20 MPH

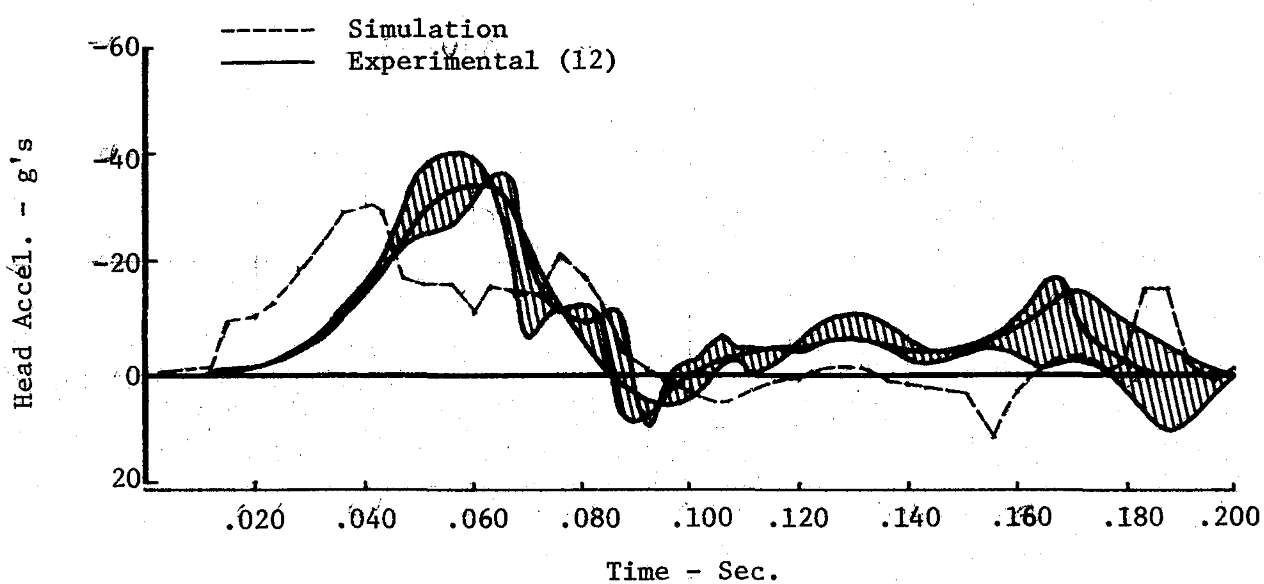

FIGURE 40.-HEAD ACCELERATION IN SEGMENT Z DIRECTION-
LAP AND TORSO RESTRAINT-CART VELOCITY 20 MPH

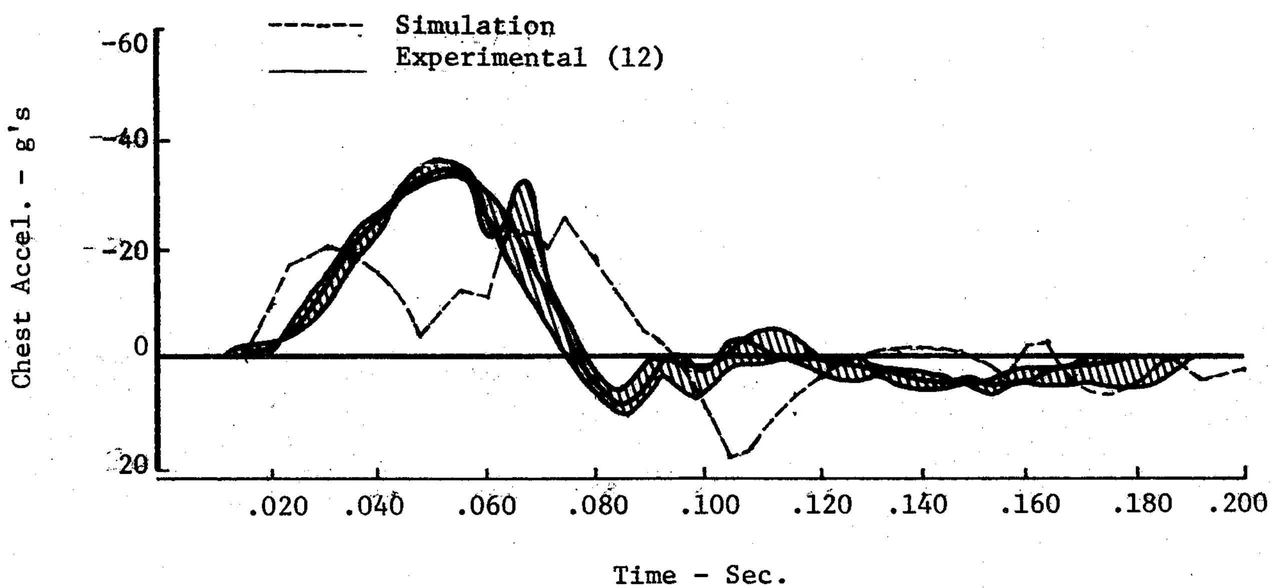

FIGURE 41.-CHEST ACCELERATION IN SEGMENT X DIRECTION-
LAP AND TORSO RESTRAINT-CART VELOCITY 20 MPH

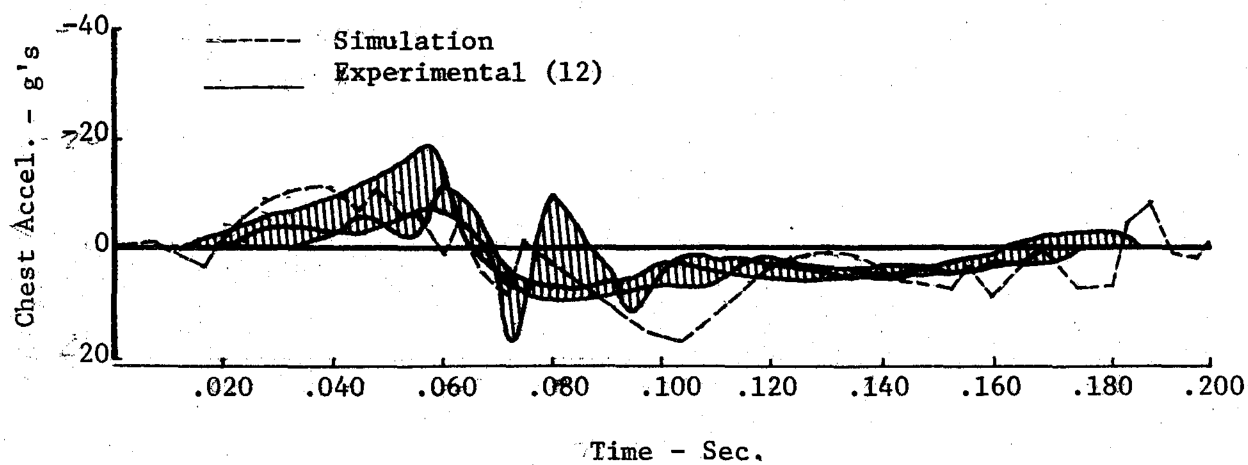

FIGURE 42.-CHEST ACCELERATION IN SEGMENT Z DIRECTION-
LAP AND TORSO RESTRAINT-CART VELOCITY 20 MPH

CONCLUSIONS

The analytical model described herein provides the engineering profession with a useful tool with which to study vehicle and roadway problems culminating in saving lives and reducing occupant injuries. Admittedly, the model was validated for the planar case only; but this in no way precludes its application to three-dimensional motion, especially if qualitative results are being sought.

More specifically the passenger model is a solution to the problem of predicting the motion, acceleration, and forces experienced by a vehicle occupant during a collision or violent maneuver of the vehicle. From this standpoint, the application of the passenger model includes:

1. the evaluation of roadway geometry, i.e., sideslopes, ditches, terrain involving a variation of vertical and horizontal alignment, etc.; roadside safety features such as the breakaway sign, energy absorbing impact cushions, etc.; roadside protective barriers such as guardrails, bridge rails, median barriers, etc.;
2. the design of the vehicle interior and restraint systems;
3. the study of the dynamic behavior of a pedestrian when struck by an automobile; and
4. the study of collisions involving more than one vehicle.

RECOMMENDATIONS FOR EXTENDED RESEARCH

To implement the passenger model to its fullest capacity, consideration should be given to the following:

1. validating the model for the general three-dimensional case utilizing controlled full-scale testing;
2. conducting a parameter study to expose measures necessary to achieve consistent correlation with testing, i.e., damping constants, spring stiffnesses, etc.;
3. compiling a source of force-deformation characteristics for various vehicle interior surfaces through testing;
4. determining the extent to which the vehicle model (9, 11, 15, 20) may be used in conjunction with the passenger model as an integral design tool;
5. extending the model to include passive safety devices, e.g., the air bag, etc.;
6. writing a computer program to predict the dynamic response of a pedestrian struck by an automobile; this would involve only a slight modification of the present passenger program.

APPENDIX I.-REFERENCES

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APPENDIX II.-NOTATION

The following symbols are used in this report:

F_L = loop force in a restraint belt;

$\bar{F}_R$ = restraining force on the body due to a restraint belt;

F_{xi}, F_{yi}, F_{zi} = components of the resultant force on "contact sphere" No. i with respect to the space-fixed coordinate system for the passenger;

g = acceleration due to gravity;

I_{xn}, I_{yn}, I_{zn} = principal mass moments of inertia of body segment No. n with respect to segment-fixed coordinates;

J_n = viscous damping coefficient for body joint No. n;

K_{B1}, K_{B2} = the two slopes of the idealized bilinear force-elongation relation for a restraint belt;

K_{c1}, K_{c2} = the two slopes of the idealized bilinear force-deformation relation for a particular surface of the vehicle interior;

K_1, K_2 = stiffness of the two rotational back springs used to simulate spinal elasticity;

L_i = length of body segment No. i;

L_{Bt} = length of a restraint belt at any time, t;

L_{BO} = initial length of a restraint belt;

L_{xm} = distance of the center of a "contact sphere" from vehicle interior surface No. m;

M_i = mass of body segment No. i;

$\bar{P}_{ij}$ = vector extending from point j to point i;

q_j = the j^{th} generalized coordinate or degree of freedom (j = 1,2...31);

$\dot{q}_j$ = first time derivative of q_j, i.e., the j^{th} generalized velocity;

$\ddot{q}_j$ = second time derivative of q_j, i.e., the j^{th} generalized acceleration;

r_i = the radius of "contact sphere" No. i;

r_{xi}, r_{yi}, r_{zi} = components of the radius vector extending from the center of "contact sphere" No. i to the point of application of a load on the sphere with respect to segment-fixed coordinates;

S_i = the position of "contact sphere" No. i on its respective segment;

U = kinetic energy;

V = potential energy;

X', Y', Z' = the space-fixed cartesian coordinate system for the passenger;

X_n, Y_n, Z_n = the cartesian coordinate system fixed in segment No. n;

X_v, Y_v, Z_v = the cartesian coordinate system fixed at the center of gravity of the vehicle;

X'_n, Y'_n, Z'_n = the coordinates of point n with respect to the space-fixed coordinate system for the passenger;

X'_v, Y'_v, Z'_v = the space-fixed cartesian coordinate system for the vehicle;

X_{vi}, Y_{vi}, Z_{vi} = the coordinates of point i with respect to the vehicle-fixed coordinate system;

$X'_{Ai}, Y'_{Ai}, Z'_{Ai}$ = coordinates of the point of application of a force on "contact sphere" No. i with respect to the space fixed coordinate system for the passenger;

$X'_{T1}, Y'_{T1}, Z'_{T1}$ = coordinates of the reference point on the articulated body (terminal end of segment No. 1) with respect to the space-fixed coordinate system for the passenger; these correspond to generalized coordinates q_1, q_2, and q_3, respectively;

$X'_{vc}, Y'_{vc}, Z'_{vc}$ = coordinates of the vehicle center of gravity with respect to the space-fixed system for the vehicle;

$\dot{X}'_n, \dot{Y}'_n, \dot{Z}'_n$ = components of velocity of the center of mass of body segment No. n with respect to the space-fixed system for the passenger;

α_i = the angle between an extreme body segment (forearm or lower leg) and its adjoining body segment; α_7, α_8, α_{11}, and α_{12} correspond to generalized coordinates q_{22}, q_{23}, q_{30}, and q_{31}, respectively;

β_i = angular displacement of body joint No. i (as in Figure 7);

$\dot{\beta}_i$ = angular velocity of body joint No. i $(\frac{d\beta}{dt})$;

Δ = deformation during a collision between a "contact sphere" and a surface of the vehicle interior;

ΔL_B = change in length (elongation) of a restraint belt;

ϕ_i, θ_i, ψ_i = the Eulerian angles of body segment No. i; these are also generalized coordinates;

ϕ_c, θ_c, ψ_c = the Eulerian angles of the vehicle;

$\dot{\phi}_i, \dot{\theta}_i, \dot{\psi}_i$ = Eulerian angular velocities;

ρ_i = distance from the reference end of body joint No. i to its center of mass;

$\bar{\rho}_i = L_i - \rho_i$

$\omega_{xn}, \omega_{yn}, \omega_{zn}$ = angular velocities of body segment No. n about principal axes fixed in the segment;

$\bar{\omega}_{xm}, \bar{\omega}_{ym}, \bar{\omega}_{zm}$ = angular velocities of the body extremities (forearm and lower leg segments) with respect to principal axes fixed in segment No. m.

APPENDIX III.-EQUATIONS OF MOTION

$$[D]\{\ddot{q}\} = \{E\} + \{F_p\} + \{F_s\} + \{Q_f\} + \{R\} \qquad (1)$$

The above matrix equation represents a set of 31 nonlinear, second-order, simultaneous differential equations describing the motion of the vehicle passenger.

Matrices D and E are strictly a result of kinetic energy considerations; D may be thought of as a pseudo-mass matrix and E as an inertial force matrix.

Column vector $\ddot{q}$ is composed of the generalized accelerations.

Column vector F_p contains the generalized forces due to potential energy of position (gravity loads).

Column vector F_s contains the generalized forces resulting from the potential energy of the rotational springs in the two back joints.

Column vector Q_f is comprised of generalized forces derived from externally applied loads, i.e., nonconservative contact forces from collisions with the vehicle interior.

Column vector R accommodates the nonconservative generalized forces due to frictional resistance in the joints (viscous damping).

In defining the elements of the matrices in Eq. 1, the following notation is used:

$$\{\dot{q}\} = \frac{d}{dt}(\{q\})$$

$$\{\ddot{q}\} = \frac{d}{dt}(\{\dot{q}\})$$

$q_1 = X'_{T1}$	$q_7 = \phi_2$	$q_{13} = \phi_4$	$q_{19} = \phi_6$	$q_{25} = \theta_9$
$q_2 = Y'_{T1}$	$q_8 = \theta_2$	$q_{14} = \theta_4$	$q_{20} = \theta_6$	$q_{26} = \psi_9$
$q_3 = Z'_{T1}$	$q_9 = \psi_2$	$q_{15} = \psi_4$	$q_{21} = \psi_6$	$q_{27} = \phi_{10}$
$q_4 = \phi_1$	$q_{10} = \phi_3$	$q_{16} = \phi_5$	$q_{22} = \alpha_7$	$q_{28} = \theta_{10}$
$q_5 = \theta_1$	$q_{11} = \theta_3$	$q_{17} = \theta_5$	$q_{23} = \alpha_8$	$q_{29} = \psi_{10}$
$q_6 = \psi_1$	$q_{12} = \psi_3$	$q_{18} = \psi_5$	$q_{24} = \phi_9$	$q_{30} = \alpha_{11}$
				$q_{31} = \alpha_{12}$

$s\phi_i = \sin\phi_i$; $c\phi_i = \cos\phi_i$

$s\theta_i = \sin\theta_i$; $c\theta_i = \cos\theta_i$

$s\psi_i = \sin\psi_i$; $c\psi_i = \cos\psi_i$

$s\alpha_i = \sin\alpha_i$; $c\alpha_i = \cos\alpha_i$

During the derivation process, it was discovered that the physical properties of the articulated body (length, mass, and mass-moment of inertia) combined in a repetitive manner thus creating these subsequent constant terms:

$$C_1 = \sum_{n=1}^{12} M_n$$

$$C_2 = -\bar{\rho}_1 M_1 - L_1(M_4 + M_9 + M_{10} + M_{11} + M_{12})$$

$$C_3 = \rho_2 M_2 + L_2(M_3 + M_5 + M_6 + M_7 + M_8)$$

$$C_4 = \rho_3 M_3$$

$$C_5 = -\bar{\rho}_4 M_4 - L_4(M_9 + M_{10} + M_{11} + M_{12})$$

$$c_6 = -\bar{\rho}_5 M_5 - L_5 M_7$$

$$c_7 = -\bar{\rho}_6 M_6 - L_6 M_8$$

$$c_8 = -\bar{\rho}_7 M_7$$

$$c_9 = -\bar{\rho}_8 M_8$$

$$c_{10} = -\bar{\rho}_9 M_9 - L_9 M_{11}$$

$$c_{11} = -\bar{\rho}_{10} M_{10} - L_{10} M_{12}$$

$$c_{12} = -\bar{\rho}_{11} M_{11}$$

$$c_{13} = -\bar{\rho}_{12} M_{12}$$

$$c_{14} = L_5 (M_5 - M_6 + M_7 - M_8)$$

$$c_{15} = L_H (M_9 - M_{10} + M_{11} - M_{12})$$

$$c_{16} = M_1 \bar{\rho}_1^2 + L_1^2 (M_4 + M_9 + M_{10} + M_{11} + M_{12})$$

$$c_{17} = M_2 \rho_2^2 + L_2^2 (M_3 + M_5 + M_6 + M_7 + M_8)$$

$$c_{18} = M_3 \rho_3^2$$

$$c_{19} = M_4 \bar{\rho}_4^2 + L_4^2 (M_9 + M_{10} + M_{11} + M_{12})$$

$$c_{20} = L_5^2 (M_5 + M_6 + M_7 + M_8)$$

$$c_{21} = M_5 \bar{\rho}_5^2 + M_7 L_5^2$$

$$c_{22} = M_6 \bar{\rho}_6^2 + M_8 L_6^2$$

$$c_{23} = M_7 \bar{\rho}_7^2$$

$$c_{24} = M_8 \bar{\rho}_8^2$$

$$c_{25} = L_H^2 (M_9 + M_{10} + M_{11} + M_{12})$$

$$c_{26} = M_9 \bar{\rho}_9^2 + M_{11} L_9^2$$

$$c_{27} = M_{10} \bar{\rho}_{10}^2 + M_{12} L_{10}^2$$

$$C_{28} = M_{11}\,\bar{\rho}_{11}^{2}$$

$$C_{29} = M_{12}\,\bar{\rho}_{12}^{2}$$

Following is the upper triangular portion of the symmetric D matrix (by rows), excluding zero terms:

$$D_{1\,1} = C_1$$

$$D_{1\,4} = -C_2\ s\phi_1 s\theta_1$$

$$D_{1\,5} = C_2\ c\phi_1\ c\theta_1$$

$$D_{1\,7} = -C_3\ s\phi_2\ s\theta_2 + C_{14}\ (s\phi_2\ c\theta_2\ s\psi_2 - c\phi_2\ c\psi_2)$$

$$D_{1\,8} = C_3\ c\phi_2\ c\theta_2 + C_{14}\ \ c\phi_2\ s\theta_2\ s\psi_2$$

$$D_{1\,9} = C_{14}\ (s\phi_2\ s\psi_2 - c\phi_2\ c\theta_2\ c\psi_2)$$

$$D_{1\,10} = -C_4\ \ s\phi_3\ s\theta_3$$

$$D_{1\,11} = C_4\ \ c\phi_3\ c\theta_3$$

$$D_{1\,13} = -C_5\ s\phi_4\ s\theta_4\ + C_{15}(s\phi_4\ c\theta_4\ s\psi_4 - c\phi_4\ c\psi_4)$$

$$D_{1\,14} = C_5\ c\phi_4\ c\theta_4 + C_{15}\ \ c\phi_4\ s\theta_4\ s\psi_4$$

$$D_{1\,15} = C_{15}\ (s\phi_4\ s\psi_4\ - c\phi_4\ c\theta_4\ c\psi_4)$$

$$D_{1\,16} = -C_6\ s\phi_5\ s\theta_5 - C_8\ (s\phi_5\ c\theta_5\ c\psi_5\ s\alpha_7 + c\phi_5\ s\psi_5\ s\alpha_7 + s\phi_5\ \ s\theta_5\ \ c\alpha_7)$$

$$D_{1\,17} = C_6\ c\phi_5\ c\theta_5\ + C_8\ (c\phi_5\ c\theta_5\ c\alpha_7 - c\phi_5\ s\theta_5\ c\psi_5\ s\alpha_7)$$

$$D_{1\,18} = -C_8\ (c\phi_5\ c\theta_5\ s\psi_5\ s\alpha_7 + s\phi_5\ c\psi_5\ s\alpha_7)$$

$$D_{1\,19} = -C_7\, s\phi_6\, s\theta_6 - C_9\,(s\phi_6\, c\theta_6\, c\psi_6\, s\alpha_8 + c\phi_6\, s\psi_6\, s\alpha_8 + s\phi_6\, s\theta_6\, c\alpha_8)$$

$$D_{1\,20} = C_7\, c\phi_6\, c\theta_6 + C_9\,(c\phi_6\, c\theta_6\, c\alpha_8 - c\phi_6\, s\theta_6\, c\psi_6\, s\alpha_8)$$

$$D_{1\,21} = -C_9\,(c\phi_6\, c\theta_6\, s\psi_6\, s\alpha_8 + s\phi_6\, c\psi_6\, s\alpha_8)$$

$$D_{1\,22} = C_8\,(c\phi_5\, c\theta_5\, c\psi_5\, c\alpha_7 - s\phi_5\, s\psi_5\, c\alpha_7 - c\phi_5\, s\theta_5\, s\alpha_7)$$

$$D_{1\,23} = C_9\,(c\phi_6\, c\theta_6\, c\psi_6\, c\alpha_8 - s\phi_6\, s\psi_6\, c\alpha_8 - c\phi_6\, s\theta_6\, s\alpha_8)$$

$$D_{1\,24} = -C_{10}\, s\phi_9\, s\theta_9 - C_{12}\,(s\phi_9\, c\theta_9\, c\psi_9\, s\alpha_{11} + c\phi_9\, s\psi_9\, s\alpha_{11} + s\phi_9\, s\theta_9\, c\alpha_{11})$$

$$D_{1\,25} = C_{10}\, c\phi_9\, c\theta_9 + C_{12}\,(c\phi_9\, c\theta_9\, c\alpha_{11} - c\phi_9\, s\theta_9\, c\psi_9\, s\alpha_{11})$$

$$D_{1\,26} = -C_{12}\,(c\phi_9\, c\theta_9\, s\psi_9\, s\alpha_{11} + s\phi_9\, c\psi_9\, s\alpha_{11})$$

$$D_{1\,27} = -C_{11}\, s\phi_{10}\, s\theta_{10} - C_{13}\,(s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\psi_{10}\, s\alpha_{12} + s\phi_{10}\, s\theta_{10}\, c\alpha_{12})$$

$$D_{1\,28} = C_{11}\, c\phi_{10}\, c\theta_{10} + C_{13}\,(c\phi_{10}\, c\theta_{10}\, c\alpha_{12} - c\phi_{10}\, s\theta_{10}\, c\psi_{10}\, s\alpha_{12})$$

$$D_{1\,29} = -C_{13}\,(c\phi_{10}\, c\theta_{10}\, s\psi_{10}\, s\alpha_{12} + s\phi_{10}\, c\psi_{10}\, s\alpha_{12})$$

$$D_{1\,30} = C_{12}\,(c\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} - s\phi_9\, s\psi_9\, c\alpha_{11} - c\phi_9\, s\theta_9\, s\alpha_{11})$$

$$D_{1\,31} = C_{13}\,(c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, c\alpha_{12} - s\phi_{10}\, s\psi_{10}\, c\alpha_{12} - c\phi_{10}\, s\theta_{10}\, s\alpha_{12})$$

$$D_{2\,2} = C_1$$

$$D_{2\,4} = C_2\, c\phi_1\, s\theta_1$$

$$D_{2\,5} = C_2\, s\phi_1\, c\theta_1$$

$$D_{2\,7} = C_3\, c\phi_2\, s\theta_2 - C_{14}(c\phi_2\, c\theta_2\, s\psi_2 + s\phi_2\, c\psi_2)$$

$$D_{2\,8} = C_3\, s\phi_2\, c\theta_2 + C_{14}\, s\phi_2\, s\theta_2\, s\psi_2$$

$$D_{2\,9} = -C_{14}(s\phi_2\, c\theta_2\, c\psi_2 + c\phi_2\, s\psi_2)$$

$$D_{2\,10} = C_4\, c\phi_3\, s\theta_3$$

$$D_{2\,11} = C_4\, s\phi_3\, c\theta_3$$

$$D_{2\,13} = C_5\, c\phi_4\, s\theta_4 - C_{15}(c\phi_4\, c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4)$$

$$D_{2\,14} = C_5\, s\phi_4\, c\theta_4 + C_{15}\, s\phi_4\, s\theta_4\, s\psi_4$$

$$D_{2\,15} = -C_{15}(s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4)$$

$$D_{2\,16} = C_6\, c\phi_5\, s\theta_5 + C_8(c\phi_5\, c\theta_5\, c\psi_5\, s\alpha_7 - s\phi_5\, s\psi_5\, s\alpha_7 + c\phi_5\, s\theta_5\, c\alpha_7)$$

$$D_{2\,17} = C_6\, s\phi_5\, c\theta_5 + C_8(s\phi_5\, c\theta_5\, c\alpha_7 - s\phi_5\, s\theta_5\, c\psi_5\, s\alpha_7)$$

$$D_{2\,18} = C_8(c\phi_5\, c\psi_5\, s\alpha_7 - s\phi_5\, c\theta_5\, s\psi_5\, s\alpha_7)$$

$$D_{2\,19} = C_7\, c\phi_6\, s\theta_6 + C_9(c\phi_6\, c\theta_6\, c\psi_6\, s\alpha_8 - s\phi_6\, s\psi_6\, s\alpha_8 + c\phi_6\, s\theta_6\, c\alpha_8)$$

$$D_{2\,20} = C_7\, s\phi_6\, c\theta_6 + C_9(s\phi_6\, c\theta_6\, c\alpha_8 - s\phi_6\, s\theta_6\, c\psi_6\, s\alpha_8)$$

$$D_{2\,21} = C_9(c\phi_6\, c\psi_6\, s\alpha_8 - s\phi_6\, c\theta_6\, s\psi_6\, s\alpha_8)$$

$$D_{2\,22} = C_8(s\phi_5\, c\theta_5\, c\psi_5\, c\alpha_7 + c\phi_5\, s\psi_5\, c\alpha_7 - s\phi_5\, s\theta_5\, s\alpha_7)$$

$$D_{2\,23} = C_9 (s\phi_6\, c\theta_6\, c\psi_6\, c\alpha_8 + c\phi_6\, s\psi_6\, c\alpha_8 - s\phi_6\, s\theta_6\, s\alpha_8)$$

$$D_{2\,24} = C_{10}\, c\phi_9\, s\theta_9 + C_{12} (c\phi_9\, c\theta_9\, c\psi_9\, s\alpha_{11} - s\phi_9\, s\psi_9\, s\alpha_{11} + c\phi_9\, s\theta_9\, c\alpha_{11})$$

$$D_{2\,25} = C_{10}\, s\phi_9\, c\theta_9 + C_{12} (s\phi_9\, c\theta_9\, c\alpha_{11} - s\phi_9\, s\theta_9\, c\psi_9\, s\alpha_{11})$$

$$D_{2\,26} = C_{12} (c\phi_9\, c\psi_9\, s\alpha_{11} - s\phi_9\, c\theta_9\, s\psi_9\, s\alpha_{11})$$

$$D_{2\,27} = C_{11}\, c\phi_{10}\, s\theta_{10} + C_{13} (c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} - s\phi_{10}\, s\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\theta_{10}\, c\alpha_{12})$$

$$D_{2\,28} = C_{11}\, s\phi_{10}\, c\theta_{10} + C_{13} (s\phi_{10}\, c\theta_{10}\, c\alpha_{12} - s\phi_{10}\, s\theta_{10}\, c\psi_{10}\, s\alpha_{12})$$

$$D_{2\,29} = C_{13} (c\phi_{10}\, c\psi_{10}\, s\alpha_{12} - s\phi_{10}\, c\theta_{10}\, s\psi_{10}\, s\alpha_{12})$$

$$D_{2\,30} = C_{12} (s\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} + c\phi_9\, s\psi_9\, c\alpha_{11} - s\phi_9\, s\theta_9\, s\alpha_{11})$$

$$D_{2\,31} = C_{13} (s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, c\alpha_{12} + c\phi_{10}\, s\psi_{10}\, c\alpha_{12} - s\phi_{10}\, s\theta_{10}\, s\alpha_{12})$$

$$D_{3\,3} = C_1$$

$$D_{3\,5} = -C_2\, s\theta_1$$

$$D_{3\,8} = -C_3\, s\theta_2 + C_{14}\, c\theta_2\, s\psi_2$$

$$D_{3\,9} = C_{14}\, s\theta_2\, c\psi_2$$

$$D_{3\,11} = -C_4\, s\theta_3$$

$$D_{3\,14} = -C_5\, s\theta_4 + C_{15}\, c\theta_4\, s\psi_4$$

$$D_{3\,15} = C_{15}\, s\theta_4\, c\psi_4$$

$$D_{3\,17} = -C_6\, s\theta_5 - C_8 (c\theta_5\, c\psi_5\, s\alpha_7 + s\theta_5\, c\alpha_7)$$

$$D_{3\,18} = C_8\, s\theta_5\, s\psi_5\, s\alpha_7$$

$$D_{3\,20} = -C_7\, s\theta_6 - C_9 (c\theta_6\, c\psi_6\, s\alpha_8 + s\theta_6\, c\alpha_8)$$

$$D_{3\,21} = C_9\, s\theta_6\, s\psi_6\, s\alpha_8$$

$$D_{3\,22} = -C_8 (s\theta_5\, c\psi_5\, c\alpha_7 + c\theta_5\, s\alpha_7)$$

$$D_{3\,23} = -C_9 (s\theta_6\, c\psi_6\, c\alpha_8 + c\theta_6\, s\alpha_8)$$

$$D_{3\,25} = -C_{10}\, s\theta_9 - C_{12} (c\theta_9\, c\psi_9\, s\alpha_{11} + s\theta_9\, c\alpha_{11})$$

$$D_{3\,26} = C_{12}\, s\theta_9\, s\psi_9\, s\alpha_{11}$$

$$D_{3\,28} = -C_{11}\, s\theta_{10} - C_{13} (c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + s\theta_{10}\, c\alpha_{12})$$

$$D_{3\,29} = C_{13}\, s\theta_{10}\, s\psi_{10}\, s\alpha_{12}$$

$$D_{3\,30} = -C_{12} (s\theta_9\, c\psi_9\, c\alpha_{11} + c\theta_9\, s\alpha_{11})$$

$$D_{3\,31} = -C_{13} (s\theta_{10}\, c\psi_{10}\, c\alpha_{12} + c\theta_{10}\, s\alpha_{12})$$

$$D_{4\,4} = s^2\theta_1 [C_{16} + I_{x_1} c^2\psi_1 + I_{y_1} s^2\psi_1] + I_{z_1} c^2\theta_1$$

$$D_{4\,5} = (I_{y_1} - I_{x_1})\, s\psi_1\, c\psi_1\, s\theta_1$$

$$D_{4\,6} = I_{z_1}\, c\theta_1$$

$$D_{4\,13} = L_1\, s\theta_1 \{C_{15} [(s\phi_4\, c\theta_4\, s\psi_4 - c\phi_4\, c\psi_4)\, s\phi_1 + (c\phi_4\, c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4)\, c\phi_1] - C_5\, s\theta_4 (s\phi_4\, s\phi_1 + c\phi_4\, c\phi_1)\}$$

$$D_{4\,14} = L_1 s\theta_1 (c\phi_4 s\phi_1 - s\phi_4 c\phi_1)[C_5 c\theta_4 + C_{15} s\theta_4 s\psi_4]$$

$$D_{4\,15} = L_1 C_{15} s\theta_1 [(s\phi_4 s\psi_4 - c\phi_4 c\theta_4 c\psi_4) s\phi_1 + (s\phi_4 c\theta_4 c\psi_4 + c\phi_4 s\psi_4) c\phi_1]$$

$$D_{4\,24} = -L_1 s\theta_1 \{C_{10} s\theta_9 (s\phi_9 s\phi_1 + c\phi_9 c\phi_1) + C_{12} [(s\phi_9 s\theta_9 c\alpha_{11} + s\phi_9 c\theta_9 c\psi_9 s\alpha_{11} + c\phi_9 s\psi_9 s\alpha_{11}) s\phi_1 + (c\phi_9 c\theta_9 c\psi_9 s\alpha_{11} - s\phi_9 s\psi_9 s\alpha_{11} + c\phi_9 s\theta_9 c\alpha_{11}) c\phi_1]\}$$

$$D_{4\,25} = L_1 s\theta_1 \{(c\phi_9 s\phi_1 - c\phi_1 s\phi_9)[C_{10} c\theta_9 + C_{12} (c\theta_9 c\alpha_{11} - s\theta_9 c\psi_9 s\alpha_{11})]\}$$

$$D_{4\,26} = -L_1 C_{12} s\alpha_{11} s\theta_1 [(c\phi_9 c\theta_9 s\psi_9 + s\phi_9 c\psi_9) s\phi_1 + (c\phi_9 c\psi_9 - s\phi_9 c\theta_9 s\psi_9) c\phi_1]$$

$$D_{4\,27} = -L_1 s\theta_1 \{C_{11} s\theta_{10} (s\phi_{10} s\phi_1 + c\phi_{10} c\phi_1) + C_{13} [(s\phi_{10} c\theta_{10} c\psi_{10} s\alpha_{12} + c\phi_{10} s\psi_{10} s\alpha_{12} + s\phi_{10} s\theta_{10} c\alpha_{12}) s\phi_1 + (c\phi_{10} c\theta_{10} c\psi_{10} s\alpha_{12} - s\phi_{10} s\psi_{10} s\alpha_{12} + c\phi_{10} s\theta_{10} c\alpha_{12}) c\phi_1]\}$$

$$D_{4\,28} = L_1 s\theta_1 \{(c\phi_{10} s\phi_1 - s\phi_{10} c\phi_1)[C_{11} c\theta_{10} + C_{13} (c\theta_{10} c\alpha_{12} - s\theta_{10} c\psi_{10} s\alpha_{12})]\}$$

$$D_{4\,29} = -L_1 C_{13} s\alpha_{12} s\theta_1 [(c\phi_{10} c\theta_{10} s\psi_{10} + s\phi_{10} c\psi_{10}) s\phi_1 + (c\phi_{10} c\psi_{10} - s\phi_{10} c\theta_{10} s\psi_{10}) c\phi_1]$$

$$D_{4\,30} = L_1 C_{12} s\theta_1 [(c\phi_9 c\theta_9 c\psi_9 c\alpha_{11} - s\phi_9 s\psi_9 c\alpha_{11} - c\phi_9 s\theta_9 s\alpha_{11}) s\phi_1 - (s\phi_9 c\theta_9 c\psi_9 c\alpha_{11} + c\phi_9 s\psi_9 c\alpha_{11} - s\phi_9 s\theta_9 s\alpha_{11}) c\phi_1]$$

$$D_{4\,31} = L_1 C_{13} s\theta_1 [(c\phi_{10} c\theta_{10} c\psi_{10} c\alpha_{12} - s\phi_{10} s\psi_{10} c\alpha_{12} - c\phi_{10} s\theta_{10} s\alpha_{12}) s\phi_1 - (s\phi_{10} c\theta_{10} c\psi_{10} c\alpha_{12} + c\phi_{10} s\psi_{10} c\alpha_{12} - s\phi_{10} s\theta_{10} s\alpha_{12}) c\phi_1)]$$

$$D_{55} = C_{16} + I_{x_1} s^2\psi_1 + I_{y_1} c^2\psi_1$$

$$D_{5\,13} = L_1 c\theta_1 \{C_5 s\theta_4 (s\phi_4 c\phi_1 - c\phi_4 s\phi_1) - C_{15}[(s\phi_4 c\theta_4 s\psi_4 - c\phi_4 c\psi_4) c\phi_1 - (c\phi_4 c\theta_4 s\psi_4 + s\phi_4 c\psi_4) s\phi_1]\}$$

$$D_{5\,14} = -L_1 \{(C_5 c\theta_4 + C_{15} s\theta_4 s\psi_4)(c\phi_1 c\theta_1 c\phi_4 + s\phi_1 c\theta_1 s\phi_4) + s\theta_1 (C_5 s\theta_4 - C_{15} c\theta_4 s\psi_4)\}$$

$$D_{5\,15} = -L_1 C_{15} \{-c\psi_4 [s\theta_4 s\theta_1 + c\theta_1 c\theta_4 (c\phi_4 c\phi_1 + s\phi_4 s\phi_1)] + s\psi_4 c\theta_1 (s\phi_4 c\phi_1 - c\phi_4 s\phi_1)\}$$

$$D_{5\,24} = L_1 c\theta_1 \{C_{10} s\theta_9 [s\phi_9 c\phi_1 - c\phi_9 s\phi_1] + C_{12} [(s\phi_9 c\theta_9 c\psi_9 s\alpha_{11} + c\phi_9 s\psi_9 s\alpha_{11} + s\phi_9 s\theta_9 c\alpha_{11}) c\phi_1 - (c\phi_9 c\theta_9 c\psi_9 s\alpha_{11} - s\phi_9 s\psi_9 s\alpha_{11} + c\phi_9 s\theta_9 c\alpha_{11}) s\phi_1]\}$$

$$D_{5\,25} = -L_1 \{c\theta_1 [C_{10} c\theta_9 + C_{12} (c\theta_9 c\alpha_{11} - s\theta_9 c\psi_9 s\alpha_{11})][c\phi_1 c\phi_9 + s\phi_1 s\phi_9] + s\theta_1 [C_{10} s\theta_9 + C_{12} (c\theta_9 c\psi_9 s\alpha_{11} + s\theta_9 c\alpha_{11})]\}$$

$$D_{5\,26} = L_1 C_{12} \{s\alpha_{11} [c\theta_1 \langle (c\phi_9 c\theta_9 s\psi_9 + s\phi_9 c\psi_9) c\phi_1 - (c\phi_9 c\psi_9 - s\phi_9 c\theta_9 s\psi_9) s\phi_1 \rangle + s\theta_9 s\psi_9 s\theta_1]\}$$

$$D_{5\,27} = L_1 c\theta_1 [C_{11} s\theta_{10} (s\phi_{10} c\phi_1 - c\phi_{10} s\phi_1) + C_{13} \langle (s\phi_{10} c\theta_{10} c\psi_{10} s\alpha_{12} + c\phi_{10} s\psi_{10} s\alpha_{12} + s\phi_{10} s\theta_{10} c\alpha_{12}) c\phi_1 - s\phi_1 (c\phi_{10} c\theta_{10} c\psi_{10} s\alpha_{12} - s\phi_{10} s\psi_{10} s\alpha_{12} + c\phi_{10} s\theta_{10} c\alpha_{12}) \rangle]$$

$$D_{5\,28} = -L_1 \{ c\theta_1 [C_{11} c\theta_{10} + C_{13} (c\theta_{10} c\alpha_{12} - s\theta_{10} c\psi_{10} s\alpha_{12})] [c\phi_{10} c\phi_1 + s\phi_{10} s\phi_1] + s\theta_1 [C_{11} s\theta_{10} + C_{13} (c\theta_{10} c\psi_{10} s\alpha_{12} + s\theta_{10} c\alpha_{12}] \}$$

$$D_{5\,29} = L_1 C_{13} s\alpha_{12} [c\theta_1 \langle (c\phi_{10} c\theta_{10} s\psi_{10} + s\phi_{10} c\psi_{10}) c\phi_1 - (c\phi_{10} c\psi_{10} - s\phi_{10} c\theta_{10} s\psi_{10}) s\phi_1 \rangle + s\theta_{10} s\psi_{10} s\theta_1]$$

$$D_{5\,30} = -L_1 C_{12} \{ c\theta_1 [(c\phi_9 c\theta_9 c\psi_9 c\alpha_{11} - s\phi_9 s\psi_9 c\alpha_{11} - c\phi_9 s\theta_9 s\alpha_{11}) c\phi_1 + (s\phi_9 c\theta_9 c\psi_9 c\alpha_{11} + c\phi_9 s\psi_9 c\alpha_{11} - s\phi_9 s\theta_9 s\alpha_{11}) s\phi_1] + (s\theta_9 c\psi_9 c\alpha_{11} + c\theta_9 s\alpha_{11}) s\theta_1 \}$$

$$D_{5\,31} = -L_1 C_{13} \{ c\theta_1 [(c\phi_{10} c\theta_{10} c\psi_{10} c\alpha_{12} - s\phi_{10} s\psi_{10} c\alpha_{12} - c\phi_{10} s\theta_{10} s\alpha_{12}) c\phi_1 + (s\phi_{10} c\theta_{10} c\psi_{10} c\alpha_{12} + c\phi_{10} s\psi_{10} c\alpha_{12} - s\phi_{10} s\theta_{10} s\alpha_{12}) s\phi_1] + (s\theta_{10} c\psi_{10} c\alpha_{12} + c\theta_{10} s\alpha_{12}) s\theta_1 \}$$

$$D_{6\,6} = I_{z_1}$$

$$D_{7\,7} = s^2\theta_2 (C_{17} + I_{x_2} c^2\psi_2 + I_{y_2} s^2\psi_2) - 2 L_2 C_{14} s\theta_2 c\theta_2 s\psi_2 + I_{z_2} c^2\theta_2 + C_{20} [c^2\theta_2 s^2\psi_2 + c^2\psi_2]$$

$$D_{7\,8} = c\psi_2 [s\theta_2 s\psi_2 (I_{y_2} - I_{x_2} - C_{20}) - L_2 C_{14} c\theta_2]$$

$$D_{7\,9} = c\theta_2 (C_{20} + I_{z_2}) - L_2 C_{14} s\theta_2 s\psi_2$$

$$D_{7\,10} = L_2 C_4 s\theta_2 s\theta_3 (s\phi_3 s\phi_2 + c\phi_3 c\phi_2)$$

$$D_{7\,11} = L_2 C_4 c\theta_3 s\theta_2 (s\phi_3 c\phi_2 - c\phi_3 s\phi_2)$$

$$D_{7\,16} = [L_2 s\phi_2 s\theta_2 - L_5 (s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2)] [C_6 s\phi_5 s\theta_5 + C_8 (s\phi_5 c\theta_5 c\psi_5 s\alpha_7 + c\phi_5 s\psi_5 s\alpha_7 + s\phi_5 s\theta_5 c\alpha_7)] + [L_2 c\phi_2 s\theta_2 - L_5 (c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)][C_6 c\phi_5 s\theta_5 + C_8 (c\phi_5 c\theta_5 c\psi_5 s\alpha_7 - s\phi_5 s\psi_5 s\alpha_7 + c\phi_5 s\theta_5 c\alpha_7)]$$

$$D_{7\,17} = [C_6 c\theta_5 + C_8 (c\theta_5 c\alpha_7 - s\theta_5 c\psi_5 s\alpha_7)] \{c\phi_5 [L_5 (s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2) - L_2 s\phi_2 s\theta_2] + s\phi_5 [L_2 c\phi_2 s\theta_2 - L_5 (c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)]\}$$

$$D_{7\,18} = C_8 s\alpha_7 \{(c\phi_5 c\theta_5 s\psi_5 + s\phi_5 c\psi_5)[L_2 s\phi_2 s\theta_2 - L_5 (s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2)] + (c\phi_5 c\psi_5 - s\phi_5 c\theta_5 s\psi_5)[L_2 c\phi_2 s\theta_2 - L_5 (c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)]\}$$

$$D_{7\,19} = [L_2 s\phi_2 s\theta_2 + L_5(s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2)][G_7 s\phi_6 s\theta_6 + G_9(s\phi_6 c\theta_6 c\psi_6 s\alpha_8 + c\phi_6 s\psi_6 s\alpha_8 + s\phi_6 s\theta_6 c\alpha_8)] + [L_2 c\phi_2 s\theta_2 + L_5(c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)][G_7 c\phi_6 s\theta_6 + G_9(c\phi_6 c\theta_6 c\psi_6 s\alpha_8 - s\phi_6 s\psi_6 s\alpha_8 + c\phi_6 s\theta_6 c\alpha_8)]$$

$$D_{7\,20} = [G_7 c\theta_6 + G_9(c\theta_6 c\alpha_8 - s\theta_6 c\psi_6 s\alpha_8)]\{c\phi_6[L_5(c\phi_2 c\psi_2 - s\phi_2 c\theta_2 s\psi_2) - L_2 s\phi_2 s\theta_2] + s\phi_6[L_2 c\phi_2 s\theta_2 + L_5(c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)]\}$$

$$D_{7\,21} = G_9 s\alpha_8\{(c\phi_6 c\theta_6 s\psi_6 + s\phi_6 c\psi_6)[L_2 s\phi_2 s\theta_2 + L_5(s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2)] + (c\phi_6 c\psi_6 - s\phi_6 c\theta_6 s\psi_6)[L_2 c\phi_2 s\theta_2 + L_5(c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)]\}$$

$$D_{7\,22} = G_8\{(c\phi_5 c\theta_5 c\psi_5 c\alpha_7 - s\phi_5 s\psi_5 c\alpha_7 - c\phi_5 s\theta_5 s\alpha_7)[-L_2 s\phi_2 s\theta_2 + L_5(s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2)] + (s\phi_5 c\theta_5 c\psi_5 c\alpha_7 + c\phi_5 s\psi_5 c\alpha_7 - s\phi_5 s\theta_5 s\alpha_7)[L_2 c\phi_2 s\theta_2 - L_5(c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)]\}$$

$$D_{7\,23} = C_9\{(c\phi_6\, c\theta_6\, c\psi_6\, c\alpha_8 - s\phi_6\, s\psi_6\, c\alpha_8 - c\phi_6\, s\theta_6\, s\alpha_8)[-L_2\, s\phi_2\, s\theta_2 - L_3(s\phi_2\, c\theta_2\, s\psi_2 - c\phi_2\, c\psi_2)] + (s\phi_6\, c\theta_6\, c\psi_6\, c\alpha_8 + c\phi_6\, s\psi_6\, c\alpha_8 - s\phi_6\, s\theta_6\, s\alpha_8)[L_2\, c\phi_2\, s\theta_2 + L_3(c\phi_2\, c\theta_2\, s\psi_2 + s\phi_2\, c\psi_2)]\}$$

$$D_{8\,8} = C_{17} + s^2\psi_2(C_{20} + I_{x_2}) + I_{y_2}\, c^2\psi_2$$

$$D_{8\,9} = -L_2\, C_{14}\, c\psi_2$$

$$D_{8\,10} = L_2\, C_4\, c\theta_2\, s\theta_3(c\phi_3\, s\phi_2 - s\phi_3\, c\phi_2)$$

$$D_{8\,11} = L_2\, C_4[c\theta_3\, c\theta_2(c\phi_3\, c\phi_2 + s\phi_3\, s\phi_2) + s\theta_3\, s\theta_2]$$

$$D_{8\,16} = (L_2\, c\theta_2 + L_3\, s\theta_2\, s\psi_2)\{-c\phi_2[C_6\, s\phi_5\, s\theta_5 + C_8(s\phi_5\, c\theta_5\, c\psi_5\, s\alpha_7 + c\phi_5\, s\psi_5\, s\alpha_7 + s\phi_5\, s\theta_5\, c\alpha_7)] + s\phi_2[C_6\, c\phi_5\, s\theta_5 + C_8(c\phi_5\, c\theta_5\, c\psi_5\, s\alpha_7 - s\phi_5\, s\psi_5\, s\alpha_7 + c\phi_5\, s\theta_5\, c\alpha_7)]\}$$

$$D_{8\,17} = (L_2\, c\theta_2 + L_3\, s\theta_2\, s\psi_2)[C_6\, c\theta_5 + C_8(c\theta_5\, c\alpha_7 - s\theta_5\, c\psi_5\, s\alpha_7)](c\phi_2\, c\phi_5 + s\phi_2\, s\phi_5) + (-L_3\, c\theta_2\, s\psi_2 + L_2\, s\theta_2)[C_6\, s\theta_5 + C_8(s\theta_5\, c\alpha_7 + c\theta_5\, c\psi_5\, s\alpha_7)]$$

$$D_{8\,18} = C_8\, s\alpha_7\{(L_2\, c\theta_2 + L_3\, s\theta_2\, s\psi_2)[-c\phi_2(c\phi_5\, c\theta_5\, s\psi_5 + s\phi_5\, c\psi_5) + s\phi_2(c\phi_5\, c\psi_5 - s\phi_5\, c\theta_5\, s\psi_5)] + s\theta_5\, s\psi_5(L_3\, c\theta_2\, s\psi_2 - L_2\, s\theta_2)\}$$

$$D_8\ 19 = (L_2 c\theta_2 - L_5 s\theta_2 s\psi_2)\{-c\phi_2[C_7 s\phi_6 s\theta_6 + C_9(s\phi_6 c\theta_6 c\psi_6 s\alpha_8 + c\phi_6 s\psi_6 s\alpha_8 + s\phi_6 s\theta_6 c\alpha_8)] + s\phi_2[C_7 c\phi_6 s\theta_6 + C_9(c\phi_6 c\theta_6 c\psi_6 s\alpha_8 - s\phi_6 s\psi_6 s\alpha_8 + c\phi_6 s\theta_6 c\alpha_8)]\}$$

$$D_8\ 20 = (L_2 c\theta_2 - L_5 s\theta_2 s\psi_2)[C_7 c\theta_6 + C_9(c\theta_6 c\alpha_8 - s\theta_6 c\psi_6 s\alpha_8)](c\phi_2 c\phi_6 + s\phi_2 s\phi_6) + [L_2 s\theta_2 + L_5 c\theta_2 s\psi_2][C_7 s\theta_6 + C_9(c\theta_6 c\psi_6 s\alpha_8 + s\theta_6 c\alpha_8)]$$

$$D_8\ 21 = C_9 s\alpha_8\{(L_2 c\theta_2 - L_5 s\theta_2 s\psi_2)[-c\phi_2(c\phi_6 c\theta_6 s\psi_6 + s\phi_6 c\psi_6) + s\phi_2(c\phi_6 c\psi_6 - s\phi_6 c\theta_6 s\psi_6)] - s\theta_6 s\psi_6(L_2 s\theta_2 + L_5 c\theta_2 s\psi_2)\}$$

$$D_8\ 22 = C_8\{(L_2 c\theta_2 + L_5 s\theta_2 s\psi_2)[c\phi_2(c\phi_5 c\theta_5 c\psi_5 c\alpha_7 - s\phi_5 s\psi_5 c\alpha_7 - c\phi_5 s\theta_5 s\alpha_7) + s\phi_2(s\phi_5 c\theta_5 c\psi_5 c\alpha_7 + c\phi_5 s\psi_5 c\alpha_7 - s\phi_5 s\theta_5 s\alpha_7)] + (s\theta_5 c\psi_5 c\alpha_7 + c\theta_5 s\alpha_7)[L_2 s\theta_2 - L_5 c\theta_2 s\psi_2]\}$$

$$D_8\ 23 = C_9\{(L_2 c\theta_2 - L_5 s\theta_2 s\psi_2)[c\phi_2(c\phi_6 c\theta_6 c\psi_6 c\alpha_8 - s\phi_6 s\psi_6 c\alpha_8 - c\phi_6 s\theta_6 s\alpha_8) + s\phi_2(s\phi_6 c\theta_6 c\psi_6 c\alpha_8 + c\phi_6 s\psi_6 c\alpha_8 - s\phi_6 s\theta_6 s\alpha_8)] + (s\theta_6 c\psi_6 c\alpha_8 + c\theta_6 s\alpha_8)[L_2 s\theta_2 + L_5 c\theta_2 s\psi_2]\}$$

$$D_{9\,9} = C_{20} + I_{Z_2}$$

$$D_{9\,16} = L_5 \{ (c\phi_2 c\theta_2 c\psi_2 - s\phi_2 s\psi_2)[C_6 s\phi_5 s\theta_5 + C_8 (s\phi_5 c\theta_5 c\psi_5 s\alpha_7 + c\phi_5 s\psi_5 s\alpha_7 + s\phi_5 s\theta_5 c\alpha_7)] - (s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2)[C_6 c\phi_5 s\theta_5 + C_8 (c\phi_5 c\theta_5 c\psi_5 s\alpha_7 - s\phi_5 s\psi_5 s\alpha_7 + c\phi_5 s\theta_5 c\alpha_7)] \}$$

$$D_{9\,17} = L_5 \{ [C_6 c\theta_5 + C_8 (c\theta_5 c\alpha_7 - s\theta_5 c\psi_5 s\alpha_7)] [c\phi_5 (s\phi_2 s\psi_2 - c\phi_2 c\theta_2 c\psi_2) - s\phi_5 (s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2)] - s\theta_2 c\psi_2 [C_6 s\theta_5 + C_8 (c\theta_5 c\psi_5 s\alpha_7 + s\theta_5 c\alpha_7)] \}$$

$$D_{9\,18} = -L_5 C_8 s\alpha_7 [(c\phi_5 c\theta_5 s\psi_5 + s\phi_5 c\psi_5)(s\phi_2 s\psi_2 - c\phi_2 c\theta_2 c\psi_2) + (c\phi_5 c\psi_5 - s\phi_5 c\theta_5 s\psi_5)(s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2) - s\theta_5 s\psi_5 s\theta_2 c\psi_2]$$

$$D_{9\,19} = -L_5 \{ (c\phi_2 c\theta_2 c\psi_2 - s\phi_2 s\psi_2)[C_7 s\phi_6 s\theta_6 + C_9 (s\phi_6 c\theta_6 c\psi_6 s\alpha_8 + c\phi_6 s\psi_6 s\alpha_8 + s\phi_6 s\theta_6 c\alpha_8)] - (s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2)[C_7 c\phi_6 s\theta_6 + C_9 (c\phi_6 c\theta_6 c\psi_6 s\alpha_8 - s\phi_6 s\psi_6 s\alpha_8 + c\phi_6 s\theta_6 c\alpha_8)] \}$$

$$D_{9\,20} = -L_5 \{ [C_7 c\theta_6 + C_9 (c\theta_6 c\alpha_8 - s\theta_6 c\psi_6 s\alpha_8)][c\phi_6 (s\phi_2 s\psi_2 - c\phi_2 c\theta_2 c\psi_2) - s\phi_6 (s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2)] - s\theta_2 c\psi_2 [C_7 s\theta_6 + C_9 (c\theta_6 c\psi_6 s\alpha_8 + s\theta_6 c\alpha_8)] \}$$

$$D_{9\,21} = L_5 C_9 s\alpha_8 [(c\phi_6 c\theta_6 s\psi_6 + s\phi_6 c\psi_6)(s\phi_2 s\psi_2 - c\phi_2 c\theta_2 c\psi_2) + (c\phi_6 c\psi_6 - s\phi_6 c\theta_6 s\psi_6)(s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2) - s\theta_6 s\psi_6 s\theta_2 c\psi_2]$$

$$D_{9\,22} = L_5 C_8 \{ (s\phi_2 s\psi_2 - c\phi_2 c\theta_2 c\psi_2)(c\phi_5 c\theta_5 c\psi_5 c\alpha_7 - s\phi_5 s\psi_5 c\alpha_7 - c\phi_5 s\theta_5 s\alpha_7) - (s\phi_5 c\theta_5 c\psi_5 c\alpha_7 + c\phi_5 s\psi_5 c\alpha_7 - s\phi_5 s\theta_5 s\alpha_7)(s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2) - (s\theta_5 c\psi_5 c\alpha_7 + c\theta_5 s\alpha_7)(s\theta_2 c\psi_2) \}$$

$$D_{9\,23} = L_5 C_9 \{ -(c\phi_6 c\theta_6 c\psi_6 c\alpha_8 - s\phi_6 s\psi_6 c\alpha_8 - c\phi_6 s\theta_6 s\alpha_8)(s\phi_2 s\psi_2 - c\phi_2 c\theta_2 c\psi_2) + (s\phi_6 c\theta_6 c\psi_6 c\alpha_8 + c\phi_6 s\psi_6 c\alpha_8 - s\phi_6 s\theta_6 s\alpha_8)(s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2) + (s\theta_6 c\psi_6 c\alpha_8 + c\theta_6 s\alpha_8) s\theta_2 c\psi_2 \}$$

$$D_{10\,10} = s^2\theta_3 [(I_{x_3} c^2\psi_3 + I_{y_3} s^2\psi_3) + C_{18}] + I_{z_3} c^2\theta_3$$

$$D_{10\,11} = (I_{y_3} - I_{x_3}) s\theta_3 s\psi_3 c\psi_3$$

$$D_{10\,12} = I_{z_3} c\theta_3$$

$$D_{11\,11} = C_{18} + I_{x_3} s^2\psi_3 + I_{y_3} c^2\psi_3$$

$$D_{12\,12} = I_{z_3}$$

$$D_{13\ 13} = s^2\theta_4 [C_{19} + I_{X_4} c^2\psi_4 + I_{Y_4} s^2\psi_4] + c^2\theta_4 [I_{Z_4} + C_{25} s^2\psi_4] + 2L_4 C_{15} s\theta_4 c\theta_4 s\psi_4 + C_{25} c^2\psi_4$$

$$D_{13\ 14} = c\psi_4 [L_4 C_{15} c\theta_4 + s\psi_4 s\theta_4 (I_{Y_4} - I_{X_4} - C_{25})]$$

$$D_{13\ 15} = L_4 C_{15} s\theta_4 s\psi_4 + C_{25} c\theta_4$$

$$D_{13\ 24} = -[s\phi_9 s\theta_9 C_{10} + C_{12}(s\phi_9 c\theta_9 c\psi_9 s\alpha_{11} + c\phi_9 s\psi_9 s\alpha_{11} + s\phi_9 s\theta_9 c\alpha_{11})][L_4 s\phi_4 s\theta_4 + L_H(s\phi_4 c\theta_4 s\psi_4 - c\phi_4 c\psi_4)] - [c\phi_9 s\theta_9 C_{10} + C_{12}(c\phi_9 c\theta_9 c\psi_9 s\alpha_{11} - s\phi_9 s\psi_9 s\alpha_{11} + c\phi_9 s\theta_9 c\alpha_{11})][L_4 c\phi_4 s\theta_4 + L_H(c\phi_4 c\theta_4 s\psi_4 + s\phi_4 c\psi_4)]$$

$$D_{13\ 25} = [C_{10} c\theta_9 + C_{12}(c\theta_9 c\alpha_{11} - s\theta_9 c\psi_9 s\alpha_{11})] \{L_4 s\theta_4 (s\phi_4 c\phi_9 - s\phi_9 c\phi_4) + L_H [c\phi_9 (s\phi_4 c\theta_4 s\psi_4 - c\phi_4 c\psi_4) - s\phi_9 (c\phi_4 c\theta_4 s\psi_4 + s\phi_4 c\psi_4)]\}$$

$$D_{13\ 26} = -C_{12} s\alpha_{11} \{(c\phi_9 c\theta_9 s\psi_9 + s\phi_9 c\psi_9)[L_4 s\phi_4 s\theta_4 + L_H(s\phi_4 c\theta_4 s\psi_4 - c\phi_4 c\psi_4)] + (c\phi_9 c\psi_9 - s\phi_9 c\theta_9 s\psi_9)[L_4 c\phi_4 s\theta_4 + L_H(c\phi_4 c\theta_4 s\psi_4 + s\phi_4 c\psi_4)]\}$$

$$
\begin{aligned}
D_{13\,27} = & -[s\phi_{10}\, s\theta_{10}\, C_{11} + C_{13}(s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} \\
& + c\phi_{10}\, s\psi_{10}\, s\alpha_{12} + s\phi_{10}\, s\theta_{10}\, c\alpha_{12})] \\
& [L_4\, s\phi_4\, s\theta_4 - L_H(s\phi_4\, c\theta_4\, s\psi_4 - c\phi_4\, c\psi_4)] \\
& -[c\phi_{10}\, s\theta_{10}\, C_{11} + C_{13}(c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} \\
& - s\phi_{10}\, s\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\theta_{10}\, c\alpha_{12})][L_4\, c\phi_4 \\
& s\theta_4 - L_H(c\phi_4\, c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4)]
\end{aligned}
$$

$$
\begin{aligned}
D_{13\,28} = & [C_{11}\, c\theta_{10} + C_{13}(c\theta_{10}\, c\alpha_{12} - s\theta_{10}\, c\psi_{10}\, s\alpha_{12})] \\
& \{L_4\, s\theta_4(s\phi_4\, c\phi_{10} - c\phi_4\, s\phi_{10}) + L_H[s\phi_{10}(c\phi_4 \\
& c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4) - c\phi_{10}(s\phi_4\, c\theta_4\, s\psi_4 - c\phi_4 \\
& c\psi_4)]\}
\end{aligned}
$$

$$
\begin{aligned}
D_{13\,29} = & -C_{13}\, s\alpha_{12}\{(c\phi_{10}\, c\theta_{10}\, s\psi_{10} + s\phi_{10}\, c\psi_{10}) \\
& [L_4\, s\phi_4\, s\theta_4 - L_H(s\phi_4\, c\theta_4\, s\psi_4 - c\phi_4\, c\psi_4)] \\
& + (c\phi_{10}\, c\psi_{10} - s\phi_{10}\, c\theta_{10}\, s\psi_{10})[L_4\, c\phi_4\, s\theta_4 \\
& - L_H(c\phi_4\, c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4)]\}
\end{aligned}
$$

$$
\begin{aligned}
D_{13\,30} = & C_{12}\{(c\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} - s\phi_9\, s\psi_9\, c\alpha_{11} \\
& - c\phi_9\, s\theta_9\, s\alpha_{11})[L_4\, s\phi_4\, s\theta_4 + L_H(s\phi_4\, c\theta_4 \\
& s\psi_4 - c\phi_4\, c\psi_4)] - (s\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} + c\phi_9 \\
& s\psi_9\, c\alpha_{11} - s\phi_9\, s\theta_9\, s\alpha_{11})[L_4\, c\phi_4\, s\theta_4 \\
& + L_H(c\phi_4\, c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4)]\}
\end{aligned}
$$

$$D_{13\;31} = C_{13}\{(c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, c\alpha_{12} - s\phi_{10}\, s\psi_{10}\, c\alpha_{12} - c\phi_{10}\, s\theta_{10}\, s\alpha_{12})[L_4\, s\phi_4\, s\theta_4 - L_H(s\phi_4\, c\theta_4\, s\psi_4 - c\phi_4\, c\psi_4)] - (s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, c\alpha_{12} + c\phi_{10}\, s\psi_{10}\, c\alpha_{12} - s\phi_{10}\, s\theta_{10}\, s\alpha_{12})[L_4\, c\phi_4\, s\theta_4 - L_H(c\phi_4\, c\theta_4\, s\psi_4 + s\phi_4\, c\psi_4)]\}$$

$$D_{14\;14} = C_{19} + (C_{25} + I_{x_4})\, s^2\psi_4 + I_{y_4}\, c^2\psi_4$$

$$D_{14\;15} = L_4\, C_{15}\, c\psi_4$$

$$D_{14\;24} = [L_4\, c\theta_4 - L_H\, s\theta_4\, s\psi_4]\{C_{10}\, s\theta_9(s\phi_9\, c\phi_4 - s\phi_4\, c\phi_9) + C_{12}[c\phi_4(s\phi_9\, c\theta_9\, c\psi_9\, s\alpha_{11} + c\phi_9\, s\psi_9\, s\alpha_{11} + s\phi_9\, s\theta_9\, c\alpha_{11}) - s\phi_4(c\phi_9\, c\theta_9\, c\psi_9\, s\alpha_{11} - s\phi_9\, s\psi_9\, s\alpha_{11} + c\phi_9\, s\theta_9\, c\alpha_{11})]\}$$

$$D_{14\;25} = (L_H\, s\theta_4\, s\psi_4 - L_4\, c\theta_4)(c\phi_9\, c\phi_4 + s\phi_9\, s\phi_4)[C_{10}\, c\theta_9 + C_{12}(c\theta_9\, c\alpha_{11} - s\theta_9\, c\psi_9\, s\alpha_{11})] - (L_H\, c\theta_4\, s\psi_4 + L_4\, s\theta_4)[C_{10}\, s\theta_9 + C_{12}(c\theta_9\, c\psi_9\, s\alpha_{11} + s\theta_9\, c\alpha_{11})]$$

$$D_{14\;26} = C_{12}\{[-c\phi_4(c\phi_9\, c\theta_9\, s\psi_9\, s\alpha_{11} + s\phi_9\, c\psi_9\, s\alpha_{11}) + s\phi_4(c\phi_9\, c\psi_9\, s\alpha_{11} - s\phi_9\, c\theta_9\, s\psi_9\, s\alpha_{11})][-L_4\, c\theta_4 + L_H\, s\theta_4\, s\psi_4] + [s\theta_9\, s\psi_9\, s\alpha_{11}][L_4\, s\theta_4 + L_H\, c\theta_4\, s\psi_4]\}$$

$$D_{14\,27} = \{ C_{11}\, s\theta_{10} [s\phi_{10}\, c\phi_4 - c\phi_{10}\, s\phi_4] + C_{13} [c\phi_4 (s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\psi_{10}\, s\alpha_{12} + s\phi_{10}\, s\theta_{10}\, c\alpha_{12}) - s\phi_4 (c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} - s\phi_{10}\, s\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\theta_{10}\, c\alpha_{12})]\}\{ L_4\, c\theta_4 + L_H\, s\theta_4\, s\psi_4 \}$$

$$D_{14\,28} = -(L_4\, c\theta_4 + L_H\, s\theta_4\, s\psi_4)(c\phi_{10}\, c\phi_4 + s\phi_{10}\, s\phi_4) [C_{11}\, c\theta_{10} + C_{13} (c\theta_{10}\, c\alpha_{12} - s\theta_{10}\, c\psi_{10}\, s\alpha_{12})] - (L_4\, s\theta_4 - L_H\, c\theta_4\, s\psi_4) [C_{11}\, s\theta_{10} + C_{13} (c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + s\theta_{10}\, c\alpha_{12})]$$

$$D_{14\,29} = C_{13}\, s\alpha_{12} \{ [L_4\, c\theta_4 + L_H\, s\theta_4\, s\psi_4][c\phi_4 (c\phi_{10}\, c\theta_{10}\, s\psi_{10} + s\phi_{10}\, c\psi_{10}) - s\phi_4 (c\phi_{10}\, c\psi_{10} - s\phi_{10}\, c\theta_{10}\, s\psi_{10})] + s\theta_{10}\, s\psi_{10} (L_4\, s\theta_4 - L_H\, c\theta_4\, s\psi_4) \}$$

$$D_{14\,30} = C_{12} \{ [c\phi_4 (c\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} - s\phi_9\, s\psi_9\, c\alpha_{11} - c\phi_9\, s\theta_9\, s\alpha_{11}) + s\phi_4 (s\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} + c\phi_9\, s\psi_9\, c\alpha_{11} - s\phi_9\, s\theta_9\, s\alpha_{11})] [-L_4\, c\theta_4 + L_H\, s\theta_4\, s\psi_4] - [(s\theta_9\, c\psi_9\, c\alpha_{11} + c\theta_9\, s\alpha_{11})(L_4\, s\theta_4 + L_H\, c\theta_4\, s\psi_4)]\}$$

$$
\begin{aligned}
D_{14\,31} = C_{13}\{&[c\phi_4(c\phi_{10}c\theta_{10}c\psi_{10}c\alpha_{12} - s\phi_{10}s\psi_{10}c\alpha_{12} \\
&- c\phi_{10}s\theta_{10}s\alpha_{12}) + s\phi_4(s\phi_{10}c\theta_{10}c\psi_{10}c\alpha_{12} \\
&+ c\phi_{10}s\psi_{10}c\alpha_{12} - s\phi_{10}s\theta_{10}s\alpha_{12})][-L_4c\theta_4 \\
&- L_Hs\theta_4s\psi_4] - [(s\theta_{10}c\psi_{10}c\alpha_{12} + c\theta_{10}s\alpha_{12}) \\
&(L_4s\theta_4 - L_Hc\theta_4s\psi_4)]\}
\end{aligned}
$$

$$D_{15\,15} = C_{25} + I_{z4}$$

$$
\begin{aligned}
D_{15\,24} = -L_H\{&(s\phi_4s\psi_4 - c\phi_4c\theta_4c\psi_4)[C_{10}s\phi_9s\theta_9 \\
&+ C_{12}(s\phi_9c\theta_9c\psi_9s\alpha_{11} + c\phi_9s\psi_9s\alpha_{11} + s\phi_9 \\
&s\theta_9c\alpha_{11})] + (s\phi_4c\theta_4c\psi_4 + c\phi_4s\psi_4)[C_{10}c\phi_9 \\
&s\theta_9 + C_{12}(c\phi_9c\theta_9c\psi_9s\alpha_{11} - s\phi_9s\psi_9s\alpha_{11} \\
&+ c\phi_9s\theta_9c\alpha_{11})]\}
\end{aligned}
$$

$$
\begin{aligned}
D_{15\,25} = L_H\{&[C_{10}c\theta_9 + C_{12}(c\theta_9c\alpha_{11} - s\theta_9c\psi_9s\alpha_{11})] \\
&[c\phi_9(s\phi_4s\psi_4 - c\phi_4c\theta_4c\psi_4) - s\phi_9(s\phi_4c\theta_4 \\
&c\psi_4 + c\phi_4s\psi_4)] - s\theta_4c\psi_4[C_{10}s\theta_9 + C_{12}(c\theta_9 \\
&c\psi_9s\alpha_{11} + s\theta_9c\alpha_{11})]\}
\end{aligned}
$$

$$
\begin{aligned}
D_{15\,26} = -L_H C_{12}s\alpha_{11}&[(c\phi_9c\theta_9s\psi_9 + s\phi_9c\psi_9) \\
&(s\phi_4s\psi_4 - c\phi_4c\theta_4c\psi_4) + (c\phi_9c\psi_9 - s\phi_9 \\
&c\theta_9s\psi_9)(s\phi_4c\theta_4c\psi_4 + c\phi_4s\psi_4) - s\theta_9 \\
&s\psi_9s\theta_4c\psi_4]
\end{aligned}
$$

$$D_{15\;27} = L_H \{ C_{11} [(s\phi_{10}\, s\theta_{10})(-c\phi_4\, c\theta_4\, c\psi_4 + s\phi_4\, s\psi_4) + (c\phi_{10}\, s\theta_{10})(s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4)] + C_{13} [(s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\psi_{10}\, s\alpha_{12} + s\phi_{10}\, s\theta_{10}\, c\alpha_{12})(-c\phi_4\, c\theta_4\, c\psi_4 + s\phi_4\, s\psi_4) + (c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, s\alpha_{12} - s\phi_{10}\, s\psi_{10}\, s\alpha_{12} + c\phi_{10}\, s\theta_{10}\, c\alpha_{12})(s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4)] \}$$

$$D_{15\;28} = L_H \{ C_{11} [c\phi_{10}\, c\theta_{10} (c\phi_4\, c\theta_4\, c\psi_4 - s\phi_4\, s\psi_4) + s\phi_{10}\, c\theta_{10} (s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4) + s\theta_{10}\, s\theta_4\, c\psi_4] + C_{13} [(c\phi_{10}\, c\theta_{10}\, c\alpha_{12} - c\phi_{10}\, s\theta_{10}\, c\psi_{10}\, s\alpha_{12})(c\phi_4\, c\theta_4\, c\psi_4 - s\phi_4\, s\psi_4) + (s\phi_{10}\, c\theta_{10}\, c\alpha_{12} - s\phi_{10}\, s\theta_{10}\, c\psi_{10}\, s\alpha_{12})(s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4) + (c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + s\theta_{10}\, c\alpha_{12})(s\theta_4\, c\psi_4)] \}$$

$$D_{15\;29} = L_H C_{13} [(c\phi_{10}\, c\theta_{10}\, s\psi_{10}\, s\alpha_{12} + s\phi_{10}\, c\psi_{10}\, s\alpha_{12})(-c\phi_4\, c\theta_4\, c\psi_4 + s\phi_4\, s\psi_4) + (c\phi_{10}\, c\psi_{10}\, s\alpha_{12} - s\phi_{10}\, c\theta_{10}\, s\psi_{10}\, s\alpha_{12})(s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4) - s\theta_{10}\, s\psi_{10}\, s\alpha_{12}\, s\theta_4\, c\psi_4]$$

$$D_{15\;30} = L_H C_{12} [(c\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} - s\phi_9\, s\psi_9\, c\alpha_{11} - c\phi_9\, s\theta_9\, s\alpha_{11})(-c\phi_4\, c\theta_4\, c\psi_4 + s\phi_4\, s\psi_4) + (s\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} + c\phi_9\, s\psi_9\, c\alpha_{11} - s\phi_9\, s\theta_9\, s\alpha_{11})(-s\phi_4\, c\theta_4\, c\psi_4 - c\phi_4\, s\psi_4) - (s\theta_9\, c\psi_9\, c\alpha_{11} + c\theta_9\, s\alpha_{11})\, s\theta_4\, c\psi_4]$$

$$D_{15\;31} = L_4 C_{13}\Big[\big(c\phi_{10}\, c\theta_{10}\, c\psi_{10}\, c\alpha_{12} - s\phi_{10}\, s\psi_{10}\, c\alpha_{12} - c\phi_{10}\, s\theta_{10}\, s\alpha_{12}\big)\big(c\phi_4\, c\theta_4\, c\psi_4 - s\phi_4\, s\psi_4\big) + \big(s\phi_{10}\, c\theta_{10}\, c\psi_{10}\, c\alpha_{12} + c\phi_{10}\, s\psi_{10}\, c\alpha_{12} - s\phi_{10}\, s\theta_{10}\, s\alpha_{12}\big)\big(s\phi_4\, c\theta_4\, c\psi_4 + c\phi_4\, s\psi_4\big) + \big(s\theta_{10}\, c\psi_{10}\, c\alpha_{12} + c\theta_{10}\, s\alpha_{12}\big)\, s\theta_4\, c\psi_4\Big]$$

$$D_{16\;16} = s^2\theta_5\Big[C_{Z1} - 2L_5 C_8\, c\alpha_7 + I_{X5}\, c^2\psi_5 + (I_{Y5} + I_{Y7})\, s^2\psi_5\Big] + I_{Z5}\, c^2\theta_5 + C_{Z3}\Big[s^2\psi_5\, s^2\alpha_7 + (s\theta_5\, c\alpha_7 + c\theta_5\, c\psi_5\, s\alpha_7)^2\Big] - 2L_5 C_8\, s\theta_5\, c\theta_5\, c\psi_5\, s\alpha_7 + I_{X7}(s\theta_5\, c\psi_5\, c\alpha_7 + c\theta_5\, s\alpha_7)^2 + I_{Z7}(c\theta_5\, c\alpha_7 - s\theta_5\, c\psi_5\, s\alpha_7)^2$$

$$D_{16\;17} = s\psi_5\Big\{c\theta_5\, s\alpha_7\Big[L_5 C_8 - c\alpha_7\,(C_{Z3} + I_{X7} - I_{Z7})\Big] + s\theta_5\, c\psi_5\Big[s^2\alpha_7\,(C_{Z3} + I_{X7} - I_{Z7}) + I_{Y5} + I_{Y7} - I_{X5} - I_{X7}\Big]\Big\}$$

$$D_{16\;18} = s\theta_5\, c\psi_5\, s\alpha_7\Big[c\alpha_7\,(C_{Z3} + I_{X7} - I_{Z7}) - L_5 C_8\Big] + c\theta_5\Big[s^2\alpha_7\,(C_{Z3} + I_{X7} - I_{Z7}) + I_{Z7} + I_{Z5}\Big]$$

$$D_{16\;22} = s\theta_5\, s\psi_5\Big[C_{Z3} - L_5 C_8\, c\alpha_7 + I_{Y7}\Big]$$

$$D_{17\;17} = C_{Z1} + c\alpha_7\Big[C_{Z3}\, c\alpha_7 - 2L_5 C_8\Big] + c^2\psi_5\Big[C_{Z3}\, s^2\alpha_7 + I_{Y5} + I_{Y7}\Big] + s^2\psi_5\Big[I_{X7}\, c^2\alpha_7 + I_{Z7}\, s^2\alpha_7 + I_{X5}\Big]$$

$$D_{17\;18} = s\psi_5\, s\alpha_7\Big[L_5 C_8 - c\alpha_7\,(C_{Z3} + I_{X7} - I_{Z7})\Big]$$

$$D_{17\;22} = c\psi_5\Big[C_{Z3} - L_5 C_8\, c\alpha_7 + I_{Y7}\Big]$$

$$D_{18\ 18} = (C_{Z3} + I_{X7} - I_{Z7})\, s^2\alpha_7 + I_{Z5} + I_{Z7}$$

$$D_{19\ 19} = s^2\theta_6 \left[C_{Z2} - 2L_6 C_9 c\alpha_8 + c^2\psi_6 I_{X6} + (I_{Y6} + I_{Y8})\, s^2\psi_6 \right] - 2L_6 C_9\, s\theta_6\, c\theta_6\, c\psi_6\, s\alpha_8 + C_{Z4}\left[s^2\psi_6\, s^2\alpha_8 + (c\theta_6\, c\psi_6\, s\alpha_8 + s\theta_6\, c\alpha_8)^2 \right] + c^2\theta_6 I_{Z6} + I_{X8}(s\theta_6\, c\psi_6\, c\alpha_8 + c\theta_6\, s\alpha_8)^2 + I_{Z8}(c\theta_6\, c\alpha_8 - s\theta_6\, c\psi_6\, s\alpha_8)^2$$

$$D_{19\ 20} = s\psi_6 \left\{ c\psi_6\, s\theta_6 \left[I_{Y6} - I_{X6} + I_{Y8} - I_{X8} + s^2\alpha_8 (C_{Z4} + I_{X8} - I_{Z8}) \right] + c\theta_6\, s\alpha_8 \left[L_6 C_9 - c\alpha_8 (C_{Z4} + I_{X8} - I_{Z8}) \right] \right\}$$

$$D_{19\ 21} = c\psi_6\, s\alpha_8\, s\theta_6 \left[c\alpha_8 (C_{Z4} + I_{X8} - I_{Z8}) - L_6 C_9 \right] + c\theta_6 \left[s^2\alpha_8 (C_{Z4} + I_{X8} - I_{Z8}) + I_{Z6} + I_{Z8} \right]$$

$$D_{19\ 23} = s\theta_6\, s\psi_6 \left[C_{Z4} - L_6 C_9\, c\alpha_8 + I_{Y8} \right]$$

$$D_{20\ 20} = C_{Z2} + c\alpha_8 \left[C_{Z4}\, c\alpha_8 - 2L_6 C_9 \right] + c^2\psi_6 \left[I_{Y6} + I_{Y8} + C_{Z4}\, s^2\alpha_8 \right] + s^2\psi_6 \left[I_{X6} + I_{X8} + s^2\alpha_8 (-I_{X8} + I_{Z8}) \right]$$

$$D_{20\ 21} = s\psi_6\, s\alpha_8 \left[c\alpha_8 (-C_{Z4} - I_{X8} + I_{Z8}) + L_6 C_9 \right]$$

$$D_{20\ 23} = c\psi_6 \left[C_{Z4} - L_6 C_9\, c\alpha_8 + I_{Y8} \right]$$

$$D_{21\ 21} = s^2\alpha_8 (C_{Z4} + I_{X8} - I_{Z8}) + I_{Z6} + I_{Z8}$$

$$D_{22\ 22} = C_{23} + I_{y7}$$

$$D_{23\ 23} = C_{24} + I_{y8}$$

$$D_{24\ 24} = s^2\theta_9 \left[C_{26} - 2L_9 C_{12}\, c\alpha_{11} + c^2\psi_9 I_{x9} + (I_{y9} + I_{y11}) s^2\psi_9 \right] + c\theta_9 \left[c\theta_9 I_{z9} - 2L_9 C_{12}\, s\theta_9\, c\psi_9\, s\alpha_{11} \right] + C_{28}\left[s^2\psi_9\, s^2\alpha_{11} + (c\theta_9\, c\psi_9\, s\alpha_{11} + s\theta_9\, c\alpha_{11})^2 \right] + I_{x11}(s\theta_9\, c\psi_9\, c\alpha_{11} + c\theta_9\, s\alpha_{11})^2 + I_{z11}(c\theta_9\, c\alpha_{11} - s\theta_9\, c\psi_9\, s\alpha_{11})^2$$

$$D_{24\ 25} = s\psi_9 \left\{ c\theta_9\, s\alpha_{11} \left[L_9 C_{12} + c\alpha_{11}(-C_{28} - I_{x11} + I_{z11}) + s\theta_9\, c\psi_9 \left[s^2\alpha_{11}(C_{28} + I_{x11} - I_{z11}) + I_{y9} + I_{y11} - I_{x9} - I_{x11} \right] \right] \right\}$$

$$D_{24\ 26} = s\theta_9\, c\psi_9\, s\alpha_{11} \left[c\alpha_{11}(C_{28} + I_{x11} - I_{z11}) - L_9 C_{12} \right] + c\theta_9 \left[I_{z9} + I_{z11} + s^2\alpha_{11}(C_{28} + I_{x11} - I_{z11}) \right]$$

$$D_{24\ 30} = s\theta_9\, s\psi_9 \left[C_{28} - L_9 C_{12}\, c\alpha_{11} + I_{y11} \right]$$

$$D_{25\ 25} = C_{26} + c\alpha_{11}\left[C_{28}\, c\alpha_{11} - 2L_9 C_{12} \right] + c^2\psi_9 (I_{y9} + I_{y11} + C_{28}\, s^2\alpha_{11}) + s^2\psi_9 \left[I_{x9} + I_{x11} + s^2\alpha_{11}(-I_{x11} + I_{z11}) \right]$$

$$D_{25\ 26} = s\psi_9\, s\alpha_{11} \left[L_9 C_{12} + c\alpha_{11}(-C_{28} - I_{x11} + I_{z11}) \right]$$

$$D_{25\ 30} = c\psi_9\ (C_{28} - L_9 C_{12}\, c\alpha_{11} + I_{y11})$$

$$D_{26\ 26} = s^2\alpha_{11}\,(C_{28} + I_{x11} - I_{z11}) + I_{z9} + I_{z11}$$

$$D_{27\;27} = s^2\theta_{10}\left[C_{27} + c^2\psi_{10} I_{x10} + (I_{y10} + I_{y12}) s^2\psi_{10} - 2 L_{10} C_{13} c\alpha_{12}\right] + C_{29}\left[s^2\psi_{10}\, s^2\alpha_{12} + (c\theta_{10}\, c\psi_{10}\, s\alpha_{12} + s\theta_{10}\, c\alpha_{12})^2\right] + c\theta_{10}\left[c\theta_{10} I_{z10} - 2 L_{10} C_{13}\, s\theta_{10}\, c\psi_{10}\, s\alpha_{12}\right] + I_{x12}(s\theta_{10}\, c\psi_{10}\, c\alpha_{12} + c\theta_{10}\, s\alpha_{12})^2 + I_{z12}(c\theta_{10}\, c\alpha_{12} - s\theta_{10}\, c\psi_{10}\, s\alpha_{12})^2$$

$$D_{27\;28} = s\psi_{10}\left\{c\theta_{10}\, s\alpha_{12}\left[L_{10} C_{13} + c\alpha_{12}(-C_{29} - I_{x12} + I_{z12})\right] + s\theta_{10}\, c\psi_{10}\left[s^2\alpha_{12}(-I_{z12} + I_{x12} + C_{29}) - I_{x12} + I_{y12} + I_{y10} - I_{x10}\right]\right\}$$

$$D_{27\;29} = s\theta_{10}\, c\psi_{10}\, s\alpha_{12}\left[c\alpha_{12}(C_{29} + I_{x12} - I_{z12}) - L_{10} C_{13}\right] + c\theta_{10}\left[I_{z10} + I_{z12} + s^2\alpha_{12}(C_{29} - I_{z12} + I_{x12})\right]$$

$$D_{27\;31} = s\theta_{10}\, s\psi_{10}\left[C_{29} + I_{y12} - L_{10} C_{13}\, c\alpha_{12}\right]$$

$$D_{28\;28} = C_{27} + c\alpha_{12}\left[C_{29}\, c\alpha_{12} - 2 L_{10} C_{13}\right] + c^2\psi_{10}\left[I_{y10} + I_{y12} + C_{29}\, s^2\alpha_{12}\right] + s^2\psi_{10}\left[I_{x10} + I_{x12} + s^2\alpha_{12}(-I_{x12} + I_{z12})\right.$$

$$D_{28\;29} = s\psi_{10}\, s\alpha_{12}\left[L_{10} C_{13} + c\alpha_{12}(-C_{29} - I_{x12} + I_{z12})\right]$$

$$D_{28\;31} = c\psi_{10}\left[C_{29} + I_{y12} - L_{10} C_{13}\, c\alpha_{12}\right]$$

$$D_{29\;29} = s^2\alpha_{12}(C_{29} + I_{x12} - I_{z12}) + I_{z10} + I_{z12}$$

$$D_{30\;30} = C_{28} + I_{y11}$$

$$D_{31\;31} = C_{29} + I_{y12}$$

To define the elements of column vector E, these parameters are necessary:

$$A_{13}^{n} = -c\phi_n\, s\theta_n\,(\dot{\phi}_n^2 + \dot{\theta}_n^2) - 2\dot{\phi}_n\dot{\theta}_n\, s\phi_n\, c\theta_n$$
$$A_{23}^{n} = -s\phi_n\, s\theta_n\,(\dot{\phi}_n^2 + \dot{\theta}_n^2) + 2\dot{\theta}_n\dot{\phi}_n\, c\phi_n\, c\theta_n$$
$$A_{33}^{n} = -\dot{\theta}_n^2\, c\theta_n$$

$$\begin{aligned} A_{12}^{n} = {} & c\phi_n\big[(\dot{\phi}_n^2 + \dot{\theta}_n^2 + \dot{\psi}_n^2)\, c\theta_n\, s\psi_n + 2\dot{\theta}_n\dot{\psi}_n\, s\theta_n \\ & c\psi_n\big] + s\phi_n\big[(\dot{\phi}_n^2 + \dot{\psi}_n^2)\, c\psi_n - 2\dot{\phi}_n\dot{\theta}_n\, s\theta_n\, s\psi_n\big] \\ & + 2\dot{\phi}_n\dot{\psi}_n\,(s\phi_n\, c\theta_n\, c\psi_n + c\phi_n\, s\psi_n) \end{aligned}$$
$$\begin{aligned} A_{22}^{n} = {} & (\dot{\phi}_n^2 + \dot{\theta}_n^2 + \dot{\psi}_n^2)\, s\phi_n\, c\theta_n\, s\psi_n - (\dot{\phi}_n^2 + \dot{\psi}_n^2)(c\phi_n\, c\psi_n) \\ & + 2\big[\dot{\theta}_n\, s\theta_n\,(\dot{\psi}_n\, s\phi_n\, c\psi_n + \dot{\phi}_n\, c\phi_n\, s\psi_n) + (\dot{\phi}_n\dot{\psi}_n) \\ & (s\phi_n\, s\psi_n - c\phi_n\, c\theta_n\, c\psi_n)\big] \end{aligned}$$
$$A_{32}^{n} = -(\dot{\theta}_n^2 + \dot{\psi}_n^2)\, s\theta_n\, s\psi_n + 2(\dot{\theta}_n\dot{\psi}_n\, c\theta_n\, c\psi_n)$$

$$\begin{aligned} \bar{A}_{13}^{nm} = {} & -(\dot{\phi}_n^2 + \dot{\theta}_n^2 + \dot{\psi}_n^2 + \dot{\alpha}_m^2)(c\phi_n\, c\theta_n\, c\psi_n\, s\alpha_m) \\ & + (\dot{\phi}_n^2 + \dot{\psi}_n^2 + \dot{\alpha}_m^2)(s\phi_n\, s\psi_n\, s\alpha_m) - (\dot{\phi}_n^2 + \dot{\theta}_n^2 \\ & + \dot{\alpha}_m^2)(c\phi_n\, s\theta_n\, c\alpha_m) + 2\big\{\dot{\phi}_n\big[\dot{\theta}_n\, s\phi_n\,(s\theta_n \\ & c\psi_n\, s\alpha_m - c\theta_n\, c\alpha_m) + \dot{\psi}_n\, s\alpha_m\,(s\phi_n\, c\theta_n\, s\psi_n \\ & - c\phi_n\, c\psi_n) - \dot{\alpha}_m\,(s\phi_n\, c\theta_n\, c\psi_n\, c\alpha_m + c\phi_n\, s\psi_n \\ & c\alpha_m - s\phi_n\, s\theta_n\, s\alpha_m)\big] - \dot{\alpha}_m\big[\dot{\theta}_n\,(c\phi_n\, s\theta_n\, c\psi_n \\ & c\alpha_m + c\phi_n\, c\theta_n\, s\alpha_m) \end{aligned}$$

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($\bar{A}_{13}^{nm}$ cont'd.)

$$+\dot{\psi}_n\,(c\phi_n\, c\theta_n\, s\psi_n\, c\alpha_m + s\phi_n\, c\psi_n\, c\alpha_m)] + \dot{\theta}_n\,\dot{\psi}_n\, c\phi_n\, s\theta_n\, s\psi_n\, s\alpha_m\}$$

$$\bar{A}_{23}^{nm} = -[(\dot{\phi}_n^2 + \dot{\theta}_n^2 + \dot{\psi}_n^2 + \dot{\alpha}_m^2)(s\phi_n\, c\theta_n\, c\psi_n\, s\alpha_m) + (\dot{\phi}_n^2 + \dot{\psi}_n^2 + \dot{\alpha}_m^2)(c\phi_n\, s\psi_n\, s\alpha_m) + (\dot{\phi}_n^2 + \dot{\theta}_n^2 + \dot{\alpha}_m^2)(s\phi_n\, s\theta_n\, c\alpha_m)] + 2\{\dot{\phi}_n[\dot{\theta}_n\, c\phi_n\,(c\theta_n\, c\alpha_m - s\theta_n\, c\psi_n\, s\alpha_m) - \dot{\psi}_n\, s\alpha_m(c\phi_n\, c\theta_n\, s\psi_n + s\phi_n\, c\psi_n) + \dot{\alpha}_m(c\phi_n\,(c\theta_n\, c\psi_n\, c\alpha_m - s\theta_n\, s\alpha_m) - s\phi_n\, s\psi_n\, c\alpha_m)] - \dot{\alpha}_m[\dot{\theta}_n\, s\phi_n\,(s\theta_n\, c\psi_n\, c\alpha_m + c\theta_n\, s\alpha_m) + \dot{\psi}_n\, c\alpha_m\,(s\phi_n\, c\theta_n\, s\psi_n - c\phi_n\, c\psi_n)] + \dot{\theta}_n\,\dot{\psi}_n\,(s\phi_n\, s\theta_n\, s\psi_n\, s\alpha_m)\}$$

$$\bar{A}_{33}^{nm} = (\dot{\theta}_n^2 + \dot{\psi}_n^2 + \dot{\alpha}_m^2)(s\theta_n\, c\psi_n\, s\alpha_m) - (\dot{\theta}_n^2 + \dot{\alpha}_m^2)(c\theta_n\, c\alpha_m) + 2\{\dot{\theta}_n\,\dot{\alpha}_m(-c\theta_n\, c\psi_n\, c\alpha_m + s\theta_n\, s\alpha_m) + \dot{\psi}_n\, s\psi_n(\dot{\theta}_n\, c\theta_n\, s\alpha_m + \dot{\alpha}_m\, s\theta_n\, c\alpha_m)\}$$

$$A_x^n = \dot{\theta}_n\, c\psi_n(\dot{\psi}_n - \dot{\phi}_n\, c\theta_n) + \dot{\phi}_n\,\dot{\psi}_n\, s\psi_n\, s\theta_n$$

$$A_y^n = \dot{\theta}_n\, s\psi_n(\dot{\phi}_n\, c\theta_n - \dot{\psi}_n) + \dot{\phi}_n\,\dot{\psi}_n\, c\psi_n\, s\theta_n$$

$$A_z^n = -\dot{\phi}_n\,\dot{\theta}_n\, s\theta_n$$

Following are the elements of column vector E:

$$E_1 = -(C_2 A_{13}^{1} + C_3 A_{13}^{2} + C_4 A_{13}^{3} + C_5 A_{13}^{4} + C_6 A_{13}^{5}$$
$$+ C_7 A_{13}^{6} + C_8 \bar{A}_{13}^{57} + C_9 \bar{A}_{13}^{68} + C_{10} A_{13}^{9} + C_{11} A_{13}^{10}$$
$$+ C_{12} \bar{A}_{13}^{911} + C_{13} \bar{A}_{13}^{1012} + C_{14} A_{12}^{2} + C_{15} A_{12}^{4})$$

$$E_2 = -(C_2 A_{23}^{1} + C_3 A_{23}^{2} + C_4 A_{23}^{3} + C_5 A_{23}^{4} + C_6 A_{23}^{5}$$
$$+ C_7 A_{23}^{6} + C_8 \bar{A}_{23}^{57} + C_9 \bar{A}_{23}^{68} + C_{10} A_{23}^{9} + C_{11} A_{23}^{10}$$
$$+ C_{12} \bar{A}_{23}^{911} + C_{13} \bar{A}_{23}^{1012} + C_{14} A_{22}^{2} + C_{15} A_{22}^{4})$$

$$E_3 = -(C_2 A_{33}^{1} + C_3 A_{33}^{2} + C_4 A_{33}^{3} + C_5 A_{33}^{4} + C_6 A_{33}^{5}$$
$$+ C_7 A_{33}^{6} + C_8 \bar{A}_{33}^{57} + C_9 \bar{A}_{33}^{68} + C_{10} A_{33}^{9} + C_{11} A_{33}^{10}$$
$$+ C_{12} \bar{A}_{33}^{911} + C_{13} \bar{A}_{33}^{1012} + C_{14} A_{32}^{2} + C_{15} A_{32}^{4})$$

$$E_4 = s\theta_1 \{ C_{16} (A_{13}^{1} s\phi_1 - A_{23}^{1} c\phi_1) + L_1 [C_5 (-A_{13}^{4}$$
$$s\phi_1 + A_{23}^{4} c\phi_1) + C_{15} (-A_{12}^{4} s\phi_1 + A_{22}^{4} c\phi_1)$$
$$+ C_{10} (-A_{13}^{9} s\phi_1 + A_{23}^{9} c\phi_1) + C_{11} (-A_{13}^{10} s\phi_1 + A_{23}^{10}$$
$$c\phi_1) + C_{12} (-\bar{A}_{13}^{911} s\phi_1 + \bar{A}_{23}^{911} c\phi_1) + C_{13} (-\bar{A}_{13}^{1012}$$
$$s\phi_1 + \bar{A}_{23}^{1012} c\phi_1)] - A_x' I_{x_1} c\psi_1 + A_y' I_{y_1} s\psi_1$$
$$+ I_{z_1} \omega_{z_1} \dot{\theta}_1 - I_{x_1} \omega_{x_1} \dot{\psi}_1 s\psi_1 - I_{y_1} \omega_{y_1} \dot{\psi}_1 c\psi_1 \}$$
$$+ c\theta_1 [A_z' I_{z_1} + \dot{\theta}_1 (I_{x_1} \omega_{x_1} c\psi_1 - I_{y_1} \omega_{y_1} s\psi_1)]$$

$$E_5 = c\theta_1 \{ c\phi_1 [-C_{16} A_{13}^{1} + L_1 (C_5 A_{13}^{4} + C_{15} A_{12}^{4}$$
$$+ C_{10} A_{13}^{9} + C_{11} A_{13}^{10} + C_{12} \bar{A}_{13}^{911} + C_{13} \bar{A}_{13}^{1012})]$$
$$+ s\phi_1 [-C_{16} A_{23}^{1} + L_1 (C_5 A_{23}^{4} + C_{15} A_{22}^{4}$$
$$+ C_{10} A_{23}^{9} + C_{11} A_{23}^{10} + C_{12} \bar{A}_{23}^{911} + C_{13} \bar{A}_{23}^{1012})]$$
$$+ \dot{\phi}_1 (I_{y_1} \omega_{y_1} s\psi_1 - I_{x_1} \omega_{x_1} c\psi_1) \} - s\theta_1 [-C_{16}$$
$$A_{33}^{1} + L_1 (C_5 A_{33}^{4}$$

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(E5 cont'd)

$$+ C_{15} A^{4}_{32} + C_{10} A^{9}_{33} + C_{11} A^{10}_{33} + C_{12} \bar{A}^{9\,11}_{33} + C_{13} \bar{A}^{10\,12}_{33}) + I_{z_1} \omega_{z_1} \dot{\phi}_1] + \dot{\psi}_1 (I_{y_1} \omega_{y_1} s\psi_1 - I_{x_1} \omega_{x_1} c\psi_1 + A^{1}_{x} I_{x_1} s\psi_1 + A^{1}_{y} I_{y_1} c\psi_1$$

$$E_6 = A^{1}_{z} I_{z_1} + \dot{\theta}_1 (I_{x_1} \omega_{x_1} c\psi_1 - I_{y_1} \omega_{y_1} s\psi_1) + \dot{\phi}_1 s\theta_1 (I_{x_1} \omega_{x_1} s\psi_1 + I_{y_1} \omega_{y_1} c\psi_1)$$

$$E_7 = s\theta_2 \{ C_{17} [A^{2}_{13} s\phi_2 - A^{2}_{23} c\phi_2] + L_2 [C_4 (A^{3}_{13} s\phi_2 - A^{3}_{23} c\phi_2) + C_{14} (A^{2}_{12} s\phi_2 - A^{2}_{22} c\phi_2) + C_6 (A^{5}_{13} s\phi_2 - A^{5}_{23} c\phi_2) + C_7 (A^{6}_{13} s\phi_2 - A^{6}_{23} c\phi_2) + C_8 (\bar{A}^{57}_{13} s\phi_2 - \bar{A}^{57}_{23} c\phi_2) + C_9 (\bar{A}^{68}_{13} s\phi_2 - \bar{A}^{68}_{23} c\phi_2)] - A^{2}_{x} I_{x_2} c\psi_2 + A^{2}_{y} I_{y_2} s\psi_2 - \dot{\psi}_2 (I_{x_2} \omega_{x_2} s\psi_2 + I_{y_2} \omega_{y_2} c\psi_2) + \dot{\theta}_2 I_{z_2} \omega_{z_2} \} + (s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2) \{ -C_{20} A^{2}_{12} + L_3 [-C_6 A^{5}_{13} + C_7 A^{6}_{13} - C_8 \bar{A}^{57}_{13} + C_9 \bar{A}^{68}_{13}] - L_2 C_{14} A^{2}_{13} \} + (c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2) \{ C_{20} A^{2}_{22} + L_3 [C_6 A^{5}_{23} - C_7 A^{6}_{23} + C_8 \bar{A}^{57}_{23} - C_9 \bar{A}^{68}_{23}] + L_2 C_{14} A^{2}_{23} \} + c\theta_2 [A^{2}_{z} I_{z_2} + \dot{\theta}_2 (I_{x_2} \omega_{x_2} c\psi_2 - I_{y_2} \omega_{y_2} s\psi_2)]$$

$$E_8 = c\theta_2 \{ -C_{17} (A^{2}_{13} c\phi_2 + A^{2}_{23} s\phi_2) + L_2 [-C_{14} (A^{2}_{33} s\phi_2 + A^{2}_{12} c\phi_2 + A^{2}_{22} s\phi_2) - C_4 (A^{3}_{13} c\phi_2 + A^{3}_{23} s\phi_2) - C_6 (A^{5}_{13} c\phi_2 + A^{5}_{23} s\phi_2) - C_7 (A^{6}_{13} c\phi_2 + A^{6}_{23} s\phi_2)$$

(Cont. on next page)

(E_8 cont'd.)

$$
\begin{aligned}
&-C_8(\bar{A}^{57}_{13}\, c\phi_2 + \bar{A}^{57}_{23}\, s\phi_2) - C_9(\bar{A}^{68}_{13}\, c\phi_2 + \bar{A}^{68}_{23} \\
&s\phi_2)] + s\psi_2 [L_5(-C_6 A^5_{33} + C_7 A^6_{33} - C_8 \bar{A}^{57}_{33} \\
&+ C_9 \bar{A}^{68}_{33}) - C_{20} A^2_{32}] + \dot{\phi}_2 (I_{y_2}\, \omega_{y_2}\, s\psi_2 - I_{x_2} \\
&\omega_{x_2}\, c\psi_2)\} + s\theta_2\{C_{17} A^2_{33} + L_2[C_4 A^3_{33} + C_{14} \\
&A^2_{32} + C_6 A^5_{33} + C_7 A^6_{33} + C_8 \bar{A}^{57}_{33} + C_9 \bar{A}^{68}_{33}] \\
&+ s\psi_2 \langle -L_2[C_{14}(A^2_{13}\, c\phi_2 + A^2_{23}\, s\phi_2)] - C_{20} \\
&[A^2_{12}\, c\phi_2 + A^2_{22}\, s\phi_2] + L_5[-C_6(A^5_{13}\, c\phi_2 + A^5_{23} \\
&s\phi_2 + C_7(A^6_{13}\, c\phi_2 + A^6_{23}\, s\phi_2) - C_8(\bar{A}^{57}_{13}\, c\phi_2 \\
&+ \bar{A}^{57}_{23}\, s\phi_2) + C_9(\bar{A}^{68}_{13}\, c\phi_2 + \bar{A}^{68}_{23}\, s\phi_2)]\rangle \\
&- \dot{\phi}_2 I_{z_2}\, \omega_{z_2}\} + c\psi_2[A^2_y I_{y_2} - \dot{\psi}_2 I_{x_2} \\
&\omega_{x_2}] + s\psi_2[A^2_x I_{x_2} + \dot{\psi}_2 I_{y_2}\, \omega_{y_2}]
\end{aligned}
$$

$$
\begin{aligned}
E_9 = &-(s\phi_2\, s\psi_2 - c\phi_2\, c\theta_2\, c\psi_2)[L_2\; C_{14} A^2_{13} \\
&+ C_{20} A^2_{12} + L_5(C_6 A^5_{13} - C_7 A^6_{13} + C_8 \bar{A}^{57}_{13} \\
&- C_9 \bar{A}^{68}_{13})] + (s\phi_2\, c\theta_2\, c\psi_2 + c\phi_2\, s\psi_2) \\
&[L_2 C_{14} A^2_{23} + C_{20} A^2_{22} + L_5(C_6 A^5_{23} - C_7 A^6_{23} \\
&+ C_8 \bar{A}^{57}_{23} - C_9 \bar{A}^{68}_{23})] - s\theta_2\{c\psi_2[L_2 C_{14} \\
&A^2_{33} + C_{20} A^2_{32} + L_5(C_6 A^5_{33} - C_7 A^6_{33} + C_8 \\
&\bar{A}^{57}_{33} - C_9 \bar{A}^{68}_{33})] - \dot{\phi}_2(I_{x_2}\, \omega_{x_2}\, s\psi_2 + I_{y_2} \\
&\omega_{y_2}\, c\psi_2)\} + \dot{\theta}_2 (I_{x_2}\, \omega_{x_2}\, c\psi_2 - I_{y_2}\, \omega_{y_2}\, s\psi_2)
\end{aligned}
$$

$$
\begin{aligned}
E_{10} = &\, s\theta_3\{L_2 C_4(A^2_{13}\, s\phi_3 - A^2_{23}\, c\phi_3) + C_{18}(A^3_{13}\, s\phi_3 \\
&- A^3_{23}\, c\phi_3)
\end{aligned}
$$

(Cont. on next page)

(E_{10} cont'd.)

$$-c\psi_3\ (A_X^3 I_{X_3} + I_{Y_3}\omega_{Y_3}\dot{\psi}_3) + s\psi_3 (A_Y^3 I_{Y_3} - \dot{\psi}_3 I_{X_3}\omega_{X_3})$$
$$+ \dot{\theta}_3 I_{Z_3}\omega_{Z_3}\} + c\theta_3 \{A_Z^3 I_{Z_3} + \dot{\theta}_3 (I_{X_3}\omega_{X_3} c\psi_3 - I_{Y_3}$$
$$\omega_{Y_3} s\psi_3)\}\}$$

$$E_{11} = -c\theta_3 [L_2 C_4 (A_{13}^2\ c\phi_3 + A_{23}^2\ s\phi_3) + C_{18}(A_{13}^3 c\phi_3 + A_{23}^3 s\phi_3)$$
$$+ \dot{\phi}_3 (I_{X_3}\ \omega_{X_3} c\psi_3 - I_{Y_3}\omega_{Y_3} s\psi_3)] + s\theta_3 (C_{18} A_{33}^3 - \dot{\phi}_3 I_{Z_3}$$
$$\omega_{Z_3} + L_2 C_4 A_{33}^2) + s\psi_3 (A_X^3 I_{X_3} + \dot{\psi}_3 I_{Y_3}\omega_{Y_3}) + c\psi_3 (A_Y^3$$
$$I_{Y_3} - \dot{\psi}_3 I_{X_3}\ \omega_{X_3})$$

$$E_{12} = A_Z^3\ I_{Z_3} - \dot{\theta}_3 (I_{Y_3}\ \omega_{Y_3} s\psi_3 - I_{X_3}\omega_{X_3} c\psi_3) + \dot{\phi}_3$$
$$s\theta_3 (I_{X_3}\ \omega_{X_3} s\psi_3 + I_{Y_3}\ \omega_{Y_3}\ c\psi_3)$$

$$E_{13} = s\theta_4 \{ s\phi_4 [-L_1 C_5 A_{13}^1 + C_{19} A_{13}^4 - L_4 (C_{15} A_{12}^4 + C_{10}$$
$$A_{13}^9 + C_{11} A_{13}^{10} + C_{12} \bar{A}_{13}^{9\,11} + C_{13} \bar{A}_{13}^{10\,12})] + c\phi_4 [L_1$$
$$C_5 A_{23}^1 - C_{19} A_{23}^4 + L_4 (C_{15} A_{22}^4 + C_{10} A_{23}^9 + C_{11}$$
$$A_{23}^{10} + C_{12} \bar{A}_{23}^{9\,11} + C_{13} \bar{A}_{23}^{10\,12})] - A_X^4 I_{X_4} c\psi_4 \} + (s\phi_4$$
$$c\theta_4 s\psi_4 - c\phi_4 c\psi_4)[C_{15} (L_1 A_{13}^1 + L_4 A_{13}^4 - C_{25}$$
$$A_{12}^4 - L_H (C_{10} A_{13}^9 - C_{11} A_{13}^{10} + C_{12} \bar{A}_{13}^{9\,11} - C_{13} \bar{A}_{13}^{10\,12})]$$
$$-(c\phi_4\ c\theta_4\ s\psi_4 + s\phi_4\ c\psi_4)[C_{15}\ (L_1 A_{23}^1 + L_4 A_{23}^4)$$
$$- C_{25} A_{22}^4 - L_H\ (C_{10} A_{23}^9 - C_{11} A_{23}^{10} + C_{12} \bar{A}_{23}^{9\,11}$$
$$+ C_{13}\ \bar{A}_{23}^{10\,12})] - \dot{\psi}_4 s\theta_4\ (I_{X_4}\ \omega_{X_4} s\psi_4 + I_{Y_4}\ \omega_{Y_4}$$
$$c\psi_4) + \dot{\theta}_4 [c\theta_4 (I_{X_4}\ \omega_{X_4} c\psi_4 - I_{Y_4}\ \omega_{Y_4} s\psi_4) + I_{Z_4}$$
$$\omega_{Z_4} s\theta_4] + A_Z^4 I_{Z_4}\ c\theta_4 + A_Y^4 I_{Y_4} s\psi_4\ s\theta_4$$

$$
\begin{aligned}
E_{14} = {} & c\theta_4\{c\phi_4[L_1 C_5 A^{1}_{13} - C_{19} A^{4}_{13} + L_4(C_{15} A^{4}_{12} \\
& + C_{10} A^{9}_{13} + C_{11} A^{10}_{13} + C_{12} \bar{A}^{9\,11}_{13} + C_{13} \bar{A}^{10\,12}_{13})] \\
& + s\phi_4[L_1 C_5 A^{1}_{23} - C_{19} A^{4}_{23} + L_4(C_{15} A^{4}_{22} \\
& + C_{10} A^{9}_{23} + C_{11} A^{10}_{23} + C_{12} \bar{A}^{9\,11}_{23} + C_{13} \bar{A}^{10\,12}_{23})] \\
& + s\psi_4[C_{15}(L_1 A^{1}_{33} + L_4 A^{4}_{33}) - C_{25} A^{4}_{32} - L_H \\
& (C_{10} A^{9}_{33} - C_{11} A^{10}_{33} + C_{12} \bar{A}^{9\,11}_{33} - C_{13} \bar{A}^{10\,12}_{33}) \\
& + I_{y_4} \omega_{y_4} \dot{\phi}_4] - \dot{\phi}_4 I_{x_4} \omega_{x_4} c\psi_4\} + s\theta_4 \\
& \{-\dot{\phi}_4 I_{z_4} \omega_{z_4} + s\psi_4 \langle c\phi_4[C_{15}(L_1 A^{1}_{13} + L_4 A^{4}_{13}) \\
& - C_{25} A^{4}_{12} - L_H(C_{10} A^{9}_{13} - C_{11} A^{10}_{13} + C_{12} \bar{A}^{9\,11}_{13} \\
& - C_{13} \bar{A}^{10\,12}_{13})] + s\phi_4[C_{15}(L_1 A^{1}_{23} + L_4 A^{4}_{23}) \\
& - C_{25} A^{4}_{22} - L_H(C_{10} A^{9}_{23} - C_{11} A^{10}_{23} + C_{12} \bar{A}^{9\,11}_{23} \\
& - C_{13} \bar{A}^{10\,12}_{23})]\rangle - [L_4 C_{15} A^{4}_{32} + L_1 C_5 A^{1}_{33} \\
& - C_{19} A^{4}_{33} + L_4(C_{10} A^{9}_{33} + C_{11} A^{10}_{33} + C_{12} \bar{A}^{9\,11}_{33} \\
& + C_{13} \bar{A}^{10\,12}_{33})]\} + A^{4}_{x} I_{x_4} s\psi_4 + A^{4}_{y} I_{y_4} c\psi_4 \\
& + \dot{\psi}_4 (I_{y_4} \omega_{y_4} s\psi_4 - I_{x_4} \omega_{x_4} c\psi_4)
\end{aligned}
$$

$$
\begin{aligned}
E_{15} = {} & (-c\phi_4 c\theta_4 c\psi_4 + s\phi_4 s\psi_4)[C_{15}(L_1 A^{1}_{13} + L_4 \\
& A^{4}_{13}) - C_{25} A^{4}_{12} - L_H (C_{10} A^{9}_{13} - C_{11} A^{10}_{13} \\
& + C_{12} \bar{A}^{9\,11}_{13} - C_{13} \bar{A}^{10\,12}_{13})] - (s\phi_4 c\theta_4 c\psi_4 \\
& + c\phi_4 s\psi_4)[C_{15}(L_1 A^{1}_{23} + L_4 A^{4}_{23}) - C_{25} A^{4}_{22} \\
& - L_H(C_{10} A^{9}_{23} - C_{11} A^{10}_{23} + C_{12} \bar{A}^{9\,11}_{23} - C_{13} \bar{A}^{10\,12}_{23})] \\
& + s\theta_4\{\dot{\phi}_4 (I_{x_4} \omega_{x_4} s\psi_4 + I_{y_4} \omega_{y_4} c\psi_4) + c\psi_4 \\
& [C_{15}(L_1 A^{1}_{33}
\end{aligned}
$$

(cont. on next page)

(E$_{15}$ cont'd)

$$+L_4 A^4_{33}) - C_{25} A^4_{32} - L_H (C_{10} A^9_{33} - C_{11} A^{10}_{33} + C_{12} \bar{A}^{911}_{33} - C_{13} \bar{A}^{1012}_{33})]\} + \dot{\theta}_4 (I_{x4} \omega_{x4} C\psi_4 - I_{y4} \omega_{y4} S\psi_4) - A^4_z I_{z4}$$

$$E_{16} = (A^2_{13} L_2 + A^2_{12} L_5)[C_6 S\phi_5 S\theta_5 + C_8 (S\phi_5 C\theta_5 C\psi_5 S\alpha_7 + C\phi_5 S\psi_5 S\alpha_7 + S\phi_5 S\theta_5 C\alpha_7)] - (A^2_{23} L_2 + A^2_{22} L_5)[C_6 C\phi_5 S\theta_5 + C_8 (C\phi_5 C\theta_5 C\psi_5 S\alpha_7 - S\phi_5 S\psi_5 S\alpha_7 + C\phi_5 S\theta_5 C\alpha_7)] + (A^5_{13} L_5 C_8 - \bar{A}^{57}_{13} C_{23})(-S\phi_5 C\theta_5 C\psi_5 S\alpha_7 - C\phi_5 S\psi_5 S\alpha_7 - S\phi_5 S\theta_5 C\alpha_7) + (A^5_{23} L_5 C_8 - \bar{A}^{57}_{23} C_{23})(C\phi_5 C\theta_5 C\psi_5 S\alpha_7 - S\phi_5 S\psi_5 S\alpha_7 + C\phi_5 S\theta_5 C\alpha_7) + S\theta_5 [(A^5_{13} C_{21} - \bar{A}^{57}_{13} L_5 C_8) S\phi_5 - (A^5_{23} C_{21} - \bar{A}^{57}_{23} L_5 C_8) C\phi_5] + S\theta_5 [+A^5_x I_{x5} C\psi_5 - A^5_y I_{y5} S\psi_5] - A^5_z I_{z5} C\theta_5 + I_{x5} (\dot{\theta}_5 S\psi_5 - \dot{\phi}_5 C\psi_5 S\theta_5)(-\dot{\psi}_5 S\psi_5 S\theta_5 + \dot{\theta}_5 C\psi_5 C\theta_5) + I_{y5} (\dot{\phi}_5 S\psi_5 S\theta_5 + \dot{\theta}_5 C\psi_5)(-\dot{\psi}_5 C\psi_5 S\theta_5 - \dot{\theta}_5 S\psi_5 C\theta_5) + I_{z5} (\dot{\phi}_5 C\theta_5 + \dot{\psi}_5) \dot{\theta}_5 S\theta_5 + I_{x7} (S\theta_5 C\psi_5 C\alpha_7 + C\theta_5 S\alpha_7) \{A^5_x C\alpha_7 + A^5_z S\alpha_7 - \dot{\alpha}_7 (\omega_{x5} S\alpha_7 + \omega_{z5} C\alpha_7)\} - I_{y7} (S\theta_5$$

(Cont. on next page)

(E16 cont'd)

$$S\psi_5)A_Y^5 + I_{z7}(-C\theta_5 C\alpha_7 + S\theta_5 C\psi_5 S\alpha_7)$$
$$[\dot{\alpha}_7(\omega_{x5} C\alpha_7 - \omega_{z5} S\alpha_7) + A_x^5 S\alpha_7$$
$$+ A_z^5 C\alpha_7] - I_{x7}\bar{\omega}_{x7}[\dot{\theta}_5(S\theta_5 S\alpha_7$$
$$- C\theta_5 C\psi_5 C\alpha_7) + \dot{\psi}_5 S\theta_5 S\psi_5 C\alpha_7 + \dot{\alpha}_7$$
$$(S\theta_5 C\psi_5 S\alpha_7 - C\theta_5 C\alpha_7)] - I_{Y7}\bar{\omega}_{Y7}$$
$$(\dot{\theta}_5 C\theta_5 S\psi_5 + \dot{\psi}_5 S\theta_5 C\psi_5) + I_{z7}\bar{\omega}_{z7}$$
$$[\dot{\theta}_5(S\theta_5 C\alpha_7 + C\theta_5 C\psi_5 S\alpha_7) - \dot{\psi}_5 S\theta_5 S\psi_5$$
$$S\alpha_7 + \dot{\alpha}_7(C\theta_5 S\alpha_7 + S\theta_5 C\psi_5 C\alpha_7)]$$

$$E_{17} = [-L_2(A_{13}^2 C\phi_5 + A_{23}^2 S\phi_5) - L_5(A_{12}^2 C\phi_5 + A_{22}^2$$
$$S\phi_5)][C_6 C\theta_5 + C_8(-S\theta_5 C\psi_5 S\alpha_7 + C\theta_5$$
$$C\alpha_7)] + [A_{33}^2 L_2 + A_{32}^2 L_5][C_6 S\theta_5 + C_8$$
$$(C\theta_5 C\psi_5 S\alpha_7 + S\theta_5 C\alpha_7)] - [A_{13}^5 C\phi_5 + A_{23}^5$$
$$S\phi_5][C_{21} C\theta_5 - L_5 C_8(-S\theta_5 C\psi_5 S\alpha_7 + C\theta_5$$
$$C\alpha_7)] - [\bar{A}_{13}^{57} C\phi_5 + \bar{A}_{23}^{57} S\phi_5][C_{23}(-S\theta_5$$
$$C\psi_5 S\alpha_7 + C\theta_5 C\alpha_7) - L_5 C_8 C\theta_5]$$
$$+ [L_5 C_8 A_{33}^5 - \bar{A}_{33}^{57} C_{23}][-C\theta_5 C\psi_5 S\alpha_7 - S\theta_5$$
$$C\alpha_7] + [A_{33}^5 C_{21} - \bar{A}_{33}^{57} L_5 C_8] S\theta_5 + I_{x5}$$
$$[-A_x^5 S\psi_5 + (\dot{\theta}_5 S\psi_5 - \dot{\phi}_5 C\psi_5 S\theta_5)(-\dot{\phi}_5$$
$$C\psi_5 C\theta_5 - \dot{\psi}_5 C\psi_5)] + I_{Y5}[-A_Y^5 C\psi_5 + (\dot{\phi}_5$$
$$S\psi_5 S\theta_5 + \dot{\theta}_5 C\psi_5)(\dot{\phi}_5 S\psi_5 C\theta_5 + \dot{\psi}_5 S\psi_5)]$$

(Cont. on next page)

(E_{17} Cont'd)

$$-I_{z5}\dot{\phi}_5 S\theta_5(\dot{\phi}_5 C\theta_5+\dot{\psi}_5)-I_{x7}(S\psi_5 C\alpha_7)$$
$$\{A_x^5 C\alpha_7+A_z^5 S\alpha_7-\dot{\alpha}_7(\omega_{x5}S\alpha_7+\omega_{z5}$$
$$C\alpha_7)\}-I_{y7}C\psi_5 A_y^5-I_{z7}(S\psi_5 S\alpha_7)$$
$$[\dot{\alpha}_7(\omega_{x5}C\alpha_7-\omega_{z5}S\alpha_7)+A_x^5 S\alpha_7$$
$$+A_z^5 C\alpha_7]+I_{x7}\bar{\omega}_{x7}[-\dot{\phi}_5(C\theta_5 C\psi_5 C\alpha_7$$
$$-S\theta_5 S\alpha_7)-\dot{\psi}_5 C\psi_5 C\alpha_7+\dot{\alpha}_7 S\psi_5 S\alpha_7]$$
$$+I_{y7}\bar{\omega}_{y7}(\dot{\phi}_5 C\theta_5 S\psi_5+\dot{\psi}_5 S\psi_5)+I_{z7}\bar{\omega}_{z7}$$
$$[-\dot{\phi}_5(C\theta_5 C\psi_5 S\alpha_7+S\theta_5 C\alpha_7)-\dot{\psi}_5 C\psi_5$$
$$S\alpha_7-\dot{\alpha}_7 S\psi_5 C\alpha_7]$$

$$E_{18}=[C_8(A_{13}^2 L_2+A_{12}^2 L_5)-L_5 C_8 A_{13}^5+A_{13}^{-57}$$
$$C_{23}](C\phi_5 C\theta_5 S\psi_5 S\alpha_7+S\phi_5 C\psi_5 S\alpha_7)$$
$$+[C_8(A_{23}^2 L_2+A_{22}^2 L_5)-L_5 C_8 A_{23}^5$$
$$+A_{23}^{-57}C_{23}](S\phi_5 C\theta_5 S\psi_5 S\alpha_7-C\phi_5 C\psi_5$$
$$S\alpha_7)-[C_8(A_{33}^2 L_2+A_{32}^2 L_5)-L_5 C_8 A_{33}^5$$
$$+A_{33}^{-57}C_{23}](S\theta_5 S\psi_5 S\alpha_7)+(I_{x5}-I_{y5})$$
$$(\dot{\theta}S\psi_5-\dot{\phi}_5 C\psi_5 S\theta_5)(\dot{\theta}_5 C\psi_5+\dot{\phi}_5 S\psi_5$$
$$S\theta_5)-A_z^5 I_{z5}+I_{x7}S\alpha_7[A_x^5 C\alpha_7+A_z^5$$
$$S\alpha_7-\dot{\alpha}_7(\omega_{x5}S\alpha_7+\omega_{z5}C\alpha_7)]$$
$$-I_{z7}C\alpha_7[\dot{\alpha}_7(\omega_{x5}C\alpha_7-\omega_{z5}S\alpha_7)$$
$$+A_x^5 S\alpha_7+A_z^5 C\alpha_7]+(I_{x7}\bar{\omega}_{x7}C\alpha_7$$

(Cont. on next page)

(E_{18} cont'd)

$$+ I_{Z7}\bar{\omega}_{Z7} S\alpha_7)(\dot{\phi}_5 S\theta_5 S\psi_5 + \dot{\theta}_5 C\psi_5 + \dot{\alpha}_7) + I_{Y7}\bar{\omega}_{Y7}(\dot{\phi}_5 S\theta_5 C\psi_5 - \dot{\theta}_5 S\psi_5)$$

$$E_{19} = S\theta_6\{-S\phi_6[-C_7(L_2 A_{13}^2 - L_5 A_{12}^2) + L_6 C_9 \bar{A}_{13}^{68} - C_{22} A_{13}^6] + C\phi_6[-C_7(L_2 A_{23}^2 - L_5 A_{22}^2) + L_6 C_9 \bar{A}_{23}^{68} - C_{22} A_{23}^6] - C\psi_6 (-A_X^6 I_{X6} + \dot{\psi}_6 I_{Y6}\omega_{Y6}) + \dot{\theta}_6 I_{Z6}\omega_{Z6} + S\psi_6 (-A_Y^6 I_{Y6} - \dot{\psi}_6 I_{X6}\omega_{X6})\} - (S\phi_6 C\theta_6 C\psi_6 S\alpha_8 + C\phi_6 S\psi_6 S\alpha_8 + S\phi_6 S\theta_6 C\alpha_8)[-C_9(L_2 A_{13}^2 - L_5 A_{12}^2 - L_6 A_{13}^6) - C_{24}\bar{A}_{13}^{68}] + (C\phi_6 C\theta_6 C\psi_6 S\alpha_8 - S\phi_6 S\psi_6 S\alpha_8 + C\phi_6 S\theta_6 C\alpha_8)[-C_9(L_2 A_{23}^2 - L_5 A_{22}^2 - L_6 A_{23}^6) - C_{24}\bar{A}_{23}^{68}] + C\theta_6[\dot{\theta}_6(I_{X6}\omega_{X6} C\psi_6 - I_{Y6}\omega_{Y6} S\psi_6) - A_Z^6 I_{Z6}] + I_{X8}(S\theta_6 C\psi_6 C\alpha_8 + C\theta_6 S\alpha_8)[A_X^6 C\alpha_8 + A_Z^6 S\alpha_8 - \dot{\alpha}_8(\omega_{X6} S\alpha_8 + \omega_{Z6} C\alpha_8)] - I_{Y8}(S\theta_6 S\psi_6) A_Y^6 - I_{Z8}(C\theta_6 C\alpha_8 - S\theta_6 C\psi_6 S\alpha_8)[\dot{\alpha}_8(\omega_{X6} C\alpha_8 - \omega_{Z6} S\alpha_8) + A_X^6 S\alpha_8 + A_Z^6 C\alpha_8] - I_{X8}\bar{\omega}_{X8}[\dot{\theta}_6(S\theta_6 S\alpha_8 - C\theta_6 C\psi_6 C\alpha_8) + \dot{\psi}_6 S\theta_6 S\psi_6 C\alpha_8 + \dot{\alpha}_8(S\theta_6 C\psi_6 S\alpha_8 - C\theta_6 C\alpha_8)] - I_{Y8}\bar{\omega}_{Y8}(\dot{\theta}_6 C\theta_6 S\psi_6 + \dot{\psi} S\theta_6 C\psi_6)$$

(Cont. on next page)

(E_{19} Cont'd.)

$$+ I_{z8}\, \bar{\omega}_{z8} [\dot{\theta}_6 (S\theta_6 C\alpha_8 + C\theta_6 C\psi_6 S\alpha_8)$$
$$- \dot{\psi}_6 (S\theta_6 S\psi_6 S\alpha_8) + \dot{\alpha}_8 (C\theta_6 S\alpha_8 + S\theta_6 C\psi_6$$
$$C\alpha_8)]$$

$$E_{20} = C\theta_6 \{ C\phi_6 [C_7 (L_5 A_{12}^2 - L_2 A_{13}^2) - C_{22} A_{13}^6$$
$$+ L_6 C_9 \bar{A}_{13}^{68}] + S\phi_6 [C_7 (L_5 A_{22}^2 - L_2 A_{23}^2)$$
$$- C_{22} A_{23}^6 + L_6 C_9 \bar{A}_{23}^{68}] + \dot{\phi}_6 (I_{y6}\, \omega_{y6} S\psi_6$$
$$- I_{x6}\, \omega_{x6} C\psi_6) \} - S\theta_6 \{ C_7 (L_5 A_{32}^2 - L_2 A_{33}^2)$$
$$- C_{22} A_{33}^6 + L_6 C_9 \bar{A}_{33}^{68} + \dot{\phi}_6 I_{z6}\, \omega_{z6} \}$$
$$- (S\theta_6 C\psi_6 S\alpha_8 - C\theta_6 C\alpha_8) \{ C\phi_6 [C_9 (L_5 A_{12}^2$$
$$- L_2 A_{13}^2 + L_6 A_{13}^6) - C_{24} \bar{A}_{13}^{68}] + S\phi_6 [C_9$$
$$(L_5 A_{22}^2 - L_2 A_{23}^2 + L_6 A_{23}^6) - C_{24} \bar{A}_{23}^{68}] \}$$
$$- (C\theta_6 C\psi_6 S\alpha_8 + S\theta_6 C\alpha_8) [C_9 (L_5 A_{32}^2$$
$$- L_2 A_{33}^2 + L_6 A_{33}^6) - C_{24} \bar{A}_{33}^{68}] + \dot{\psi}_6 (I_{y6}$$
$$\omega_{y6} S\psi_6 - I_{x6}\, \omega_{x6} C\psi_6) - A_x^6 I_{x6} S\psi_6$$
$$- A_y^6 I_{y6} C\psi_6 - I_{x8} S\psi_6 C\alpha_8 [A_x^6 C\alpha_8$$
$$+ A_z^6 S\alpha_8 - \dot{\alpha}_8 (\omega_{x6} S\alpha_8 + \omega_{z6} C\alpha_8)]$$
$$- I_{y8} C\psi_6 A_y^6 - I_{z8} S\psi_6 S\alpha_8 [\dot{\alpha}_8 (\omega_{x6} C\alpha_8$$
$$- \omega_{z6} S\alpha_8) + A_x^6 S\alpha_8 + A_z^6 C\alpha_8]$$
$$+ I_{x8}\, \bar{\omega}_{x8} [-\dot{\phi}_6 (C\theta_6 C\psi_6 C\alpha_8 - S\theta_6 S\alpha_8)$$
$$- \dot{\psi}_6 C\psi_6 C\alpha_8 + \dot{\alpha}_8 S\psi_6 S\alpha_8] + I_{y8}\, \bar{\omega}_{y8}$$

(Cont. on next page)

(E_{20} Cont'd.)

$$[\dot{\phi}_6 C\theta_6 S\psi_6 + \dot{\psi}_6 S\psi_6] + I_{Z8}\,\bar{\omega}_{Z8}[-\dot{\phi}_6 (C\theta_6 C\psi_6 S\alpha_8 + S\theta_6 C\alpha_8) - \dot{\psi}_6 C\psi_6 S\alpha_8 - \dot{\alpha}_8 S\psi_6 C\alpha_8]$$

$$E_{21} = -S\alpha_8\{(C\phi_6 C\theta_6 S\psi_6 + S\phi_6 C\psi_6)[C_9 (L_5 A_{12}^2 - L_2 A_{13}^2 + L_6 A_{13}^6) - C_{24}\bar{A}_{13}^{68}] + (S\phi_6 C\theta_6 S\psi_6 - C\phi_6 C\psi_6)[C_9 (L_5 A_{22}^2 - L_2 A_{23}^2 + L_6 A_{23}^6) - C_{24}\bar{A}_{23}^{68}] - S\theta_6 S\psi_6 [C_9 (L_5 A_{32}^2 - L_2 A_{33}^2 + L_6 A_{33}^6) - C_{24}\bar{A}_{33}^{68}]\} - A_Z^6 I_{Z6} + \dot{\phi}_6 S\theta_6 (I_{X6}\omega_{X6} S\psi_6 + I_{Y6}\omega_{Y6} C\psi_6) + \dot{\theta}_6 (I_{X6}\omega_{X6} C\psi_6 - I_{Y6}\omega_{Y6} S\psi_6) + I_{X8} S\alpha_8 [A_X^6 C\alpha_8 + A_Z^6 S\alpha_8 - \dot{\alpha}_8 (\omega_{X6} S\alpha_8 + \omega_{Z6} C\alpha_8)] - I_{Z8} C\alpha_8 [\dot{\alpha}_8 (\omega_{X6} C\alpha_8 - \omega_{Z6} S\alpha_8) + A_X^6 S\alpha_8 + A_Z^6 C\alpha_8] + [I_{X8}\bar{\omega}_{X8} C\alpha_8 + I_{Z8}\bar{\omega}_{Z8} S\alpha_8][\dot{\phi}_6 S\theta_6 S\psi_6 + \dot{\theta}_6 C\psi_6 + \dot{\alpha}_8] + I_{Y8}\bar{\omega}_{Y8} (\dot{\phi}_6 S\theta_6 C\psi_6 - \dot{\theta}_6 S\psi_6)$$

$$E_{22} = (C\phi_5 C\theta_5 C\psi_5 C\alpha_7 - S\phi_5 S\psi_5 C\alpha_7 - C\phi_5 S\theta_5 S\alpha_7)\{-C_8 [L_2 A_{13}^2 + L_5 A_{12}^2 - L_5 A_{13}^5] - C_{23}\bar{A}_{13}^{57}\} + (S\phi_5 C\theta_5 C\psi_5 C\alpha_7 + C\phi_5 S\psi_5 C\alpha_7 - S\phi_5 S\theta_5 S\alpha_7)[-C_8 (L_2 A_{23}^2 + L_5 A_{22}^2$$

(Cont. on next page)

(E_{22} Cont'd)

$$-L_5 A_{23}^5) - C_{23}\bar{A}_{23}^{57}] - (S\theta_5 C\psi_5 C\alpha_7 + C\theta_5 S\alpha_7)[-C_8(L_2 A_{33}^2 + L_5 A_{32}^2 - L_5 A_{33}^5) - C_{23} \bar{A}_{33}^{57}] - I_{Y7} A_Y^5 + I_{X7}\bar{\omega}_{X7}(-\omega_{X5} S\alpha_7 - \omega_{Z5} C\alpha_7) + I_{Z7}\bar{\omega}_{Z7}(\omega_{X5} C\alpha_7 - \omega_{Z5} S\alpha_7)$$

$$E_{23} = (C\phi_6 C\theta_6 C\psi_6 C\alpha_8 - S\phi_6 S\psi_6 C\alpha_8 - C\phi_6 S\theta_6 S\alpha_8)[C_9(L_5 A_{12}^2 - L_2 A_{13}^2 + L_6 A_{13}^6) - C_{24}\bar{A}_{13}^{68}] + (S\phi_6 C\theta_6 C\psi_6 C\alpha_8 + C\phi_6 S\psi_6 C\alpha_8 - S\phi_6 S\theta_6 S\alpha_8)[C_9(L_5 A_{22}^2 - L_2 A_{23}^2 + L_6 A_{23}^6) - C_{24}\bar{A}_{23}^{68}] - (S\theta_6 C\psi_6 C\alpha_8 + C\theta_6 S\alpha_8)[C_9(L_5 A_{32}^2 - L_2 A_{33}^2 + L_6 A_{33}^6) - C_{24}\bar{A}_{33}^{68}] - A_Y^6 I_{Y8} + I_{X8}\bar{\omega}_{X8}(-\omega_{X6} S\alpha_8 - \omega_{Z6} C\alpha_8) + I_{Z8}\bar{\omega}_{Z8}(\omega_{X6} C\alpha_8 - \omega_{Z6} S\alpha_8)$$

$$E_{24} = [A_{13}^1 L_1 + A_{13}^4 L_4 - A_{12}^4 L_{11}][-C_{10} S\phi_9 S\theta_9 - C_{12}(S\phi_9 C\theta_9 C\psi_9 S\alpha_{11} + C\phi_9 S\psi_9 S\alpha_{11} + S\phi_9 S\theta_9 C\alpha_{11})] + [A_{23}^1 L_1 + A_{23}^4 L_4 - A_{22}^4 L_{11}][C_{10} C\phi_9 S\theta_9 + C_{12}(C\phi_9 C\theta_9 C\psi_9 S\alpha_{11} - S\phi_9 S\psi_9 S\alpha_{11} + C\phi_9 S\theta_9 C\alpha_{11})] + A_{13}^9[C_{26} S\phi_9 S\theta_9 + L_9 C_{12}(-S\phi_9 C\theta_9 C\psi_9 S\alpha_{11} - C\phi_9 S\psi_9 S\alpha_{11} - S\phi_9 S\theta_9 C\alpha_{11})] - A_{23}^9[C_{26} C\phi_9 S\theta_9$$

(Cont. on next page)

(E_{z4} Cont'd.)

$$-L_9 C_{12}(C\phi_9 C\theta_9 C\psi_9 S\alpha_{11} - S\phi_9 S\psi_9 S\alpha_{11}$$
$$+ C\phi_9 S\theta_9 C\alpha_{11})] + \bar{A}_{13}^{911}[C_{ZB}(S\phi_9 C\theta_9 C\psi_9$$
$$S\alpha_{11} + C\phi_9 S\psi_9 S\alpha_{11} + S\phi_9 S\theta_9 C\alpha_{11}) - L_9 C_{12}$$
$$S\phi_9 S\theta_9] - \bar{A}_{23}^{911}[C_{ZB}(C\phi_9 C\theta_9 C\psi_9 S\alpha_{11}$$
$$- S\phi_9 S\psi_9 S\alpha_{11} + C\phi_9 S\theta_9 C\alpha_{11}) - L_9 C_{12} C\phi_9$$
$$S\theta_9] + S\theta_9(+\Delta_x^9 I_{x9} C\psi_9 - \Delta_y^9 I_{y9} S\psi_9)$$
$$- \dot{A}_z^9 I_{z9} C\theta_9 + I_{x9}(\dot{\theta}_9 S\psi_9 - \dot{\phi}_9 C\psi_9 S\theta_9)$$
$$(\dot{\theta}_9 C\psi_9 C\theta_9 - \dot{\psi}_9 S\psi_9 S\theta_9) + I_{y9}(\dot{\phi}_9 S\psi_9$$
$$S\theta_9 + \dot{\theta}_9 C\psi_9)(-\dot{\psi}_9 C\psi_9 S\theta_9 - \dot{\theta}_9 S\psi_9 C\theta_9)$$
$$+ I_{z9}(\dot{\phi}_9 C\theta_9 + \dot{\psi}_9)(\dot{\theta}_9 S\theta_9) + I_{x11}(S\theta_9$$
$$C\psi_9 C\alpha_{11} + C\theta_9 S\alpha_{11})[\Delta_x^9 C\alpha_{11} + \Delta_z^9 S\alpha_{11}$$
$$- \dot{\alpha}_{11}(\omega_{x9} S\alpha_{11} + \omega_{z9} C\alpha_{11})] - I_{y11}(S\theta_9$$
$$S\psi_9)\Delta_y^9 + I_{z11}(-C\theta_9 C\alpha_{11} + S\theta_9 C\psi_9 S\alpha_{11})$$
$$[\dot{\alpha}_{11}(\omega_{x9} C\alpha_{11} - \omega_{z9} S\alpha_{11}) + \Delta_x^9 S\alpha_{11} + \Delta_z^9$$
$$C\alpha_{11}] + I_{x11}\bar{\omega}_{x11}[-\dot{\theta}_9(S\theta_9 S\alpha_{11} - C\theta_9 C\psi_9$$
$$C\alpha_{11}) - \dot{\psi}_9 S\theta_9 S\psi_9 C\alpha_{11} - \dot{\alpha}_{11}(S\theta_9 C\psi_9 S\alpha_{11}$$
$$- C\theta_9 C\alpha_{11})] + I_{y11}\bar{\omega}_{y11}(-\dot{\theta}_9 C\theta_9 S\psi_9 - \dot{\psi}_9$$
$$S\theta_9 C\psi_9) + I_{z11}\bar{\omega}_{z11}[\dot{\theta}_9(S\theta_9 C\alpha_{11} + C\theta_9$$
$$C\psi_9 S\alpha_{11}) - \dot{\psi}_9 S\theta_9 S\psi_9 S\alpha_{11} + \dot{\alpha}_{11}(C\theta_9$$
$$S\alpha_{11} + S\theta_9 C\psi_9 C\alpha_{11})]$$

$$
\begin{aligned}
E_{z5} = & (C\phi_9 C\theta_9)[C_{10}(L_1 A'_{13} + L_4 A^4_{13} - L_H A^4_{12}) \\
& - C_{z6} A^9_{13} + \bar{A}^{911}_{13} L_9 C_{12}] + C\phi_9(-S\theta_9 C\psi_9 S\alpha_{11} \\
& + C\theta_{10} C\alpha_{12})[C_{12}(L_1 A'_{13} + L_4 A^4_{13} - L_H \\
& A^4_{12} + L_9 A^9_{13}) - \bar{A}^{911}_{13} C_{z8}] + S\phi_9 C\theta_9 [(L_1 \\
& A'_{23} + L_4 A^4_{23} - L_H A^4_{22}) C_{10} - C_{z6} A^9_{23} + L_9 \\
& C_{12} \bar{A}^{911}_{23}] + S\phi_9(-S\theta_9 C\psi_9 S\alpha_{11} + C\theta_9 \\
& C\alpha_{11})[C_{12}(L_1 A'_{23} + L_4 A^4_{23} - L_H A^4_{22} + L_9 \\
& A^9_{23}) - C_{z8} \bar{A}^{911}_{23}] - S\theta_9 [C_{10}(L_1 A'_{33} + L_4 A^4_{33} \\
& - L_H A^4_{32}) - C_{z6} A^9_{33} + L_9 C_{12} \bar{A}^{911}_{33}] + (-C\theta_9 \\
& C\psi_9 S\alpha_{11} - S\theta_9 C\alpha_{11})[C_{12}(L_1 A'_{33} + L_4 A^4_{33} \\
& - L_H A^4_{32} + L_9 A^9_{33} - C_{z8} \bar{A}^{911}_{33}] - A^9_x I_{x9} \\
& S\psi_9 - A^9_y I_{y9} C\psi_9 - I_{x9} \omega_{x9} C\psi_9 (\dot{\phi}_9 C\theta_9 \\
& + \dot{\psi}_9) + I_{y9} \omega_{y9} S\psi_9 (\dot{\phi}_9 C\theta_9 + \dot{\psi}_9) - I_{z9} \\
& \omega_{z9} \dot{\phi}_9 S\theta_9 - I_{x11} S\psi_9 C\alpha_{11} [A^9_x C\alpha_{11} + A^9_z \\
& S\alpha_{11} - \dot{\alpha}_{11}(\omega_{x9} S\alpha_{11} + \omega_{z9} C\alpha_{11})] - I_{y11} C\psi_9 \\
& A^9_y - I_{z11} S\psi_9 S\alpha_{11} [\dot{\alpha}_{11}(\omega_{x9} C\alpha_{11} - \omega_{z9} S\alpha_{11}) \\
& + A^9_x S\alpha_{11} + A^9_z C\alpha_{11}] + I_{x11} \bar{\omega}_{x11} [-\dot{\phi}_9 (C\theta_9 \\
& C\psi_9 C\alpha_{11} - S\theta_9 S\alpha_{11}) - \dot{\psi}_9 C\psi_9 C\alpha_{11} + \dot{\alpha}_{11} \\
& S\psi_9 S\alpha_{11}] + I_{y11} \bar{\omega}_{y11} (\dot{\phi}_9 C\theta_9 S\psi_9 + \dot{\psi}_9 \\
& S\psi_9) + I_{z11} \bar{\omega}_{z11} [-\dot{\phi}_9 (C\theta_9 C\psi_9 S\alpha_{11} +
\end{aligned}
$$

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(E_{25} Cont'd)

$$S\theta_9 C\alpha_{11}) - \dot{\psi}_9 C\psi_9 S\alpha_{11} - \dot{\alpha}_{11} S\psi_9 C\alpha_{11}]$$

$$E_{26} = S\alpha_{11}\{[C_{12}(A^1_{13}L_1 + A^4_{13}L_4 - A^4_{12}L_H + A^9_{13}$$
$$L_9) - \bar{A}^{911}_{13} C_{28}][-C\phi_9 C\theta_9 S\psi_9 - S\phi_9 C\psi_9]$$
$$+[C_{12}(A^1_{23}L_1 + A^4_{23}L_4 - A^4_{22}L_H + A^9_{23}$$
$$L_9) - \bar{A}^{911}_{23}C_{28}][-S\phi_9 C\theta_9 S\psi_9 + C\phi_9 C\psi_9]$$
$$+ S\theta_9 S\psi_9[C_{12}(A^1_{33}L_1 + A^4_{33}L_4 - A^4_{32}$$
$$L_H + A^9_{33}L_9) - \bar{A}^{911}_{33}C_{28}]\} + (I_{x9} - I_{y9})$$
$$(\dot{\theta}_9 S\psi_9 - \dot{\phi} C\psi_9 S\theta_9)(\dot{\theta}_9 C\psi_9 + \dot{\phi}_9 S\psi_9 S\theta_9)$$
$$-A^9_z I_{z9} + I_{x11} S\alpha_{11}[A^9_x C\alpha_{11} + A^9_z S\alpha_{11}$$
$$-\dot{\alpha}_{11}(\omega_{x9} S\alpha_{11} + \omega_{z9}C\alpha_{11})] - I_{z11}C\alpha_{11}$$
$$[\dot{\alpha}_{11}(\omega_{x9}C\alpha_{11} - \omega_{z9}S\alpha_{11}) + A^9_x S\alpha_{11}$$
$$+A^9_z C\alpha_{11}] + (I_{x11}\bar{\omega}_{x11}C\alpha_{11} + I_{z11}\bar{\omega}_{z11}$$
$$S\alpha_{11})(\dot{\phi}_9 S\theta_9 S\psi_9 + \dot{\theta}_9 C\psi_9 + \dot{\alpha}_{11}) + I_{y11}$$
$$\bar{\omega}_{y11}(\dot{\phi}_9 S\theta_9 C\psi_9 - \dot{\theta}_9 S\psi_9)$$

$$E_{27} = (-S\phi_{10} S\theta_{10})[(A^1_{13}L_1 + A^4_{13}L_4 + A^4_{12}L_H)$$
$$C_{11} - A^{10}_{13}C_{27} + \bar{A}^{1012}_{13}L_{10}C_{13}] + (-S\phi_{10}$$
$$C\theta_{10} C\psi_{10} S\alpha_{12} - C\phi_{10} S\psi_{10} S\alpha_{12} - S\phi_{10}$$
$$S\theta_{10} C\alpha_{12})[C_{13}(A^1_{13}L_1 + A^4_{13}L_4 + A^4_{12}L_H$$
$$+ A^{10}_{13}L_{10}) - \bar{A}^{1012}_{13}C_{29}] + (C\phi_{10} S\theta_{10})$$
$$[(A^1_{23}L_1 + A^4_{23}L_4 + A^4_{22}L_H)C_{11} - A^{10}_{23}C_{27}$$

(Cont. on next page)

(E_{27} Cont'd.)

$$+\bar{A}_{z3}^{1012} L_{10} C_{13}] + (C\phi_{10} C\theta_{10} C\psi_{10} S\alpha_{12} - S\phi_{10} S\psi_{10} S\alpha_{12} + C\phi_{10} S\theta_{10} C\alpha_{12}) [(A_{z3}^{1} L_1 + A_{z3}^{4} L_4 + A_{z2}^{4} L_H + A_{z3}^{10} L_{10}) C_{13} - \bar{A}_{z3}^{1012} C_{29}] - A_x^{10} I_{x10}(-C\psi_{10} S\theta_{10}) - A_y^{10} I_{y10}(S\psi_{10} S\theta_{10}) - A_z^{10} I_{z10} C\theta_{10} + I_{x10}\omega_{x10}(-\dot{\psi}_{10} S\psi_{10} S\theta_{10} + \dot{\theta}_{10} C\psi_{10} C\theta_{10}) + I_{y10}\omega_{y10}(-\dot{\psi}_{10} C\psi_{10} S\theta_{10} - \dot{\theta}_{10} S\psi_{10} C\theta_{10}) + I_{z10}\omega_{z10}(\dot{\theta}_{10} S\theta_{10}) + I_{x12}(S\theta_{10} C\psi_{10} C\alpha_{12} + C\theta_{10} S\alpha_{12})[A_x^{10} C\alpha_{12} + A_z^{10} S\alpha_{12} - \dot{\alpha}_{12}(\omega_{x10} S\alpha_{12} + \omega_{z10} C\alpha_{12})] - I_{y12} S\theta_{10} S\psi_{10} A_y^{10} - I_{z12}(C\theta_{10} C\alpha_{12} - S\theta_{10} C\psi_{10} S\alpha_{12})[\dot{\alpha}_{12}(\omega_{x10} C\alpha_{12} - \omega_{z10} S\alpha_{12}) + A_x^{10} S\alpha_{12} + A_z^{10} C\alpha_{12}] + I_{x12}\bar{\omega}_{x12}[-\dot{\theta}_{10}(S\theta_{10} S\alpha_{12} - C\theta_{10} C\psi_{10} C\alpha_{12}) - \dot{\psi}_{10} S\theta_{10} S\psi_{10} C\alpha_{12} - \dot{\alpha}_{12}(S\theta_{10} C\psi_{10} S\alpha_{12} - C\theta_{10} C\alpha_{12})] - I_{y12}\bar{\omega}_{y12}(\dot{\theta}_{10} C\theta_{10} S\psi_{10} + \dot{\psi}_{10} S\theta_{10} C\psi_{10}) + I_{z12}\bar{\omega}_{z12}[\dot{\theta}_{10}(S\theta_{10} C\alpha_{12} + C\theta_{10} C\psi_{10} S\alpha_{12}) - \dot{\psi}_{10} S\theta_{10} S\psi_{10} S\alpha_{12} + \dot{\alpha}_{12}(C\theta_{10} S\alpha_{12} + S\theta_{10} C\psi_{10} C\alpha_{12})]$$

$$
\begin{aligned}
E_{Z8} = {} & (c\phi_{10}c\theta_{10})\left[C_{11}\left(A^{1}_{13}L_1 + A^{4}_{13}L_4 + A^{4}_{12}L_H\right) - A^{10}_{13}C_{27} + \bar{A}^{1012}_{13}L_{10}C_{13}\right] \\
& + \left(-c\phi_{10}s\theta_{10}c\psi_{10}s\alpha_{12} + c\phi_{10}c\theta_{10}c\alpha_{12}\right)\left[C_{13}\left(A^{1}_{13}L_1 + A^{4}_{13}L_4 + A^{4}_{12}L_H + A^{10}_{13}L_{10}\right) - \bar{A}^{1012}_{13}C_{29}\right] \\
& + (s\phi_{10}c\theta_{10})\left[C_{11}\left(A^{1}_{23}L_1 + A^{4}_{23}L_4 + A^{4}_{22}L_H\right) - A^{10}_{23}C_{27} + \bar{A}^{1012}_{23}L_{10}C_{13}\right] \\
& + \left(-s\phi_{10}s\theta_{10}c\psi_{10}s\alpha_{12} + s\phi_{10}c\theta_{10}c\alpha_{12}\right)\left[C_{13}\left(A^{1}_{23}L_1 + A^{4}_{23}L_4 + A^{4}_{22}L_H + A^{10}_{23}L_{10}\right) - \bar{A}^{1012}_{23}C_{29}\right] \\
& + (-s\theta_{10})\left[C_{11}\left(A^{1}_{33}L_1 + A^{4}_{33}L_4 + A^{4}_{32}L_H\right) - A^{10}_{33}C_{27} + \bar{A}^{1012}_{33}L_{10}C_{13}\right] \\
& + \left(-c\theta_{10}c\psi_{10}s\alpha_{12} - s\theta_{10}c\alpha_{12}\right)\left[C_{13}\left(A^{1}_{33}L_1 + A^{4}_{33}L_4 + A^{4}_{32}L_H + A^{10}_{33}L_{10}\right) - \bar{A}^{1012}_{33}C_{29}\right] - A^{10}_{x}I_{x10}s\psi_{10} \\
& - A^{10}_{y}I_{y10}c\psi_{10} + I_{x10}\omega_{x10}\left(-\dot{\phi}_{10}c\psi_{10}c\theta_{10} - \dot{\psi}_{10}c\psi_{10}\right) \\
& + I_{y10}\omega_{y10}\left(\dot{\phi}_{10}s\psi_{10}c\theta_{10} + \dot{\psi}_{10}s\psi_{10}\right) + I_{z10}\omega_{z10}\left(-\dot{\phi}_{10}s\theta_{10}\right) - I_{x12}\left(s\psi_{10}c\alpha_{12}\right)\left[A^{10}_{x}c\alpha_{12} + A^{10}_{z}s\alpha_{12}\right. \\
& \left. - \dot{\alpha}_{12}\left(\omega_{x10}s\alpha_{12} + \omega_{z10}c\alpha_{12}\right)\right] - I_{y12}c\psi_{10}A^{10}_{y} \\
& - I_{z12}s\psi_{10}s\alpha_{12}\left[\dot{\alpha}_{12}\left(\omega_{x10}c\alpha_{12} - \omega_{z10}s\alpha_{12}\right) + A^{10}_{x}s\alpha_{12} + A^{10}_{z}c\alpha_{12}\right] \\
& + I_{x12}\bar{\omega}_{x12}\left[-\dot{\phi}_{10}\left(c\theta_{10}c\psi_{10}c\alpha_{12} - s\theta_{10}s\alpha_{12}\right) - \dot{\psi}_{10}c\psi_{10}c\alpha_{12} + \dot{\alpha}_{12}s\psi_{10}s\alpha_{12}\right] \\
& + I_{y12}\bar{\omega}_{y12}\left(\dot{\phi}_{10}c\theta_{10}s\psi_{10} + \dot{\psi}_{10}s\psi_{10}\right) \\
& + I_{z12}\bar{\omega}_{z12}\left[-\dot{\phi}_{10}\left(c\theta_{10}c\psi_{10}s\alpha_{12} + s\theta_{10}c\alpha_{12}\right) - \dot{\psi}_{10}c\psi_{10}s\alpha_{12} - \dot{\alpha}_{12}s\psi_{10}c\alpha_{12}\right]
\end{aligned}
$$

$$
\begin{aligned}
E_{29} = {} & (-c\phi_{10}\, c\theta_{10}\, s\psi_{10}\, s\alpha_{12} - s\phi_{10}\, c\psi_{10}\, s\alpha_{12}) \\
& [C_{13}(A^{1}_{13} L_1 + A^{4}_{13} L_4 + A^{4}_{12} L_H + A^{10}_{13} L_{10}) - \bar{A}^{10\,12}_{13} \\
& C_{29}] + (-s\phi_{10}\, c\theta_{10}\, s\psi_{10}\, s\alpha_{12} + c\phi_{10}\, c\psi_{10}\, s\alpha_{12}) \\
& [C_{13}(A^{1}_{23} L_1 + A^{4}_{23} L_4 + A^{4}_{22} L_H + A^{10}_{23} L_{10}) \\
& - \bar{A}^{10\,12}_{23} C_{29}] + (s\theta_{10}\, s\psi_{12}\, s\alpha_{12})[C_{13}(A^{1}_{33} L_1 \\
& + A^{4}_{33} L_4 + A^{4}_{32} L_H + A^{10}_{33} L_{10}) - \bar{A}^{10\,12}_{33} C_{29}] - A^{10}_{z} \\
& I_{z10} + I_{x10}\, \omega_{x10}(\dot{\theta}_{10}\, c\psi_{10} + \dot{\phi}_{10}\, s\psi_{10}\, s\theta_{10}) \\
& + I_{y10}\, \omega_{y10}(\dot{\phi}_{10}\, c\psi_{10}\, s\theta_{10} - \dot{\theta}_{10}\, s\psi_{10}) + I_{x12} \\
& s\alpha_{12}[A^{10}_{x}\, c\alpha_{12} + A^{10}_{z}\, s\alpha_{12} - \dot{\alpha}_{12}(\omega_{x10} \\
& s\alpha_{12} + \omega_{z10}\, c\alpha_{12})] - I_{z12}\, c\alpha_{12}[\dot{\alpha}_{12}(\omega_{x10} \\
& c\alpha_{12} - \omega_{z10}\, s\alpha_{12}) + A^{10}_{x}\, s\alpha_{12} + A^{10}_{z}\, c\alpha_{12}] \\
& + [I_{x12}\, \bar{\omega}_{x12}\, c\alpha_{12} + I_{z12}\, \bar{\omega}_{z12}\, s\alpha_{12}][\dot{\phi}_{10}\, s\theta_{10} \\
& s\psi_{10} + \dot{\theta}_{10}\, c\psi_{10} + \dot{\alpha}_{12}] + I_{y12}\, \bar{\omega}_{y12}(\dot{\phi}_{10}\, s\theta_{10}\, c\psi_{10} - \\
& \dot{\theta}_{10}\, s\psi_{10})
\end{aligned}
$$

$$
\begin{aligned}
E_{30} = {} & [C_{12}(A^{1}_{13} L_1 + A^{4}_{13} L_4 - A^{4}_{12} L_H + A^{9}_{13} L_9) - \bar{A}^{9\,11}_{13} \\
& C_{28}][c\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} - s\phi_9\, s\psi_9\, c\alpha_{11} - c\phi_9\, s\theta_9 \\
& s\alpha_{11}] + [C_{12}(A^{1}_{23} L_1 + A^{4}_{23} L_4 - A^{4}_{22} L_H + A^{9}_{23} L_9) \\
& - \bar{A}^{9\,11}_{23} C_{28}][s\phi_9\, c\theta_9\, c\psi_9\, c\alpha_{11} + c\phi_9\, s\psi_9\, c\alpha_{11} \\
& - s\phi_9\, s\theta_9\, s\alpha_{11}] + [C_{12}(A^{1}_{33} L_1 + A^{4}_{33} L_4 - A^{4}_{32} L_H \\
& + A^{9}_{33} L_9) - \bar{A}^{9\,11}_{33} C_{28}][-s\theta_9\, c\psi_9\, c\alpha_{11} - c\theta_9\, s\alpha_{11}] \\
& - A^{9}_{y} I_{y11} + I_{x11}\, \bar{\omega}_{x11}(-\omega_{x9}\, s\alpha_{11} - \omega_{z9}\, c\alpha_{11}) + I_{z11} \\
& \bar{\omega}_{z11}(\omega_{x9}\, c\alpha_{11} - \omega_{z9}\, s\alpha_{11})
\end{aligned}
$$

$$
\begin{aligned}
E_{31} = {} & \left[C_{13}\left(A^{1}_{13}L_1 + A^{4}_{13}L_4 + A^{4}_{12}L_H + A^{10}_{13}L_{10}\right) - \bar{A}^{1012}_{13}C_{29}\right] \\
& \left[\left(c\phi_{10}c\theta_{10}c\psi_{10}c\alpha_{12} - s\phi_{10}s\psi_{10}c\alpha_{12} - c\phi_{10}s\theta_{10}\right.\right. \\
& \left.\left. s\alpha_{12}\right)\right] + \left[C_{13}\left(A^{1}_{23}L_1 + A^{4}_{23}L_4 + A^{4}_{22}L_H + A^{10}_{23}L_{10}\right)\right. \\
& \left. - \bar{A}^{1012}_{23}C_{29}\right]\left[s\phi_{10}c\theta_{10}c\psi_{10}c\alpha_{12} + c\phi_{10}s\psi_{10}c\alpha_{12}\right. \\
& \left. - s\phi_{10}s\theta_{10}s\alpha_{12}\right] + \left[C_{13}\left(A^{1}_{33}L_1 + A^{4}_{33}L_4 + A^{4}_{32}L_H\right.\right. \\
& \left.\left. + A^{10}_{33}L_{10}\right) - \bar{A}^{1012}_{33}C_{29}\right]\left[-s\theta_{10}c\psi_{10}c\alpha_{12} - c\theta_{10}\right. \\
& \left. s\alpha_{12}\right] - A^{10}_{Y}I_{Y12} + I_{X12}\bar{\omega}_{X12}\left(-\omega_{X10}s\alpha_{12} - \omega_{Z10}\right. \\
& \left. c\alpha_{12}\right) + I_{Z12}\bar{\omega}_{Z12}\left(\omega_{X10}c\alpha_{12} - \omega_{Z10}s\alpha_{12}\right)
\end{aligned}
$$

The elements of column vector F_P are:

$$F_{P1} = 0$$

$$F_{P2} = 0$$

$$F_{P3} = -gC_1$$

$$F_{P4} = 0$$

$$F_{P5} = gC_2\,s\theta_1$$

$$F_{P6} = 0$$

$$F_{P7} = 0$$

$$F_{P8} = g\left(C_3\,s\theta_2 - C_{14}\,c\theta_2\,s\psi_2\right)$$

$$F_{P9} = -g\,C_{14}\,s\theta_2\,c\psi_2$$

$$F_{P10} = 0$$

$$F_{P11} = g\,C_4\,s\theta_3$$

$$F_{P12} = 0$$

$$F_{P13} = 0$$

$$F_{P14} = g(C_5 s\theta_4 - C_{15} c\theta_4 s\psi_4)$$

$$F_{P15} = -g\, C_{15} s\theta_4 c\psi_4$$

$$F_{P16} = 0$$

$$F_{P17} = g[C_6 s\theta_5 + C_8(c\theta_5 c\psi_5 s\alpha_7 + s\theta_5 c\alpha_7)]$$

$$F_{P18} = -g\, C_8 s\theta_5 s\psi_5 s\alpha_7$$

$$F_{P19} = 0$$

$$F_{P20} = g[C_7 s\theta_6 + C_9(c\theta_6 c\psi_6 s\alpha_8 + s\theta_6 c\alpha_8)]$$

$$F_{P21} = -g\, C_9 s\theta_6 s\psi_6 s\alpha_8$$

$$F_{P22} = g\, C_8(s\theta_5 c\psi_5 c\alpha_7 + c\theta_5 s\alpha_7)$$

$$F_{P23} = g\, C_9(s\theta_6 c\psi_6 c\alpha_8 + c\theta_6 s\alpha_8)$$

$$F_{P24} = 0$$

$$F_{P25} = g[C_{10} s\theta_9 + C_{12}(c\theta_9 c\psi_9 s\alpha_{11} + s\theta_9 c\alpha_{11})]$$

$$F_{P26} = -g\, C_{12} s\theta_9 s\psi_9 s\alpha_{11}$$

$$F_{P27} = 0$$

$$F_{P28} = g[C_{11} s\theta_{10} + C_{13}(c\theta_{10} c\psi_{10} s\alpha_{12} + s\theta_{10} c\alpha_{12})]$$

$$F_{P29} = -g\, C_{13} s\theta_{10} s\psi_{10} s\alpha_{12}$$

$$F_{P30} = g\, C_{12}(s\theta_9 c\psi_9 c\alpha_{11} + c\theta_9 s\alpha_{11})$$

$$F_{P31} = g\, C_{13}(s\theta_{10} c\psi_{10} c\alpha_{12} + c\theta_{10} s\alpha_{12})$$

The constant g represents the acceleration due to gravity.

To define column vector F_s, the following terms must also be defined:

β_1 = angle between the upper and middle torso segments

β_{10} = initial value of β_1

β_2 = angle between the middle and lower torso segment

β_{20} = initial value of β_2

K_1 = stiffness of radial spring (in.-lb./rad.) located between upper and middle torso segments

K_2 = stiffness of radial spring (in.-lb./rad.) located between middle and lower torso segments

$$\mu_1 = s\theta_1 s\theta_2 (s\phi_1 s\phi_2 + c\phi_1 c\phi_2) + c\theta_1 c\theta_2$$

$$\beta_1 = \cos^{-1}\mu_1$$

$$\mu_2 = s\theta_1 s\theta_4 (c\phi_1 c\phi_4 + s\phi_1 s\phi_4) + c\theta_1 c\theta_4$$

$$\beta_2 = \cos^{-1}\mu_2$$

Column vector F_s is composed of the following nonzero elements:

$$F_{s4} = \frac{K_1(\beta_1 - \beta_{10})}{\sqrt{1-\mu_1^2}} [-s\phi_1 s\theta_1 c\phi_2 s\theta_2 + c\phi_1 s\theta_1 s\phi_2 s\theta_2]$$

$$+ \frac{K_2(\beta_2 - \beta_{20})}{\sqrt{1-\mu_2^2}} [-s\phi_1 s\theta_1 c\phi_4 s\theta_4 + c\phi_1 s\theta_1 s\phi_4 s\theta_4]$$

$$F_{55} = \frac{K_1(\beta_1 - \beta_{10})}{\sqrt{1-\mu_1^2}} [c\phi_1 c\theta_1 c\phi_2 s\theta_2 + s\phi_1 c\theta_1 s\phi_2 s\theta_2 - s\theta_1 c\theta_2]$$

$$+ \frac{K_2(\beta_2 - \beta_{20})}{\sqrt{1-\mu_2^2}} [c\phi_1 c\theta_1 c\phi_4 s\theta_4 + s\phi_1 c\theta_1 s\phi_4 s\theta_4 - s\theta_1 c\theta_4]$$

$$F_{57} = \frac{K_1(\beta_1 - \beta_{10})}{\sqrt{1-\mu_1^2}} [-c\phi_1 s\theta_1 s\phi_2 s\theta_2 + s\phi_1 s\theta_1 c\phi_2 s\theta_2]$$

$$F_{58} = \frac{K_1(\beta_1 - \beta_{10})}{\sqrt{1-\mu_1^2}} [c\phi_1 s\theta_1 c\phi_2 c\theta_2 + s\phi_1 s\theta_1 s\phi_2 c\theta_2 - c\theta_1 s\theta_2]$$

$$F_{513} = \frac{K_1(\beta_2 - \beta_{20})}{\sqrt{1-\mu_2^2}} [-c\phi_1 s\theta_1 s\phi_4 s\theta_4 + s\phi_1 s\theta_1 c\phi_4 s\theta_4]$$

$$F_{514} = \frac{K_2(\beta_2 - \beta_{20})}{\sqrt{1-\mu_2^2}} [c\phi_1 s\theta_1 c\phi_4 c\theta_4 + s\phi_1 s\theta_1 s\phi_4 c\theta_4 - c\theta_1 s\theta_4]$$

The elements of column vector Q_f are as follows:

$$Q_{f_1} = \sum_{n=1}^{17} F_{x_n}$$

$$Q_{f_2} = \sum_{n=1}^{17} F_{y_n}$$

$$Q_{f_3} = \sum_{n=1}^{17} F_{z_n}$$

$$\varphi_{f_4} = L_1 s\phi_1 s\theta_1 \left[F_{x_3} + \sum_{n=10}^{17} F_{x_n} \right] - L_1 c\phi_1 c\theta_1 \left[F_{y_3} + \sum_{n=10}^{17} F_{y_n} \right]$$

$$\varphi_{f_5} = -L_1 c\phi_1 c\theta_1 \left[F_{x_3} + \sum_{n=10}^{17} F_{x_n} \right] - L_1 s\phi_1 c\theta_1 \left[F_{y_3} + \sum_{n=10}^{17} F_{y_n} \right] + L_1 s\theta_1 \left[F_{z_3} + \sum_{n=10}^{17} F_{z_n} \right]$$

$$\varphi_{f_6} = 0$$

$$\begin{aligned}
\varphi_{f_7} = & -\left[s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2 \right]\left[F_{x_2} r_{x_2} + F_{x_4} r_{x_4} + F_{x_5} r_{x_5} \right] + \left[s\phi_2 c\theta_2 s\psi_2 - c\phi_2 c\psi_2 \right]\left[F_{x_2} r_{y_2} + F_{x_4} r_{y_4} + F_{x_5} r_{y_5} + L_3 (-F_{x_4} + F_{x_5} \right. \\
& \left. - F_{x_6} + F_{x_7} - F_{x_8} + F_{x_9}) \right] - s\phi_2 s\theta_2 \left[F_{x_2} (s_2 + r_{z_2}) + F_{x_4} r_{z_4} + F_{x_5} r_{z_5} + L_2 \left(F_{x_1} + \sum_{n=4}^{9} F_{x_n} \right) \right] + \left[c\phi_2 c\theta_2 c\psi_2 - s\phi_2 s\psi_2 \right] \\
& \left[F_{y_2} r_{x_2} + F_{y_4} r_{x_4} + F_{y_5} r_{x_5} \right] - \left[c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2 \right]\left[F_{y_2} r_{y_2} + F_{y_4} r_{y_4} + F_{y_5} r_{y_5} + L_3 (-F_{y_4} + F_{y_5} - F_{y_6} + F_{y_7} - F_{y_8} + F_{y_9}) \right] \\
& + c\phi_2 s\theta_2 \left[F_{y_2} (s_2 + r_{z_2}) + F_{y_4} r_{z_4} + F_{y_5} r_{z_5} + L_2 \left(F_{y_1} + \sum_{n=4}^{9} F_{y_n} \right) \right]
\end{aligned}$$

$$\begin{aligned}
\varphi_{f_8} = & -c\phi_2 s\theta_2 c\psi_2 \left[F_{x_2} r_{x_2} + F_{x_4} r_{x_4} + F_{x_5} r_{x_5} \right] + c\phi_2 s\theta_2 s\psi_2 \left[F_{x_2} r_{y_2} + F_{x_4} r_{y_4} + F_{x_5} r_{y_5} + L_3 (-F_{x_4} + F_{x_5} - F_{x_6} + F_{x_7} - F_{x_8} + F_{x_9}) \right] \\
& + c\phi_2 c\theta_2 \left[F_{x_2} (s_2 + r_{z_2}) + F_{x_4} r_{z_4} + F_{x_5} r_{z_5} + L_2 \left(F_{x_1} + \sum_{n=4}^{9} F_{x_n} \right) \right] - s\phi_2 s\theta_2 c\psi_2 \left[F_{y_2} r_{x_2} + F_{y_4} r_{x_4} + F_{y_5} r_{x_5} \right]
\end{aligned}$$

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(φ_{f8} cont'd.)

$$+ s\phi_2 s\theta_2 s\psi_2 [F_{y_2} r_{y_2} + F_{y_4} r_{y_4} + F_{y_5} r_{y_5} + L_5 (-F_{y_4} + F_{y_5} - F_{y_6} + F_{y_7} - F_{y_8} + F_{y_9})] + s\phi_2 c\theta_2 [F_{y_2} (s_2 + r_{z_2}) + F_{y_4} r_{z_4} + F_{y_5} r_{z_5} + L_2 (F_{y_1} + \sum_{n=4}^{9} F_{y_n})] - c\theta_2 c\psi_2 [F_{z_2} r_{x_2} + F_{z_4} r_{x_4} + F_{z_5} r_{x_5}] + c\theta_2 s\psi_2 [F_{z_2} r_{y_2} + F_{z_4} r_{y_4} + F_{z_5} r_{y_5} + L_5 (-F_{z_4} + F_{z_5} - F_{z_6} + F_{z_7} - F_{z_8} + F_{z_9})] - s\theta_2 [F_{z_2} (s_2 + r_{z_2}) + F_{z_4} r_{z_4} + F_{z_5} r_{z_5} + L_2 (F_{z_1} + \sum_{n=4}^{9} F_{z_n})]$$

$$\varphi_{f9} = -(c\phi_2 c\theta_2 s\psi_2 + s\phi_2 c\psi_2)[F_{x_2} r_{x_2} + F_{x_4} r_{x_4} + F_{x_5} r_{x_5}] + (-c\phi_2 c\theta_2 c\psi_2 + s\phi_2 s\psi_2)[F_{x_2} r_{y_2} + F_{x_4} r_{y_4} + F_{x_5} r_{y_5} + L_5 (-F_{x_4} + F_{x_5} - F_{x_6} + F_{x_7} - F_{x_8} + F_{x_9})] + (-s\phi_2 c\theta_2 s\psi_2 + c\phi_2 c\psi_2)[F_{y_2} r_{x_2} + F_{y_4} r_{x_4} + F_{y_5} r_{x_5}] - (s\phi_2 c\theta_2 c\psi_2 + c\phi_2 s\psi_2)[F_{y_2} r_{y_2} + F_{y_4} r_{y_4} + F_{y_5} r_{y_5} + L_5 (-F_{y_4} + F_{y_5} - F_{y_6} + F_{y_7} - F_{y_8} + F_{y_9})] + s\theta_2 s\psi_2 [F_{z_2} r_{x_2} + F_{z_4} r_{x_4} + F_{z_5} r_{x_5}] + s\theta_2 c\psi_2 [F_{z_2} r_{y_2} + F_{z_4} r_{y_4} + F_{z_5} r_{y_5} + L_5 (-F_{z_4} + F_{z_5} - F_{z_6} + F_{z_7} - F_{z_8} + F_{z_9})]$$

$$\varphi_{f10} = F_{x_1} [-r_{x_1} (s\phi_3 c\theta_3 c\psi_3 + c\phi_3 s\psi_3) + r_{y_1} (s\phi_3 c\theta_3 s\psi_3 - c\phi_3 c\psi_3) - s\phi_3 s\theta_3 (s_1 + r_{z_1})] + F_{y_1} [r_{x_1} (c\phi_3 c\theta_3 c\psi_3 - s\phi_3 s\psi_3) - r_{y_1} (c\phi_3 c\theta_3 s\psi_3 + s\phi_3 c\psi_3) + c\phi_3 s\theta_3 (s_1 + r_{z_1})]$$

$$Q_{f_{11}} = F_{x_1}[-r_{x_1} c\phi_3 s\theta_3 c\psi_3 + r_{y_1} c\phi_3 s\theta_3 s\psi_3 + c\phi_3 c\theta_3 (s_1 + r_{z_1})] + F_{y_1}[-r_{x_1} s\phi_3 s\theta_3 c\psi_3 + r_{y_1} s\phi_3 s\theta_3 s\psi_3 + s\phi_3 c\theta_3 (s_1 + r_{z_1})] + F_{z_1}[-r_{x_1} c\theta_3 c\psi_3 + r_{y_1} c\theta_3 s\psi_3 - s\theta_3 (s_1 + r_{z_1})]$$

$$Q_{f_{12}} = F_{x_1}[-r_{x_1}(c\phi_3 c\theta_3 s\psi_3 + s\phi_3 c\psi_3) + r_{y_1}(-c\phi_3 c\theta_3 c\psi_3 + s\phi_3 s\psi_3)] + F_{y_1}[r_{x_1}(-s\phi_3 c\theta_3 s\psi_3 + c\phi_3 c\psi_3) - r_{y_1}(s\phi_3 c\theta_3 c\psi_3 + c\phi_3 s\psi_3)] + F_{z_1}[r_{x_1} s\theta_3 s\psi_3 + r_{y_1} s\theta_3 c\psi_3]$$

$$Q_{f_{13}} = -F_{x_3} r_{x_3}(s\phi_4 c\theta_4 c\psi_4 + c\phi_4 s\psi_4) + (s\phi_4 c\theta_4 s\psi_4 - c\phi_4 c\psi_4)[F_{x_3} r_{y_3} + L_H(-F_{x_{10}} + F_{x_{11}} - F_{x_{12}} + F_{x_{13}} - F_{x_{14}} + F_{x_{15}} - F_{x_{16}} + F_{x_{17}})] - s\phi_4 s\theta_4 [F_{x_3}(r_{z_3} - s_3) - L_4 \sum_{n=10}^{17} F_{x_n}] + F_{y_3} r_{x_3}(c\phi_4 c\theta_4 c\psi_4 - s\phi_4 s\psi_4) - (c\phi_4 c\theta_4 s\psi_4 + s\phi_4 c\psi_4)[F_{y_3} r_{y_3} + L_H(-F_{y_{10}} + F_{y_{11}} - F_{y_{12}} + F_{y_{13}} - F_{y_{14}} + F_{y_{15}} - F_{y_{16}} + F_{y_{17}})] + c\phi_4 s\theta_4 [F_{y_3}(r_{z_3} - s_3) - L_4 \sum_{n=10}^{17} F_{y_n}]$$

$$Q_{f_{14}} = -F_{x_3} r_{x_3} c\phi_4 s\theta_4 c\psi_4 + c\phi_4 s\theta_4 s\psi_4 [F_{x_3} r_{y_3} + L_H(-F_{x_{10}} + F_{x_{11}} - F_{x_{12}} + F_{x_{13}} - F_{x_{14}} + F_{x_{15}} - F_{x_{16}} + F_{x_{17}})] + c\phi_4 c\theta_4 [F_{x_3}(r_{z_3} - s_3) - L_4 \sum_{n=10}^{17} F_{x_n}] - F_{y_3} r_{x_3} s\phi_4 s\theta_4 c\psi_4 + s\phi_4 s\theta_4 s\psi_4 [F_{y_3} r_{y_3} + L_H(-F_{y_{10}} + F_{y_{11}} - F_{y_{12}} + F_{y_{13}} - F_{y_{14}} + F_{y_{15}} - F_{y_{16}} + F_{y_{17}})]$$

(cont. on next page)

(φ_{f14} cont'd.)

$$+ s\phi_4 c\theta_4 \left[F_{y_3}(r_{z_3} - s_3) - L_4 \sum_{n=10}^{17} F_{y_n} \right] - F_{z_3} r_{x_3} c\theta_4 c\psi_4 + c\theta_4 s\psi_4 \left[F_{z_3} r_{y_3} + L_H (-F_{z_{10}} + F_{z_{11}} - F_{z_{12}} + F_{z_{13}} - F_{z_{14}} + F_{z_{15}} - F_{z_{16}} + F_{z_{17}}) \right] - s\theta_4 \left[F_{z_3}(r_{z_3} - s_3) - L_4 \sum_{n=10}^{17} F_{z_n} \right]$$

$$\varphi_{f15} = -F_{x_3} r_{x_3} \left[c\phi_4 c\theta_4 s\psi_4 + s\phi_4 c\psi_4 \right] + (-c\phi_4 c\theta_4 c\psi_4 + s\phi_4 s\psi_4) \left[F_{x_3} r_{y_3} + L_H (-F_{x_{10}} + F_{x_{11}} - F_{x_{12}} + F_{x_{13}} - F_{x_{14}} + F_{x_{15}} - F_{x_{16}} + F_{x_{17}}) \right] + F_{y_3} r_{x_3} (-s\phi_4 c\theta_4 s\psi_4 + c\phi_4 c\psi_4) - (s\phi_4 c\theta_4 c\psi_4 + c\phi_4 s\psi_4) \left[F_{y_3} r_{y_3} + L_H (-F_{y_{10}} + F_{y_{11}} - F_{y_{12}} + F_{y_{13}} - F_{y_{14}} + F_{y_{15}} - F_{y_{16}} + F_{y_{17}}) \right] + F_{z_3} r_{x_3} s\theta_4 s\psi_4 + s\theta_4 c\psi_4 \left[F_{z_3} r_{y_3} + L_H (-F_{z_{10}} + F_{z_{11}} - F_{z_{12}} + F_{z_{13}} - F_{z_{14}} + F_{z_{15}} - F_{z_{16}} + F_{z_{17}}) \right]$$

$$\varphi_{f16} = -F_{x_7} r_{x_7} (s\phi_5 c\theta_5 c\psi_5 + c\phi_5 s\psi_5) + (s\phi_5 c\theta_5 s\psi_5 - c\phi_5 c\psi_5) \left[F_{x_7} r_{y_7} + F_{x_9} r_{y_9} \right] - s\phi_5 s\theta_5 \left[F_{x_7}(r_{z_7} - L_5) - F_{x_9} L_5 \right] + F_{y_7} r_{x_7} (c\phi_5 c\theta_5 c\psi_5 - s\phi_5 s\psi_5) - (c\phi_5 c\theta_5 s\psi_5 + s\phi_5 c\psi_5) \left[F_{y_7} r_{y_7} + F_{y_9} r_{y_9} \right] + c\phi_5 s\theta_5 \left[F_{y_7}(r_{z_7} - L_5) - F_{y_9} L_5 \right] + F_{x_9} r_{x_9} (-s\phi_5 c\theta_5 c\psi_5 c\alpha_7 - c\phi_5 s\psi_5 c\alpha_7 + s\phi_5 s\theta_5 s\alpha_7) - F_{x_9} (r_{z_9} - s_9) \left[s\phi_5 c\theta_5 c\psi_5 s\alpha_7 + c\phi_5 s\psi_5 s\alpha_7 + s\phi_5 s\theta_5 c\alpha_7 \right]$$

(Cont. on next page)

(Q_{f16} cont'd.)

$$+ F_{y9}\, r_{x9}[c\phi_5\, c\theta_5\, c\psi_5\, c\alpha_7 - s\phi_5\, s\psi_5\, c\alpha_7$$
$$- c\phi_5\, s\theta_5\, s\alpha_7] + F_{y9}\,(r_{z9} - S_9)[c\phi_5\, c\theta_5$$
$$c\psi_5\, s\alpha_7 - s\phi_5\, s\psi_5\, s\alpha_7 + c\phi_5\, s\theta_5\, c\alpha_7]$$

$$Q_{f17} = - F_{x7}\, r_{x7}\, c\phi_5\, s\theta_5\, c\psi_5 + c\phi_5\, s\theta_5\, s\psi_5[F_{x7}$$
$$r_{y7} + F_{x9}\, r_{y9}] + c\phi_5\, c\theta_5[F_{x7}(r_{z7} - L_5) - F_{x9}$$
$$L_5] - F_{y7}\, r_{x7}\, s\phi_5\, s\theta_5\, c\psi_5 + s\phi_5\, s\theta_5\, s\psi_5[F_{y7}$$
$$r_{y7} + F_{y9}\, r_{y9}] + s\phi_5\, c\theta_5[F_{y7}\, r_{z7} - L_5(F_{y7}$$
$$+ F_{y9})] - F_{z7}\, r_{x7}\, c\theta_5\, c\psi_5 + c\theta_5\, s\psi_5[F_{z7}\, r_{y7}$$
$$+ F_{z9}\, r_{y9}] - s\theta_5[F_{z7}\, r_{z7} - L_5(F_{z7} + F_{z9})]$$
$$- F_{x9}\, r_{x9}(c\phi_5\, s\theta_5\, c\psi_5\, c\alpha_7 + c\phi_5\, c\theta_5\, s\alpha_7)$$
$$+ (r_{z9} - S_9)[+ F_{x9}(-c\phi_5\, s\theta_5\, c\psi_5\, s\alpha_7 + c\phi_5$$
$$c\theta_5\, c\alpha_7) + F_{y9}(-s\phi_5\, s\theta_5\, c\psi_5\, s\alpha_7 + s\phi_5\, c\theta_5$$
$$c\alpha_7) - F_{z9}\,(c\theta_5\, c\psi_5\, s\alpha_7 + s\theta_5\, c\alpha_7)] - F_{y9}\, r_{x9}$$
$$(s\phi_5\, s\theta_5\, c\psi_5\, c\alpha_7 + s\phi_5\, c\theta_5\, s\alpha_7) + F_{z9}\, r_{x9}$$
$$(-c\theta_5\, c\psi_5\, c\alpha_7 + s\theta_5\, s\alpha_7)$$

$$Q_{f18} = -(c\phi_5\, c\theta_5\, s\psi_5 + s\phi_5\, c\psi_5)\{F_{x7}\, r_{x7} + F_{x9}[r_{x9}\, c\alpha_7$$
$$+ (r_{z9} - S_9)\, s\alpha_7]\} + (-c\phi_5\, c\theta_5\, c\psi_5 + s\phi_5\, s\psi_5)[F_{x7}$$
$$r_{y7} + F_{x9}\, r_{y9}] + (-s\phi_5\, c\theta_5\, s\psi_5 + c\phi_5\, c\psi_5)\{F_{y7}\, r_{x7}$$
$$+ F_{y9}[r_{x9}\, c\alpha_7 + s\alpha_7\,(r_{z9} - S_9)]\} - (s\phi_5\, c\theta_5\, c\psi_5 + c\phi_5$$
$$s\psi_5)[F_{y7}\, r_{y7} + F_{y9}\, r_{y9}] + s\theta_5\, s\psi_5\{F_{z7}\, r_{x7} + F_{z9}[r_{x9}$$
$$c\alpha_7 + (r_{z9} - S_9)\, s\alpha_7]\} + s\theta_5\, c\psi_5[F_{z7}\, r_{y7} + F_{z9}\, r_{y9}]$$

$$
\begin{aligned}
Q_{f_{19}} = {} & -F_{x_6}\, r_{x_6} (s\phi_6\, c\theta_6\, c\psi_6 + c\phi_6\, s\psi_6) + (s\phi_6\, c\theta_6\, s\psi_6 \\
& - c\phi_6\, c\psi_6)[F_{x_6}\, r_{y_6} + F_{x_8}\, r_{y_8}] - s\phi_6\, s\theta_6 [F_{x_6} \\
& (r_{z_6} - L_6) - F_{x_8} L_6] + F_{y_6}\, r_{x_6} (c\phi_6\, c\theta_6\, c\psi_6 - s\phi_6 \\
& s\psi_6) - c\phi_6\, c\theta_6\, s\psi_6 + s\phi_6\, c\psi_6)[F_{y_6}\, r_{y_6} + F_{y_8} \\
& r_{y_8}] + c\phi_6\, s\theta_6 [F_{y_6}(r_{z_6} - L_6) - F_{y_8} L_6] + F_{x_8} \\
& r_{x_8}(-s\phi_6\, c\theta_6\, c\psi_6\, c\alpha_8 - c\phi_6\, s\psi_6\, c\alpha_8 + s\phi_6\, s\theta_6 \\
& s\alpha_8) - F_{x_8}(r_{z_8} - S_8)(s\phi_6\, c\theta_6\, c\psi_6\, s\alpha_8 + c\phi_6\, s\psi_6 \\
& s\alpha_8 + s\phi_6\, s\theta_6\, c\alpha_8) + F_{y_8}\, r_{x_8}(c\phi_6\, c\theta_6\, c\psi_6\, c\alpha_8 \\
& - s\phi_6\, s\psi_6\, c\alpha_8 - c\phi_6\, s\theta_6\, s\alpha_8) + F_{y_8}(r_{z_8} - S_8) \\
& (c\phi_6\, c\theta_6\, c\psi_6\, s\alpha_8 - s\phi_6\, s\psi_6\, s\alpha_8 + c\phi_6\, s\theta_6\, c\alpha_8)
\end{aligned}
$$

$$
\begin{aligned}
Q_{f_{20}} = {} & -F_{x_6}\, r_{x_6}\, c\phi_6\, s\theta_6\, c\psi_6 + c\phi_6\, s\theta_6\, s\psi_6 [F_{x_6}\, r_{y_6} \\
& + F_{x_8}\, r_{y_8}] + c\phi_6\, c\theta_6 [F_{x_6}(r_{z_6} - L_6) - F_{x_8} L_6] \\
& - F_{y_6}\, r_{x_6}\, s\phi_6\, s\theta_6\, c\psi_6 + s\phi_6\, s\theta_6\, s\psi_6 [F_{y_6}\, r_{y_6} \\
& + F_{y_8}\, r_{y_8}] + s\phi_6\, c\theta_6 [F_{y_6}(r_{z_8} - L_6) - F_{y_8} L_6] \\
& - F_{z_6}\, r_{x_6}\, c\theta_6\, c\psi_6 + c\theta_6\, s\psi_6 [F_{z_6}\, r_{y_6} + F_{z_8}\, r_{y_8}] \\
& - s\theta_6 [F_{z_6}(r_{z_6} - L_6) - F_{z_8} L_6] - F_{x_8}\, r_{x_8}(c\phi_6 \\
& s\theta_6\, c\psi_6\, c\alpha_8 + c\phi_6\, c\theta_6\, s\alpha_8) + F_{x_8}(r_{z_8} - S_8) \\
& (-c\phi_6\, s\theta_6\, c\psi_6\, s\alpha_8 + c\phi_6\, c\theta_6\, c\alpha_8) - F_{y_8}\, r_{x_8} \\
& (s\phi_6\, s\theta_6\, c\psi_6\, c\alpha_8 + s\phi_6\, c\theta_6\, s\alpha_8) + F_{y_8}(r_{z_8} - S_8) \\
& (-s\phi_6\, s\theta_6\, c\psi_6\, s\alpha_8 + s\phi_6\, c\theta_6\, c\alpha_8) + F_{z_8}\, r_{x_8} \\
& (-c\theta_6\, c\psi_6\, c\alpha_8 + s\theta_6\, s\alpha_8) - F_{z_8}(r_{z_8} - S_8)(c\theta_6 \\
& c\psi_6\, s\alpha_8 + s\theta_6\, c\alpha_8)
\end{aligned}
$$

$$\varphi_{F21} = -(c\phi_6 c\theta_6 s\psi_6 + s\phi_6 c\psi_6)\{F_{x_6} r_{x_6} + F_{x_8}[r_{x_8} c\alpha_8 + (r_{z_8} - S_8) s\alpha_8]\} + (-c\phi_6 c\theta_6 c\psi_6 + s\phi_6 s\psi_6)[F_{x_6} r_{y_6} + F_{x_8} r_{y_8}] + (-s\phi_6 c\theta_6 s\psi_6 + c\phi_6 c\psi_6)\{F_{y_6} r_{x_6} + F_{y_8}[r_{x_8} c\alpha_8 + (r_{z_8} - S_8) s\alpha_8]\} - (s\phi_6 c\theta_6 c\psi_6 + c\phi_6 s\psi_6)[F_{y_6} r_{y_6} + F_{y_8} r_{y_8}] + s\theta_6 s\psi_6 \{F_{z_6} r_{x_6} + F_{z_8}[r_{x_8} c\alpha_8 + (r_{z_8} - S_8) s\alpha_8]\} + s\theta_6 c\psi_6 [F_{z_6} r_{y_6} + F_{z_8} r_{y_8}]$$

$$\varphi_{F22} = F_{x_9} r_{x_9}[s\alpha_7(-c\phi_5 c\theta_5 c\psi_5 + s\phi_5 s\psi_5) - c\phi_5 s\theta_5 c\alpha_7] - F_{y_9} r_{x_9}[s\alpha_7(s\phi_5 c\theta_5 c\psi_5 + c\phi_5 s\psi_5) + s\phi_5 s\theta_5 c\alpha_7] + F_{z_9} r_{x_9}[s\theta_5 c\psi_5 s\alpha_7 - c\theta_5 c\alpha_7] + (r_{z_9} - S_9)\{F_{x_9}[c\alpha_7(c\phi_5 c\theta_5 c\psi_5 - s\phi_5 s\psi_5) - c\phi_5 s\theta_5 s\alpha_7] + F_{y_9}[c\alpha_7(s\phi_5 c\theta_5 c\psi_5 + c\phi_5 s\psi_5) - s\phi_5 s\theta_5 s\alpha_7] - F_{z_9}[s\theta_5 c\psi_5 c\alpha_7 - c\theta_5 s\alpha_7]\}$$

$$\varphi_{F23} = F_{x_8} r_{x_8}[s\alpha_8(-c\phi_6 c\theta_6 c\psi_6 + s\phi_6 s\psi_6) - c\phi_6 s\theta_6 c\alpha_8] - F_{y_8} r_{x_8}[s\alpha_8(s\phi_6 c\theta_6 c\psi_6 + c\phi_6 s\psi_6) + s\phi_6 s\theta_6 c\alpha_8] + F_{z_8} r_{x_8}[s\theta_6 c\psi_6 s\alpha_8 - c\theta_6 c\alpha_8] + (r_{z_8} - S_8)\{F_{x_8}[c\alpha_8(c\phi_6 c\theta_6 c\psi_6 - s\phi_6 s\psi_6) - c\phi_6 s\theta_6 s\alpha_8] + F_{y_8}[c\alpha_8(s\phi_6 c\theta_6 c\psi_6 + c\phi_6 s\psi_6) - s\phi_6 s\theta_6 s\alpha_8] - F_{z_8}[s\theta_6 c\psi_6 c\alpha_8 - c\theta_6 s\alpha_8]\}$$

$$\varphi_{f24} = -(s\phi_9\, c\theta_9\, c\psi_9 + c\phi_9\, s\psi_9)[F_{x_{11}} r_{x_{11}} + F_{x_{13}} r_{x_{13}}] + (s\phi_9\, c\theta_9\, s\psi_9 - c\phi_9\, c\psi_9)[F_{x_{11}} r_{y_{11}} + F_{x_{13}} r_{y_{13}} + F_{x_{15}} r_{y_{15}} + F_{x_{17}} r_{y_{17}}] - s\phi_9\, s\theta_9 [F_{x_{11}} (r_{z_{11}} - S_{11}) + F_{x_{13}} (r_{z_{13}} - L_9) - L_9 (F_{x_{15}} + F_{x_{17}})] + (c\phi_9\, c\theta_9\, c\psi_9 - s\phi_9\, s\psi_9)[F_{y_{11}} r_{x_{11}} + F_{y_{13}} r_{x_{13}}] - (c\phi_9\, c\theta_9\, s\psi_9 + s\phi_9\, c\psi_9)[F_{y_{11}} r_{y_{11}} + F_{y_{13}} r_{y_{13}} + F_{y_{15}} r_{y_{15}} + F_{y_{17}} r_{y_{17}}] + c\phi_9\, s\theta_9 [F_{y_{11}} (r_{z_{11}} - S_{11}) + F_{y_{13}} (r_{z_{13}} - L_9) - L_9 (F_{y_{15}} + F_{y_{17}})] + [c\alpha_{11} (-s\phi_9\, c\theta_9\, c\psi_9 - c\phi_9\, s\psi_9) + s\phi_9\, s\theta_9\, s\alpha_{11}][F_{x_{15}} r_{x_{15}} + F_{x_{17}} r_{x_{17}}] - [s\alpha_{11} (s\phi_9\, c\theta_9\, c\psi_9 + c\phi_9\, s\psi_9) + s\phi_9\, s\theta_9\, c\alpha_{11}][F_{x_{15}} (r_{z15} - S_{15}) + F_{x_{17}} (r_{z_{17}} - S_{17})] + [c\alpha_{11} (c\phi_9\, c\theta_9\, c\psi_9 - s\phi_9\, s\psi_9) - c\phi_9\, s\theta_9\, s\alpha_{11}][F_{y_{15}} r_{x_{15}} + F_{y_{17}} r_{x_{17}}] + [s\alpha_{11} (c\phi_9\, c\theta_9\, c\psi_9 - s\phi_9\, s\psi_9) + c\phi_9\, s\theta_9\, c\alpha_{11}][F_{y_{15}} (r_{z_{15}} - S_{15}) + F_{y_{17}} (r_{z_{17}} - S_{17})]$$

$$\varphi_{f25} = c\phi_9 \{ s\theta_9 [-c\psi_9 (F_{x_{11}} r_{x_{11}} + F_{x_{13}} r_{x_{13}}) + s\psi_9 (F_{x_{11}} r_{y_{11}} + F_{x_{13}} r_{y_{13}} + F_{x_{15}} r_{y_{15}} + F_{x_{17}} r_{y_{17}})] + c\theta_9 [F_{x_{11}} (r_{z_{11}} - S_{11}) + F_{x_{13}} r_{z_{13}} - L_9 (F_{x_{13}} + F_{x_{15}} + F_{x_{17}})] - (s\theta_9\, c\psi_9\, c\alpha_{11} + c\theta_9\, s\alpha_{11}) [F_{x_{15}} r_{x_{15}} + F_{x_{17}} r_{x_{17}}] + (-s\theta_9\, c\psi_9\, s\alpha_{11} + c\theta_9\, c\alpha_{11}) [F_{x_{15}} (r_{z15} - S_{16}) + F_{x_{17}} (r_{z_{17}} - S_{17})] \}$$

(cont. on next page)

(φ_{f25} Cont'd.)

$$+ s\phi_9\{s\theta_9[-c\psi_9(F_{y_{11}} r_{x_{11}} + F_{y_{13}} r_{x_{13}}) + s\psi_9 (F_{y_{11}} r_{y_{11}} + F_{y_{13}} r_{y_{13}} + F_{y_{15}} r_{y_{15}} + F_{y_{17}} r_{y_{17}})] + c\theta_9[F_{y_{11}}(r_{z_{11}} - s_{11}) + F_{y_{13}} r_{z_{13}} - L_9(F_{y_{13}} + F_{y_{15}} + F_{y_{17}})] - (s\theta_9 c\psi_9 c\alpha_{11} + c\theta_9 s\alpha_{11}) [F_{y_{15}} r_{x_{15}} + F_{y_{17}} r_{x_{17}}] + (-s\theta_9 c\psi_9 s\alpha_{11} + c\theta_9 c\alpha_{11})[F_{y_{15}}(r_{z_{15}} - s_{15}) + F_{y_{17}}(r_{z_{17}} - s_{17})]\} + c\theta_9[-c\psi_9(F_{z_{11}} r_{x_{11}} + F_{z_{13}} r_{x_{13}}) + s\psi_9(F_{z_{11}} r_{y_{11}} + F_{z_{13}} r_{y_{13}} + F_{z_{15}} r_{y_{15}} + F_{z_{17}} r_{y_{17}})] - s\theta_9 [F_{z_{11}}(r_{z_{11}} - s_{11}) + F_{z_{13}} r_{z_{13}} - L_9(F_{z_{13}} + F_{z_{15}} + F_{z_{17}})] + (-c\theta_9 c\psi_9 c\alpha_{11} + s\theta_9 s\alpha_{11})[F_{z_{15}} r_{x_{15}} + F_{z_{17}} r_{x_{17}}] - (c\theta_9 c\psi_9 s\alpha_{11} + s\theta_9 c\alpha_{11})[F_{z_{15}}(r_{z_{15}} - s_{15}) + F_{z_{17}}(r_{z_{17}} - s_{17})]$$

$$\varphi_{f26} = -(c\phi_9 c\theta_9 s\psi_9 + s\phi_9 c\psi_9)\{F_{x_{11}} r_{x_{11}} + F_{x_{13}} r_{x_{13}} + c\alpha_{11}[F_{x_{15}} r_{x_{15}} + F_{x_{17}} r_{x_{17}}] + s\alpha_{11}[F_{x_{15}}(r_{z_{15}} - s_{15}) + F_{x_{17}}(r_{z_{17}} - s_{17})]\} + (-c\phi_9 c\theta_9 c\psi_9 + s\phi_9 s\psi_9)[F_{x_{11}} r_{y_{11}} + F_{x_{13}} r_{y_{13}} + F_{x_{15}} r_{y_{15}} + F_{x_{17}} r_{y_{17}}] + (-s\phi_9 c\theta_9 s\psi_9 + c\phi_9 c\psi_9) \{F_{y_{11}} r_{x_{11}} + F_{y_{13}} r_{x_{13}} + c\alpha_{11}[F_{y_{15}} r_{x_{15}} + F_{y_{17}} r_{x_{17}}] + s\alpha_{11}[F_{y_{15}}(r_{z_{15}} - s_{15}) + F_{y_{17}}(r_{z_{17}} - s_{17})]\} - (s\phi_9 c\theta_9 c\psi_9 + c\phi_9 s\psi_9)[F_{y_{11}} r_{y_{11}} + F_{y_{13}} r_{y_{13}} + F_{y_{15}} r_{y_{15}} + F_{y_{17}} r_{y_{17}}]$$

(cont. on next page)

(q_{f26} cont'd.)

$$
\begin{aligned}
&+ s\theta_9\, s\psi_9 \{ F_{z_{11}} r_{x_{11}} + F_{z_{13}} r_{x_{13}} + c\alpha_{11} [F_{z_{15}} r_{x_{15}} \\
&+ F_{z_{17}} r_{x_{17}}] + s\alpha_{11} [F_{z_{15}} (r_{z_{15}} - S_{15}) + F_{z_{17}} (r_{z_{17}} \\
&- S_{17})] \} + s\theta_9\, c\psi_9 [F_{z_{11}} r_{y_{11}} + F_{z_{13}} r_{y_{13}} + F_{z_{15}} r_{y_{15}} \\
&+ F_{z_{17}} r_{y_{17}}]
\end{aligned}
$$

$$
\begin{aligned}
q_{f27} = & -(s\phi_{10}\, c\theta_{10}\, c\psi_{10} + c\phi_{10}\, s\psi_{10}) [F_{x_{10}} r_{x_{10}} + F_{x_{12}} \\
& r_{x_{12}}] + (s\phi_{10}\, c\theta_{10}\, s\psi_{10} - c\phi_{10}\, c\psi_{10}) [F_{x_{10}} r_{y_{10}} \\
& + F_{x_{12}} r_{y_{12}} + F_{x_{14}} r_{y_{14}} + F_{x_{16}} r_{y_{16}}] - (s\phi_{10} \\
& s\theta_{10}) [F_{x_{10}} (r_{z_{10}} - S_{10}) + F_{x_{12}} r_{z_{12}} - L_{10} (F_{x_{12}} \\
& + F_{x_{14}} + F_{x_{16}})] + (c\phi_{10}\, c\theta_{10}\, c\psi_{10} - s\phi_{10}\, s\psi_{10}) \\
& [F_{y_{10}} r_{x_{10}} + F_{y_{12}} r_{x_{12}}] - (c\phi_{10}\, c\theta_{10}\, s\psi_{10} + s\phi_{10} \\
& c\psi_{10}) [F_{y_{10}} r_{y_{10}} + F_{y_{12}} r_{y_{12}} + F_{y_{14}} r_{y_{14}} + F_{y_{16}} \\
& r_{y_{16}}] + c\phi_{10}\, s\theta_{10} [F_{y_{10}} (r_{z_{10}} - S_{10}) + F_{y_{12}} \\
& r_{z_{12}} - L_{10} (F_{y_{12}} + F_{y_{14}} + F_{y_{16}})] + [c\alpha_{12} \\
& (- s\phi_{10}\, c\theta_{10}\, c\psi_{10} - c\phi_{10}\, s\psi_{10}) + s\phi_{10}\, s\theta_{10} \\
& s\alpha_{12}] [F_{x_{14}} r_{x_{14}} + F_{x_{16}} r_{x_{16}}] - [s\alpha_{12} (s\phi_{10}\, c\theta_{10} \\
& c\psi_{10} + c\phi_{10}\, s\psi_{10}) + s\phi_{10}\, s\theta_{10}\, c\alpha_{12}] [F_{x_{14}} \\
& (r_{z_4} - S_{14}) + F_{x_{16}} (r_{z_{16}} - S_{16})] + [c\alpha_{12} (c\phi_{10} \\
& c\theta_{10}\, c\psi_{10} - s\phi_{10}\, s\psi_{10}) - c\phi_{10}\, s\theta_{10}\, s\alpha_{12}] [F_{x_{14}} \\
& r_{x_{14}} + F_{y_{16}} r_{x_{16}}] + [s\alpha_{12} (c\phi_{10}\, c\theta_{10}\, c\psi_{10} - s\phi_{10} \\
& s\psi_{10}) + c\phi_{10}\, s\theta_{10}\, c\alpha_{12}] [F_{y_{14}} (r_{z_{14}} - S_{14}) \\
& + F_{y_{16}} (r_{z_{16}} - S_{16})]
\end{aligned}
$$

$$
\begin{aligned}
Q_{f28} = {} & c\phi_{10}\{s\theta_{10}[-c\psi_{10}(F_{x10}r_{x10}+F_{x12}r_{x12}) \\
& + s\psi_{10}(F_{x10}r_{y10}+F_{x12}r_{y12}+F_{x14}r_{y14} \\
& + F_{x16}r_{y16})] + c\theta_{10}[F_{x10}(r_{z10}-S_{10})+F_{x12} \\
& r_{z12} - L_{10}(F_{x12}+F_{x14}+F_{x16})] - (s\theta_{10}c\psi_{10} \\
& c\alpha_{12} + c\theta_{10}s\alpha_{12})[F_{x14}r_{x14}+F_{x16}r_{x16}] \\
& + (-s\theta_{10}c\psi_{10}s\alpha_{12}+c\theta_{10}c\alpha_{12})[F_{x14}(r_{z14} \\
& - S_{14}) + F_{x16}(r_{z16}-S_{16})]\} + s\phi_{10}\{s\theta_{10} \\
& [-c\psi_{10}(F_{y10}r_{x10}+F_{y12}r_{x12}) + s\psi_{10}(F_{y10} \\
& r_{y10}+F_{y12}r_{y12}+F_{y14}r_{y14}+F_{y16}r_{y16})] \\
& + c\theta_{10}[F_{y10}(r_{z10}-S_{10})+F_{y12}r_{z12}-L_{10}(F_{y12} \\
& + F_{y14}+F_{y16})] - (s\theta_{10}c\psi_{10}c\alpha_{12}+c\theta_{10}s\alpha_{12}) \\
& [F_{y14}r_{x14}+F_{y16}r_{x16}] + (-s\theta_{10}c\psi_{10}s\alpha_{12} \\
& + c\theta_{10}c\alpha_{12})[F_{y14}(r_{z14}-S_{14})+F_{y16}(r_{z16} \\
& - S_{16})]\} + c\theta_{10}[-c\psi_{10}(F_{z10}r_{x10}+F_{z12}r_{x12}) \\
& + s\psi_{10}(F_{z10}r_{y10}+F_{z12}r_{y12}+F_{z14}r_{y14}+F_{z16} \\
& r_{y16})] - s\theta_{10}[F_{z10}(r_{z10}-S_{10})+F_{z12}r_{z12} \\
& - L_{10}(F_{z12}+F_{z14}+F_{z16})] + (-c\theta_{10}c\psi_{10}c\alpha_{12} \\
& + s\theta_{10}s\alpha_{12})[F_{z14}r_{x14}+F_{z16}r_{x16}] - (c\theta_{10} \\
& c\psi_{10}s\alpha_{12}+s\theta_{10}c\alpha_{12})[F_{z14}(r_{z14}-S_{14}) \\
& + F_{z16}(r_{z16}-S_{16})]
\end{aligned}
$$

$$
\begin{aligned}
Q_{f29} = {} & -(c\phi_{10}c\theta_{10}s\psi_{10}+s\phi_{10}c\psi_{10})\{[F_{x10}r_{x10} \\
& + F_{x12}r_{x12}]
\end{aligned}
$$

(cont. on next page)

(φ_{f29} cont'd.)

$$+c\alpha_{12}[F_{x_{14}} r_{x_{14}} + F_{x_{16}} r_{x_{16}}] + s\alpha_{12}[F_{x_{14}} (r_{z_{14}} - S_{14}) + F_{x_{16}} (r_{z_{16}} - S_{16})]\} + (-c\phi_{10} c\theta_{10} c\psi_{10} + s\phi_{10} s\psi_{10})[F_{x_{10}} r_{y_{10}} + F_{x_{12}} r_{y_{12}} + F_{x_{14}} r_{y_{14}} + F_{x_{16}} r_{y_{16}}] + (-s\phi_{10} c\theta_{10} s\psi_{10} + c\phi_{10} c\psi_{10})\{[F_{y_{10}} r_{x_{10}} + F_{y_{12}} r_{x_{12}}] + c\alpha_{12} [F_{y_{14}} r_{x_{14}} + F_{y_{16}} r_{x_{16}}] + s\alpha_{12} [F_{y_{14}}(r_{z_{14}} - S_{14}) + F_{y_{16}} (r_{z_{16}} - S_{16})]\} - (s\phi_{10} c\theta_{10} c\psi_{10} + c\phi_{10} s\psi_{10})[F_{y_{10}} r_{y_{10}} + F_{y_{12}} r_{y_{12}} + F_{y_{14}} r_{y_{14}} + F_{y_{16}} r_{y_{16}}] + s\theta_{10} s\psi_{10}\{[F_{z_{10}} r_{x_{10}} + F_{z_{12}} r_{x_{12}}] + c\alpha_{12} [F_{z_{14}} r_{x_{14}} + F_{z_{16}} r_{x_{16}}] + s\alpha_{12} [F_{z_{14}}(r_{z_{14}} - S_{14}) + F_{z_{16}} (r_{z_{16}} - S_{16})]\} + s\theta_{10} c\psi_{10} [F_{z_{10}} r_{y_{10}} + F_{z_{12}} r_{y_{12}} + F_{z_{14}} r_{y_{14}} + F_{z_{16}} r_{y_{16}}]$$

$$\varphi_{f30} = (-c\phi_9 c\theta_9 c\psi_9 s\alpha_{11} + s\phi_9 s\psi_9 s\alpha_{11} - c\phi_9 s\theta_9 c\alpha_{11})[F_{x_{15}} r_{x_{15}} + F_{x_{17}} r_{x_{17}}] + (c\phi_9 c\theta_9 c\psi_9 c\alpha_{11} - s\phi_9 s\psi_9 c\alpha_{11} - c\phi_9 s\theta_9 s\alpha_{11}) [F_{x_{15}} (r_{z_{15}} - S_{15}) + F_{x_{17}} (r_{z_{17}} - S_{17})] - (s\phi_9 c\theta_9 c\psi_9 s\alpha_{11} + c\phi_9 s\psi_9 s\alpha_{11} + s\phi_9 s\theta_9 c\alpha_{11})[F_{y_{15}} r_{x_{15}} + F_{y_{17}} r_{x_{17}}] + (s\phi_9 c\theta_9 c\psi_9 c\alpha_{11} + c\phi_9 s\psi_9 c\alpha_{11} - s\phi_9 s\theta_9 s\alpha_{11})$$

(cont. on next page)

(Φ_{f30} cont'd.)

$$[F_{y15}(r_{z15}-s_{15}) + F_{y17}(r_{z17}-s_{17})] + (s\theta_9 c\psi_9 s\alpha_{11} - c\theta_9 c\alpha_{11})[F_{z15} r_{x15} + F_{z17} r_{x17}] - (s\theta_9 c\psi_9 c\alpha_{11} + c\theta_9 s\alpha_{11})[F_{z15}(r_{z15}-s_{15}) + F_{z17}(r_{z17}-s_{17})]$$

$$\Phi_{f31} = (-c\phi_{10} c\theta_{10} c\psi_{10} s\alpha_{12} + s\phi_{10} s\psi_{10} s\alpha_{12} - c\phi_{10} s\theta_{10} c\alpha_{12})[F_{x14} r_{x14} + F_{x16} r_{x16}] + (c\phi_{10} c\theta_{10} c\alpha_{12} - s\phi_{10} s\psi_{10} c\alpha_{12} - c\phi_{10} s\theta_{10} s\alpha_{12})[F_{x14}(r_{z14}-s_{14}) + F_{x16}(r_{z16}-s_{16})] - (s\phi_{10} c\theta_{10} c\psi_{10} s\alpha_{12} + c\phi_{10} s\psi_{10} s\alpha_{12} + s\phi_{10} s\theta_{10} c\alpha_{12})[F_{y14} r_{x14} + F_{y16} r_{x16}] + (s\phi_{10} c\theta_{10} c\psi_{10} c\alpha_{12} + c\phi_{10} s\psi_{10} c\alpha_{12} - s\phi_{10} s\theta_{10} s\alpha_{12})[F_{y14}(r_{z14}-s_{14}) + F_{y16}(r_{z16}-s_{16})] + (s\theta_{10} c\psi_{10} s\alpha_{12} - c\theta_{10} c\alpha_{12})[F_{z14} r_{x14} + F_{z16} r_{x16}] - (s\theta_{10} c\psi_{10} c\alpha_{12} + c\theta_{10} s\alpha_{12})[F_{z14}(r_{z14}-s_{14}) + F_{z16}(r_{z16}-s_{16})]$$

In defining the column vector R the following terms are necessary:

$$H_1 = T_{13}^{2} T_{13}^{3} + T_{23}^{2} T_{23}^{3} + T_{33}^{2} T_{33}^{3}$$

$$G_1 = T_{13}^{2} \dot{T}_{13}^{3} + \dot{T}_{13}^{2} T_{13}^{3} + T_{23}^{2} \dot{T}_{23}^{3} + \dot{T}_{23}^{2} T_{23}^{3} + T_{33}^{2} \dot{T}_{33}^{3} + \dot{T}_{33}^{2} T_{33}^{3}$$

$$H_2 = T^2_{13} T^6_{13} + T^2_{23} T^6_{23} + T^2_{33} T^6_{33}$$

$$G_2 = T^2_{13} \dot{T}^6_{13} + \dot{T}^2_{13} T^6_{13} + T^2_{23} \dot{T}^6_{23} + \dot{T}^2_{23} T^6_{23} + T^2_{33} \dot{T}^6_{33} + \dot{T}^2_{33} T^6_{33}$$

$$H_3 = T^2_{13} T^5_{13} + T^2_{23} T^5_{23} + T^2_{33} T^5_{33}$$

$$G_3 = T^2_{13} \dot{T}^5_{13} + \dot{T}^2_{13} T^5_{13} + T^2_{23} \dot{T}^5_{23} + \dot{T}^2_{23} T^5_{23} + T^2_{33} \dot{T}^5_{33} + \dot{T}^2_{33} T^5_{33}$$

$$H_4 = T^1_{13} T^2_{13} + T^1_{23} T^2_{23} + T^1_{33} T^2_{33}$$

$$G_4 = T^1_{13} \dot{T}^2_{13} + \dot{T}^1_{13} T^2_{13} + T^1_{23} \dot{T}^2_{23} + \dot{T}^1_{23} T^2_{23} + T^1_{33} \dot{T}^2_{33} + \dot{T}^1_{33} T^2_{33}$$

$$H_7 = T^1_{13} T^4_{13} + T^1_{23} T^4_{23} + T^1_{33} T^4_{33}$$

$$G_7 = T^1_{13} \dot{T}^4_{13} + \dot{T}^1_{13} T^4_{13} + T^1_{23} \dot{T}^4_{23} + \dot{T}^1_{23} T^4_{23} + T^1_{33} \dot{T}^4_{33} + \dot{T}^1_{33} T^4_{33}$$

$$H_8 = T^4_{13} T^{10}_{13} + T^4_{23} T^{10}_{23} + T^4_{33} T^{10}_{33}$$

$$G_8 = T^4_{13} \dot{T}^{10}_{13} + \dot{T}^4_{13} T^{10}_{13} + T^4_{23} \dot{T}^{10}_{23} + \dot{T}^4_{23} T^{10}_{23} + T^4_{33} \dot{T}^{10}_{33} + \dot{T}^4_{33} T^{10}_{33}$$

$$H_9 = T^4_{13} T^9_{13} + T^4_{23} T^9_{23} + T^4_{33} T^9_{33}$$

$$G_9 = T^4_{13} \dot{T}^9_{13} + \dot{T}^4_{13} T^9_{13} + T^4_{23} \dot{T}^9_{23} + \dot{T}^4_{23} T^9_{23} + T^4_{33} \dot{T}^9_{33} + \dot{T}^4_{33} T^9_{33}$$

$$T^n_{13} = c\phi_n \, s\theta_n$$

$$\dot{T}^n_{13} = -\dot{\phi}_n s\phi_n s\theta_n + \dot{\theta}_n c\phi_n c\theta_n$$

$$T^n_{23} = s\phi_n \, s\theta_n$$

$$\dot{T}^n_{23} = \dot{\phi}_n c\phi_n s\theta_n + \dot{\theta}_n s\phi_n c\theta_n$$

$$T_{33}^{n} = c\theta_n$$

$$\dot{T}_{33}^{n} = -\dot{\theta}_n s\theta_n$$

$J_n =$ viscous damping coefficient of body joint number n (in.-lb.-sec.)

Column vector R contains the ensuing elements:

$$R_1 = 0$$

$$R_2 = 0$$

$$R_3 = 0$$

$$R_4 = -\left[\frac{J_4 G_4}{1-(H_4)^2}\right]\left\{T_{13}^{2}(-s\phi_1 s\theta_1) + T_{23}^{2}(c\phi_1 s\theta_1)\right\}$$

$$-\left[\frac{J_7 G_7}{1-(H_7)^2}\right]\left\{T_{13}^{4}(-s\phi_1 s\theta_1) + T_{23}^{4}(c\phi_1 s\theta_1)\right\}$$

$$R_5 = -\left[\frac{J_4 G_4}{1-(H_4)^2}\right]\left\{T_{13}^{2}(c\phi_1 c\theta_1) + T_{23}^{2}(s\phi_1 c\theta_1) + T_{33}^{2}(-s\theta_1)\right\}$$

$$-\left[\frac{J_7 G_7}{1-(H_7)^2}\right]\left\{T_{13}^{4}(c\phi_1 c\theta_1) + T_{23}^{4}(s\phi_1 c\theta_1) + T_{33}^{4}(-s\theta_1)\right\}$$

$$R_6 = 0$$

$$R_7 = -\left[\frac{J_1 G_1}{1-(H_1)^2}\right]\left\{T_{13}^{3}(-s\phi_2 s\theta_2) + T_{23}^{3}(c\phi_2 s\theta_2)\right\}$$

(cont. on next page)

(R_7 cont'd.)

$$-\left[\frac{J_2 G_2}{1-(H_2)^2}\right]\left\{T_{13}^{6}(-s\phi_2\, s\theta_2) + T_{23}^{6}(c\phi_2\, s\theta_2)\right\}$$

$$-\left[\frac{J_3 G_3}{1-(H_3)^2}\right]\left\{T_{13}^{5}(-s\phi_2\, s\theta_2) + T_{23}^{5}(c\phi_2\, s\theta_2)\right\}$$

$$-\left[\frac{J_4 G_4}{1-(H_4)^2}\right]\left\{T_{13}^{1}(-s\phi_2\, s\theta_2) + T_{23}^{1}(c\phi_2\, s\theta_2)\right\}$$

$$R_8 = -\left[\frac{J_1 G_1}{1-(H_1)^2}\right]\left\{T_{13}^{3}(c\phi_2\, c\theta_2) + T_{23}^{3}(s\phi_2\, c\theta_2) + T_{33}^{3}(-s\theta_2)\right\}$$

$$-\left[\frac{J_2 G_2}{1-(H_2)^2}\right]\left\{T_{13}^{6}(c\phi_2\, c\theta_2) + T_{23}^{6}(s\phi_2\, c\theta_2)\; T_{33}^{6}(-s\theta_2)\right\}$$

$$-\left[\frac{J_3 G_3}{1-(H_3)^2}\right]\left\{T_{13}^{5}(c\phi_2\, c\theta_2) + T_{23}^{5}(s\phi_2\, c\theta_2) + T_{33}^{5}(-s\theta_2)\right\}$$

$$-\left[\frac{J_4 G_4}{1-(H_4)^2}\right]\left\{T_{13}^{1}(c\phi_2\, c\theta_2) + T_{23}^{1}(s\phi_2\, c\theta_2) + T_{33}^{1}(-s\theta_2)\right\}$$

$$R_9 = 0$$

$$R_{10} = -\left[\frac{J_1 G_1}{1-(H_1)^2}\right]\left\{T_{13}^{2}(-s\phi_3\, s\theta_3) + T_{23}^{2}(c\phi_3\, s\theta_3)\right\}$$

$$R_{11} = -\left[\frac{J_1 G_1}{1-(H_1)^2}\right]\left\{T_{13}^{2}(c\phi_3\, c\theta_3) + T_{23}^{2}(s\phi_3\, c\theta_3) + T_{33}^{2}(-s\theta_3)\right\}$$

$$R_{12} = 0$$

$$R_{13} = -\left[\frac{J_7 G_7}{1-(H_7)^2}\right]\left\{T_{13}^{1}(-s\phi_4 s\theta_4) + T_{23}^{1}(c\phi_4 s\theta_4)\right\}$$

$$-\left[\frac{J_8 G_8}{1-(H_8)^2}\right]\left\{T_{13}^{10}(-s\phi_4 s\theta_4) + T_{23}^{10}(c\phi_4 s\theta_4)\right\}$$

$$-\left[\frac{J_9 G_9}{1-(H_9)^2}\right]\left\{T_{13}^{9}(-s\phi_4 s\theta_4) + T_{23}^{9}(c\phi_4 s\theta_4)\right\}$$

$$R_{14} = -\left[\frac{J_7 G_7}{1-(H_7)^2}\right]\left\{T_{13}^{1}(c\phi_4 c\theta_4) + T_{23}^{1}(s\phi_4 c\theta_4) + T_{33}^{1}(-s\theta_4)\right\}$$

$$-\left[\frac{J_8 G_8}{1-(H_8)^2}\right]\left\{T_{13}^{10}(c\phi_4 c\theta_4) + T_{23}^{10}(s\phi_4 c\theta_4) + T_{33}^{10}(-s\theta_4)\right\}$$

$$-\left[\frac{J_9 G_9}{1-(H_8)^2}\right]\left\{T_{13}^{9}(c\phi_4 c\theta_4) + T_{23}^{9}(s\phi_4 c\theta_4) + T_{33}^{9}(-s\theta_4)\right\}$$

$$R_{15} = 0$$

$$R_{16} = -\left[\frac{J_3 G_3}{1-(H_3)^2}\right]\left\{T_{13}^{2}(-s\phi_5 s\theta_5) + T_{23}^{2}(c\phi_5 s\theta_5)\right\}$$

$$R_{17} = -\left[\frac{J_3 G_3}{1-(H_3)^2}\right]\left\{T_{13}^{2}(c\phi_5 c\theta_5) + T_{23}^{2}(s\phi_5 c\theta_5) + T_{33}^{2}(-s\theta_5)\right\}$$

$$R_{18} = 0$$

$$R_{19} = -\left[\frac{J_2 G_2}{1-(H_2)^2}\right]\left\{T_{13}^{2}(-s\phi_6 s\theta_6) + T_{23}^{2}(c\phi_6 s\theta_6)\right\}$$

$$R_{20} = -\left[\frac{J_2 G_2}{1-(H_2)^2}\right]\left\{T_{13}^{2}(c\phi_6 c\theta_6) + T_{23}^{2}(s\phi_6 c\theta_6) + T_{33}^{2}(-s\theta_6)\right\}$$

$$R_{21} = 0$$

$$R_{22} = -J_5 \; \dot{\alpha}_7$$

$$R_{23} = -J_6 \; \dot{\alpha}_8$$

$$R_{24} = -\left[\frac{J_9 \; G_9}{1-(H_9)^2}\right]\left\{T_{13}^4(-s\phi_9 \; s\theta_9) + T_{23}^4(c\phi_9 \, s\theta_9)\right\}$$

$$R_{25} = -\left[\frac{J_9 \; G_9}{1-(H_9)^2}\right]\left\{T_{13}^4(c\phi_9 \, c\theta_9) + T_{23}^4(s\phi_9 \, c\theta_9) + T_{33}^4(-s\theta_9)\right\}$$

$$R_{26} = 0$$

$$R_{27} = -\left[\frac{J_8 \; G_8}{1-(H_8)^2}\right]\left\{T_{13}^4(-s\phi_{10} \; s\theta_{10}) + T_{23}^4(c\phi_{10} \, s\theta_{10})\right\}$$

$$R_{28} = -\left[\frac{J_8 \; G_8}{1-(H_8)^2}\right]\left\{T_{13}^4(c\phi_{10} \, c\theta_{10}) + T_{23}^4(s\phi_{10} \, c\theta_{10}) + T_{33}^4(-s\theta_{10})\right\}$$

$$R_{29} = 0$$

$$R_{30} = -J_{11} \; \dot{\alpha}_{11}$$

$$R_{31} = -J_{10} \; \dot{\alpha}_{12}$$

APPENDIX IV.-DESCRIPTION OF INPUT TO THE COMPUTER PROGRAM

First Card, Format(20A4)

Col. No.	Program Variable	Description
1-80	HED	alphameric information for the purpose of identification, to be printed at top of each output page

Second Card, Format(20A4)

Col. No.	Program Variable	Description
1-80	HED	a continuation of the first card

Third Card, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	T1	initial time (sec.)
11-20	TF	final time (sec.)
21-30	DT	increment of time for integration (sec.)
31-40	DTPRNT	print interval (sec.)
41-50	MREAD	= 0.0, signifies that vehicle position data will be supplied on cards rather than read from disk, hence twelfth series of cards must be included ≠ 0.0, represents the total number of consecutive positions or time stations for which vehicle position data will be read from the disk, hence omit

		twelfth series of cards (this number is punched with a decimal)
51-60	NPOS	= 0.0, program computes passenger's initial conditions based on data from fourth card and tenth series of cards (sixth series may be left out)
		= 1.0, initial conditions are to be supplied in sixth series of cards (fourth card and tenth series may be left out)
61-70	NDISK	= 0.0, signifies that data necessary for plotting will be put on disk
		= 1.0, signifies that no data will be put on disk, hence no plotting
71-79		leave blank
80	NCARD	= 3 (for third card)

Fourth Card, Format(7F10.0,I10)

Initial Translational Velocity of Passenger
(Not needed if NPOS=1.0 on third card)

Col. No.	Program Variable	Description
1-10	VXVC	velocity of passenger in vehicle-coordinate X direction (in./sec.)
11-20	VYVC	velocity of passenger in vehicle-coordinate Y direction (in./sec.)
21-30	VZVC	velocity of passenger in vehicle-coordinate Z direction (in./sec.)
31-79		leave blank
80		= 4

Fifth Series of Cards (13 Cards)

Body-Segment Properties

a. First card of fifth series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	XLS	length of shoulder or half of shoulder width (inches)
11-20	XLH	length of hip or half of hip width (inches)
21-79		leave blank
80	NCARD	= 5

b. Next twelve cards of fifth series, Format(I2,8X,6F10.0)

Col. No.	Program Variable	Description
1-2	NSEG	body-segment number
3-10		leave blank
11-20	XLE(NSEG)	length of segment (inches)
21-30	RHO(NSEG)	distance from reference end of segment to center of gravity of segment (inches)
31-40	XM(NSEG)	mass of segment ($lb.-sec.^2/in.$)
41-50	XIN(NSEG)	I_X (mass moment of inertia about X axis of segment ($in.-lb.-sec.^2$)
51-60	YIN(NSEG)	I_Y
61-70	ZIN(NSEG)	I_Z

Note: Part b is repeated for twelve segments.

Sixth Series of Cards (9 Cards)

Initial Conditions on Generalized Coordinates, Generalized Velocities, and Other Pertinent Data
(Needed only when NPOS=1.0 on third card)

a. First card of sixth series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	BETA4	initial value of the angle between middle and upper back segments (radians)
11-20	BETA7	initial value of the angle between middle and lower back segments (radians)
21-30	BLTLEO(1)*	initial length of lap belt if in use (inches)
31-40	BLTLEO(2)*	initial length of shoulder belt if in use (inches)
41-79		leave blank
80	NCARD	= 6

b. Second card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QO(1)	X'_{T1}, Y'_{T1}, Z'_{T1}, respectively, representing the coordinates of the reference point on the articulated body (terminal end of segment No. 1) in space-fixed coordinate system (inches)
11-20	QO(2)	
21-30	QO(3)	

*Leave blank when belt is not to be used.

31-40	QO(4)	ϕ_1, θ_1, ψ_1, respectively, representing the Euler angles of body segment No. 1 (radians)*
41-50	QO(5)	
51-60	QO(6)	
61-70	QO(7)	ϕ_2
71-80	QO(8)	θ_2

c. Third card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QO(9)	ψ_2
11-20	QO(10)	ϕ_3
21-30	QO(11)	θ_3
31-40	QO(12)	ψ_3
41-50	QO(13)	ϕ_4
51-60	QO(14)	θ_4
61-70	QO(15)	ψ_4
71-80	QO(16)	ϕ_5

*For the remainder of this sixth series, ϕ_n, θ_n, and ψ_n will represent the Euler angles of body segment n in radians.

d. Fourth card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QO(17)	θ_5
11-20	QO(18)	ψ_5
21-30	QO(19)	ϕ_6
31-40	QO(20)	θ_6
41-50	QO(21)	ψ_6
51-60	QO(22)	α_7, position angle of segment No. 7 relative to segment No. 5 (radians)*
61-70	QO(23)	α_8
71-80	QO(24)	ϕ_9

e. Fifth card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QO(25)	θ_9
11-20	QO(26)	ψ_9
21-30	QO(27)	ϕ_{10}
31-40	QO(28)	θ_{10}
41-50	QO(29)	ψ_{10}
51-60	QO(30)	α_{11}
61-70	QO(31)	α_{12}
71-80	blank	

---

*For the remainder of this sixth series, α_i will represent the position angle of segment No. i relative to segment No. i-2 in radians.

f. Sixth card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QDO(1)	$\dot{X}'_{T1}$, $\dot{Y}'_{T1}$, $\dot{Z}'_{T1}$, respectively, representing the components of velocity of the reference point on the articulated body in space-fixed coordinates (in./sec.)
11-20	QDO(2)	
21-30	QDO(3)	
31-40	QDO(4)	$\dot{\phi}_1$, $\dot{\theta}_1$, $\dot{\psi}_1$, respectively, representing the Euler angular velocities of body segment No. 1 (rad./sec.)*
41-50	QDO(5)	
51-60	QDO(6)	
61-70	QDO(7)	$\dot{\phi}_2$
71-80	QDO(8)	$\dot{\theta}_2$

g. Seventh card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QDO(9)	$\dot{\psi}_2$
11-20	QDO(10)	$\dot{\phi}_3$
21-30	QDO(11)	$\dot{\theta}_3$
31-40	QDO(12)	$\dot{\psi}_3$
41-50	QDO(13)	$\dot{\phi}_4$
51-60	QDO(14)	$\dot{\theta}_4$
61-70	QDO(15)	$\dot{\psi}_4$
71-80	QDO(16)	$\dot{\phi}_5$

*For the remainder of this sixth series, $\dot{\phi}_n$, $\dot{\theta}_n$, and $\dot{\psi}_n$ will represent the Euler angular velocities of body segment No. n in radians per second.

h. Eighth card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QDO(17)	$\dot{\theta}_5$
11-20	QDO(18)	$\dot{\psi}_5$
21-30	QDO(19)	$\dot{\phi}_6$
31-40	QDO(20)	$\dot{\theta}_6$
41-50	QDO(21)	$\dot{\psi}_6$
51-60	QDO(22)	$\dot{\alpha}_7$, representing the angular velocity of segment No. 7 relative to segment No. 5 (rad./sec.)*
61-70	QDO(23)	$\dot{\alpha}_8$
71-80	QDO(24)	$\dot{\phi}_9$

i. Ninth card of sixth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	QDO(25)	$\dot{\theta}_9$
11-20	QDO(26)	$\dot{\psi}_9$
21-30	QDO(27)	$\dot{\phi}_{10}$
31-40	QDO(28)	$\dot{\theta}_{10}$
41-50	QDO(29)	$\dot{\psi}_{10}$
51-60	QDO(30)	$\dot{\alpha}_{11}$
61-70	QDO(31)	$\dot{\alpha}_{12}$
71-80		leave blank

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*For the remainder of this sixth series, $\dot{\alpha}_i$ will represent the angular velocity of segment No. i relative to segment No. i-2 in radians per second.

Seventh Series of Cards (4 Cards)

Radii of Contact Spheres (inches)

a. First card of seventh series, Format(7F10.0,I8)

Col. No.	Program Variable	Description
1-79		leave blank
80	NCARD	= 7

b. Second card of seventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	RS(1)	radius of contact sphere No. 1 (inches)*
11-20	RS(2)	
21-30	RS(3)	
31-40	RS(4)	
41-50	RS(5)	
51-60	RS(6)	
61-70	RS(7)	
71-80	RS(8)	

*For the remainder of the seventh series, RS(N) will represent the radius of contact sphere No. N in inches.

c. Third card of seventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	RS(9)	
11-20	RS(10)	
21-30	RS(11)	
31-40	RS(12)	
41-50	RS(13)	
51-60	RS(14)	
61-70	RS(15)	
71-80	RS(16)	

d. Fourth card of seventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	RS(17)	
11-80		leave blank

Eighth Series of Cards (7 Cards)

Dimensions and Coordinates for Idealized Passenger Compartment

a. First card of eighth series, Format(7F10.0,I8)

Col. No.	Program Variable	Description
1-10	VCW	half of vehicle compartment width (positive number) (inches)

11-20	VRW	half of vehicle roof width (positive number) (inches)
21-30	SWP	steering wheel position (distance from center line of vehicle to center of wheel) (inches)
31-40	DSW	diameter of steering wheel (inches)
41-79		leave blank
80	NCARD	= 8

b. Second card of eighth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XV(1)	X and Z coordinates, respectively, of point No. 1 of passenger compartment in vehicle-fixed coordinate system (inches)*
11-20	ZV(1)	
21-30	XV(2)	
31-40	ZV(2)	
41-50	XV(3)	
51-60	ZV(3)	
61-70	XV(4)	
71-80	ZV(4)	

*For the remainder of this eighth series, XV(N) and ZV(N) will represent the X and Z coordinates of passenger-compartment point No. 1 in vehicle-fixed coordinates (at c.g. of vehicle) in inches.

c. Third card of eighth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XV(5)	
11-20	ZV(5)	
21-30	XV(6)	
31-40	ZV(6)	
41-50	XV(7)	
51-60	ZV(7)	
61-70	XV(8)	
71-80	ZV(8)	

d. Fourth card of eighth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XV(9)	
11-20	ZV(9)	
21-30	XV(10)	
31-40	ZV(10)	
41-50	XV(11)	
51-60	ZV(11)	
61-70	XV(12)	
71-80	ZV(12)	

e. Fifth card of eighth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XV(13)	
11-20	ZV(13)	
21-30	XV(14)	
31-40	ZV(14)	
41-50	XV(15)	
51-60	ZV(15)	
61-70	XV(16)	
71-80	ZV(16)	

f. Sixth card of eighth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XV(17)	
11-20	ZV(17)	
21-30	XV(18)	
31-40	ZV(18)	
41-50	XV(19)	
51-60	ZV(19)	
61-70	XV(20)	
71-80	ZV(20)	

g. Seventh card of eighth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XV(21)	
11-20	ZV(21)	
21-30	XV(22)	
31-40	ZV(22)	
41-80		leave blank

Ninth Series of Cards (26 Cards)

Deformation Properties and Friction Coefficients of the Twenty-five Contact Surfaces of the Idealized Passenger Compartment

a. First card of ninth series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-79		leave blank
80	NCARD	= 9

b. Second through twenty-sixth cards of ninth series, Format(I2,8X,6F10.0)

Col. No.	Program Variable	Description
1-2	J	contact surface number (1-25), punched right justified
3-10		leave blank

11-20	STIF1(J)	first and smallest of spring coefficients for the dissipative bi-linear spring contact surface J (lbs./in.)
21-30	STIF2(J)	second and largest of spring coefficients for the dissipative bi-linear spring of contace surface J (lbs./in.)
31-40	DELB(J)	the value of total deformation (that of contact sphere plus contact surface J, at which STIF2(J) comes into effect
41-50	AMU1(J)	fraction of strain energy to be conserved during a collision between passenger and contact surface J, when the total deformation is less than or equal to DELB(J)
51-60	AMU2(J)	fraction of strain energy to be conserved during a collision between passenger and contact surface J, when the total deformation is greater than DELB(J)
61-70	COFR(J)	coefficient of friction for contact surface J, i.e., friction force = (normal force) x COFR(J)
71-80		leave blank

Include one card as described above for each of the 25 contact surfaces of the idealized vehicle interior. These cards do not follow any particular numbering sequence, but J must be specified on each card.

Tenth Series of Cards (2 Cards)

Passenger Seating Information (see figure)
(Not needed if NPOS=1.0 on third card)

a. First card of tenth series, Format(7F10.0,I8)

Col. No.	Program Variable	Description
1-10	NOPTSP	= 1.0, driver position = 2.0, right front = 3.0, left rear = 4.0, right rear
11-20	GAMA1	γ_1 - position of middle back segment (degrees)
21-30	GAMA2	γ_2 - position of upper back segment (degrees)
31-40	GAMA3	γ_3 - position of head segment (degrees)
41-50	GAMA4	γ_4 - position of lower back segment (degrees)
51-60	GAMA5	γ_5 - position of upper arm segment (degrees)
61-70	ALPHA1	α_1 - position of lower arm segment (degrees)
71-78		leave blank
79-80	NCARD	= 10

b. Second card of tenth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XHPV	X_{VH} - X coordinate of passenger's heels in vehicle-coordinate system (inches)

11-20	ZHPV	Z_{VH} - Z coordinate of passenger's heels in vehicle-coordinate system (inches)
21-80		leave blank

Eleventh Series of Cards (5 Cards)

Joint Information (Viscous Damping and Elasticity) and Acceleration due to Gravity

a. First card of eleventh series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	XJNN	viscous damping coefficient of neck joint (lb.sec.)
11-20	AGSN	limiting value of angular travel of neck with reference to upper torso (radians)
21-30	XJSN	amount of viscous damping at AGSN (lb.-sec.)
31-40	AGNN	value of angular neck travel at which viscous damping begins increasing linearly toward XJSN (radians)
41-50	XJNS	viscous damping coefficient of a shoulder joint (lb.-sec.)
51-60	AGSS	limiting value of angular travel of either upper arm with reference to upper torso (radians)
61-70	XJSS	amount of viscous damping at AGSS (lb.-sec.)
71-78		leave blank
79-80	NCARD	= 11

b. Second card of eleventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	AGNS	value of angular travel of an upper arm at which viscous damping begins increasing linearly toward XJSS (radians)
11-20	XJNUB	viscous damping coefficient in upper back joint (lb.-sec.)
21-30	AGSUB	limiting value of relative angular travel between upper and middle torso segments (radians)
31-40	XJSUB	amount of viscous damping at AGSUB (lb.-sec.)
41-50	AGNUB	value of angular travel in upper back joint at which viscous damping begins to increase linearly toward XJSUB (radians)
51-60	XJNLB	viscous damping coefficient in lower back joint (lb.-sec.)
61-70	AGSLB	limiting value of relative angular travel between middle and lower torso segments (radians)
71-80	XJSLB	amount of viscous damping at AGSLB (lb.-sec.)

c. Third card of eleventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	AGNLB	value of angular travel in lower back joint at which viscous damping begins to increase linearly toward XJSLB (radians)
11-20	XJNH	viscous damping coefficient of a hip joint (lb.-sec.)
21-30	AGSH	limiting value of angular travel of either upper leg with reference to lower torso (radians)
31-40	XJSH	amount of viscous damping at AGSH (lb.-sec.)
41-50	AGNH	value of angular travel of an upper leg at which viscous damping begins increasing linearly toward XJSH (radians)
51-60	XJNE	viscous damping coefficient of elbow joint (lb.-sec.)
61-70	AGSE(1)	limiting value of angular travel of forearm relative to upper arm approaching the closed-elbow position (approx. minus 130 deg.) (radians)
71-80	AGSE(2)	limiting value of angular travel of forearm relative to upper arm approaching the straight-arm position (approx. minus 10 deg.) (radians)

d. Fourth card of eleventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	XJSE	amount of viscous damping at AGSE(1) and/or AGSE(2) (lb.-sec.)
11-20	AGNE(1)	value of angular elbow travel (approaching closed-elbow position) at which viscous damping begins increasing linearly toward XJSE (radians)
21-30	AGNE(2)	value of angular elbow travel (approaching straight-arm position) at which viscous damping begins increasing linearly toward XJSE (radians)
31-40	XJNK	viscous damping coefficient of knee joint (lb.-sec.)
41-50	AGSK(1)	limiting value of angular travel of foreleg relative to upper leg approaching the straight-leg position (approx. zero deg. from the positive side) (radians)
51-60	AGSK(2)	limiting value of angular travel of foreleg relative to upper leg approaching the closed-knee position (approx. plus 140 deg.) (radians)
61-70	XJSK	amount of viscous damping at AGSK(1) and/or AGSK(2) (lb.-sec.)
71-80	AGNK(1)	value of angular knee travel (approaching straight-leg position) at which viscous damping begins increasing linearly toward XJSK (radians)

e. Fifth card of eleventh series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	AGNK(2)	value of angular knee travel (approaching closed-knee position) at which viscous damping begins increasing linearly toward XJSK (radians)
11-20	SPJNT4	stiffness of radial spring in upper back joint (in.-lb./rad.)
21-30	SPJNT7	stiffness of radial spring in lower back joint (in.-lb./rad.)
31-40	GR	acceleration due to gravity if potential energy of position is desired (386.04 in./sec./sec.)
41-80		leave blank

Twelfth Series of Cards (No. of Cards=NVPOS+1)

Vehicle-Position versus Time Data
(Included when MREAD=0.0 on third card)

a. First card of twelfth series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	NVPOS	exactly the number of cards to be included in part b of this series (punch decimal)
11-78		leave blank
79-80	NCARD	= 12

b. Second through (NVPOS+1)th card of twelfth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10	CALT(J)	time at which the vehicle position is being specified
11-20	CALX(J)	X coordinate of vehicle center of gravity in the space-fixed coordinate system defined by CAL* for vehicle model (inches)
21-30	CALY(J)	Y coordinate of vehicle center of gravity in CAL space-fixed system (inches)
31-40	CALZ(J)	Z coordinate of vehicle center of gravity in CAL space-fixed system (inches)
41-50	CALPH(J)	the angle ϕ of the Euler angle system used by CAL (roll angle) (degrees)
51-60	CALTH(J)	the angle θ of the Euler angle system used by CAL (pitch angle) (degrees)
61-70	CALPS(J)	the angle ψ of the Euler angle system used by CAL (yaw angle) (degrees)
71-80		leave blank

Note: J = 1,NVPOS, hence NVPOS number of cards must be included in part b.

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*Cornell Aeronautical Laboratory (CAL) uses a different space-fixed coordinate system as well as a different Euler system to derive their mathematical vehicle model. This twelfth series corresponds to the CAL system.

Thirteenth Series of Cards (3 Cards)

Lap and Shoulder Belt Information
(This series if left out for unrestrained case)

a. First card of thirteenth series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	NOPBLT	= 1.0, lap belt only = 2.0, shoulder belt only = 3.0, lap and shoulder belts
11-20**	P1(1,1)	X coordinate* of lap-belt anchor point No. 1 (inches)
21-30**	P1(1,2)	Y coordinate* of lap-belt anchor point No. 1 (inches)
31-40**	P1(1,3)	Z coordinate* of lap-belt anchor point No. 1 (inches)
41-50**	P2(1,1)	X coordinate* of lap-belt anchor point No. 2 (inches)
51-60**	P2(1,2)	Y coordinate* of lap-belt anchor point No. 2 (inches)
61-70**	P2(1,3)	Z coordinate* of lap-belt anchor point No. 2 (inches)
71-78		leave blank
79-80	NCARD	= 13

*Vehicle-fixed coordinate system
**Leave blank if NOPBLT=2.0

b. Second card of thirteenth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10*+	P1(2,1)	X coordinate* of shoulder-belt anchor point No. 1 (inches)
11-20*+	P1(2,2)	Y coordinate* of shoulder-belt anchor point No. 1 (inches)
21-30*+	P1(2,3)	Z coordinate* of shoulder-belt anchor point No. 1 (inches)
31-40*+	P2(2,1)	X coordinate* of shoulder-belt anchor point No. 2 (inches)
41-50*+	P2(2,2)	Y coordinate* of shoulder-belt anchor point No. 2 (inches)
51-60*+	P2(2,3)	Z coordinate* of shoulder-belt anchor point No. 2 (inches)
61-70**	SLACK(1)	amount of initial slack (looseness of fit) in lap-belt (inches)
71-80*+	SLACK(2)	amount of initial slack in shoulder-belt (inches)

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*Vehicle-fixed coordinate system
**Leave blank if NOPBLT=2.0
*+Leave blank if NOPBLT=1.0

c. Third card of thirteenth series, Format(8F10.0)

Col. No.	Program Variable	Description
1-10**	SK1(1)	the first slope of the idealized* force-elongation curve for the lap-belt (lb./in.)
11-20**	SK2(1)	the second slope of the idealized* force-elongation curve for the lap-belt (lb./in.)
21-30**	DELC(1)	the value of lap-belt elongation at which the slope of its force-elongation curve* changes from SK1(1) to SK1(2) (inches)
31-40*+	SK1(1)	the first slope of the idealized* force-elongation curve for the shoulder-belt (lb./in.)
41-50*+	SK2(2)	the second slope of the idealized* force-elongation curve for the shoulder-belt (lb./in.)
51-60*+	DELC(2)	the value of shoulder-belt elongation at which the slope of its force-elongation curve* changes from SK1(2) to SK2(2) (inches)
61-80		leave blank

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*A bi-linear representation
**Leave blank if NOPBLT=2.0
*+Leave blank if NOPBLT=1.0

Fourteenth Series of Cards (NOCRD+1 Cards)

Information* Necessary to Extend an Existing Computer Solution without Duplication of Effort
(Needed only for extending an existing solution)

a. First card of fourteenth series, Format(7F10.0,I10)

Col. No.	Program Variable	Description
1-10	NOCRD	exactly the number of cards to be included in part b of this series (punch decimal)
11-78		leave blank
79-80	NCARD	= 14

b. Second through (NOCRD+1)th card of fourteenth series, Format(2I2,F6.0)

Col. No.	Program Variable	Description
1-2	J	refer to last printout sheet of computer solution in question for numerical value (punch right justified without a decimal)
3-4	K	refer to last printout sheet of computer solution in question for numerical value (punch right justified without a decimal)

*"Contact Sphere No. J" and "Contact Surface No. K" for which NARRY(J,K)=0, initially. This information is given on last printout sheet of existing computer solution in question.

5-10	NARRY(J,K)	= 0.0 or leave blank
11-80		leave blank

Note: A separate card as shown above is required for each NARRY(J,K) to be initialized.

Fifteenth Series of Cards (1 card)

Terminates Reading of Numerical Input

Col. No.	Program Variable	Description
1-76		leave blank
77-80	NCARD	= 1000

Sixteenth Series of Cards (1 card)

X-Y Plot Option

Col. No.	Program Variable	Description
1-4	HED(1)	= STAN, yields X-Y plots of 14 selected variables vs. time
		= NONE, no plots desired
5-80		leave blank

Final Card

Col. No.	Program Variable	Description
1-4	HED(1)	= FINI
5-80		leave blank