AN EMPIRICAL EQUATION
for
CALCULATING DEFLECTIONS
on the
SURFACE OF A TWO-LAYER ELASTIC SYSTEM

by

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PREFACE

This is the fourth report issued under Research Study 2-8-69-136, Design and Evaluation of Flexible Pavements, being conducted at the Texas Transportation Institute as part of the cooperative research program with the Texas Highway Department and the Department of Transportation, Federal Highway Administration.

Previous reports from this study are as follows:

"Seasonal Variations of Pavement Deflections in Texas: by Rudell Poehl and Frank H. Scrivner, Research Report 136-1, Texas Transportation Institute, January, 1971.

"A Technique for Measuring the Displacement Vector throughout the Body of a Pavement Structure Subjected to Cyclic Loading" by William M. Moore and Gilbert Swift, Research Report 136-2, Texas Transportation Institute, August 1971.

"A Graphical Technique for Determining the Elastic Moduli of a Two-Layered Structure from Measured Surface Deflections", by Gilbert Swift, Research Report 136-3, Texas Transportation Institute, November, 1972.

The author wishes to thank all members of the Institute who assisted in this research, particularly Mr. F. H. Scrivner and Dr. W. M. Moore whose help in the work leading to this report has been especially valuable.

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ABSTRACT

An empirically derived algebraic expression has been developed for computing surface deflections of a two-layer elastic structure loaded vertically at a point on its surface (or on a sufficiently small area of the surface). Calculated deflections from the empirical equation are in substantial agreement with those of previous more laborious methods.

Key words: Two-Layer Elastic Systems, Calculation of Deflections, Pavement Structures

SUMMARY

The equation presented in this report makes it possible to compute the deflections on the surface of a two-layer elastic system very rapidly at low cost. Use of this equation is also economically attractive for performing the large number of computations involved in solving the reverse problem, which is that of determining the elastic properties of a structure from knowledge of its surface deflections.

IMPLEMENTATION STATEMENT

The equation presented in this report will be of interest primarily to researchers engaged in the study of layered elastic structures. However, the development of practical methods for interpreting deflection data obtained on pavement structures may be furthered by use of this equation in appropriate computational procedures.

TABLE OF CONTENTS

1. INTRODUCTION

Computation of the vertical deflections on the surface of an elastic structure having more than one layer has heretofore involved costly, highly laborious computational procedures. The previously available methods, "CHEVRON" (1), "BISTRO" (2) and "ELASTIC MODULUS" (3) are all either derived from - or are similar to - the solution first published by Burmister (4) and hence require the integration of complex expressions involving Bessel functions.

Upon plotting and examining a substantial set of deflection data, computed by Scrivner, Michalak and Moore using the method given in their report (3), it appeared possible that an empirical expression could be found which would fit this data. Accordingly, such an equation was hypothesized. Upon testing this equation in comparison with deflections computed by the "BISTRO" and " ELASTIC MODULUS" programs, it has been found to give results which are in substantial agreement.

Since the "ELASTIC MODULUS" program is limited to cases in which Poisson's ratio is equal to 0.5, the present equation is similarly restricted. However it is evident that the same technique which was used to formulate this equation could be applied to similar data-sets calculated by the "BISTRO" program, using other values of Poisson's ratio, and thus derive appropriate empirical equations for these cases. From a series of such cases it should require relatively little effort to obtain a general expression applicable to any case in which Poisson's ratio is equal in both layers.

The technique used to formulate the present equation consisted of plotting the data in the form, wr, as a function of $\frac{r}{h}$, then subtracting the wr value applicable to homogeneous half spaces having moduli equal to those of the upper layer in each layered case, next determining the coefficients of the set of terms of the type $\frac{rx^n}{L_{n+1}}$ required to fit the remainders at any arbitrary multiple of $\frac{r}{h}$, and finally, determining the value of x required in each case for closest fit at the correct values of $\frac{r}{h}$. These values of x, when plotted against $\frac{E_1}{E_2}$, suggested the cube-root relationship given beneath equations 1 and 2.

In its present form the equation will be of interest primarily to researchers engaged in the study of layered structures such as highway and airport pavements, earth dams, foundation slabs and other layered structures having an accessible upper surface. However, the development of practical methods for interpreting deflection data obtained on pavement surfaces may be furthered by use of this equation in suitable computational procedures to solve for the elastic properties when given the surface deflections.

2. DEFLECTION EQUATION

The empirical equation has been derived for the specific case of Poisson's ratio equal to 0.5 in both layers. The physical situation to which it applies is, as shown in Figure 1, that of a known vertical load applied at a point on the surface of a two-layer elastic system having an upper layer of thickness, h, containing material of modulus E_1 in "rough" contact with a lower layer extending infinitely downward containing material of modulus E_2. The equation gives the amount of vertical deflection, w, on the surface at any finite distance, r, from the point at which the load, P, is applied.

The equation is:

$$w = \frac{3P}{4\pi}\ \frac{1}{r}\left[\frac{1}{E_1} + \left(\frac{1}{E_2} - \frac{1}{E_1}\right)\left(\frac{r}{L} + \frac{rx^2}{2L^3} + \frac{3rx^4}{2L^5}\right)\right] \qquad \text{Eq. 1}$$

In which $L = \sqrt{r^2 + x^2}$

and $x^2 = 4h^2\left(\frac{E_1 + 2E_2}{3E_2}\right)^{2/3}$

or $x = 2h\sqrt[3]{\frac{E_1 + 2E_2}{3E_2}}$

This equation can be expressed in various forms. One particular form, which permits ordinary tables of trigonometric functions to be used in performing the calculations, is:

$$w = \frac{3P}{4\pi}\ \frac{1}{r}\left[\frac{1}{E_1} + \left(\frac{1}{E_2} - \frac{1}{E_1}\right)\left([\sin\theta]\left\{1 + \frac{\cos^2\theta}{2}[1 + 3\cos^2\theta]\right\}\right)\right] \qquad \text{Eq. 2}$$

where $\theta = \tan^{-1}\frac{r}{x}$

and $x = 2h\sqrt[3]{\frac{E_1 + 2E_2}{3E_2}}$

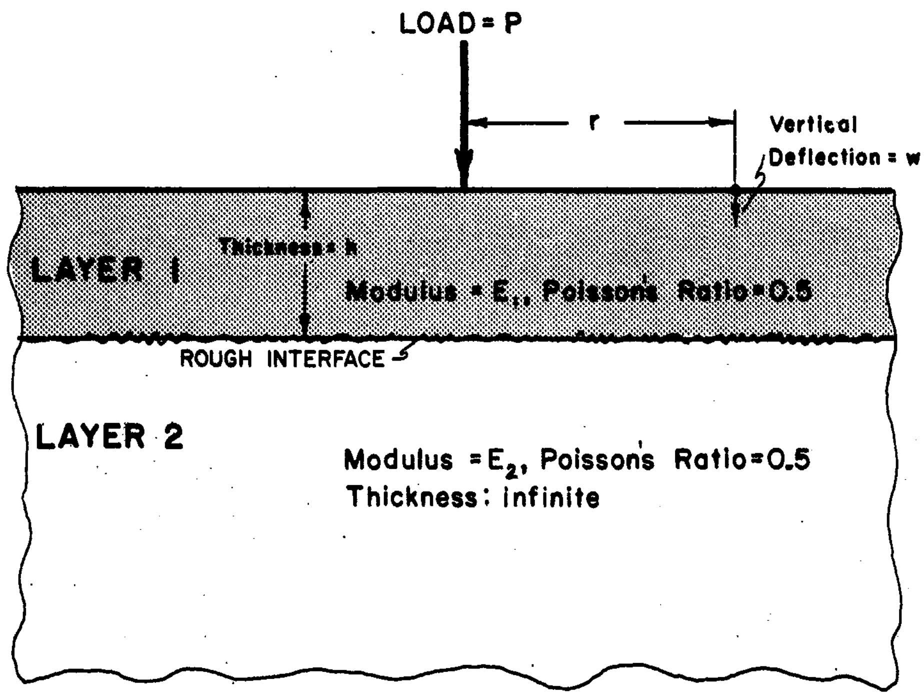

Figure 1: Two Layer Elastic System

3. VERIFICATION OF THE EQUATION

Comparisons have been made between the deflections computed by Equation 1 and those computed by the "BISTRO" and "ELASTIC MODULUS" programs. Values of w were obtained for cases comprising all combinations of the following values of the parameters:

E_1 = 1000, 10,000, 100,000, 1 million and 10 million (psi)

E_2 = 10,000 (psi)

h = 5, 10, 20, and 40 (inches)

r = 10, 15.6, 26, 37.4 and 49 (inches)

The detailed results are given in Table 1. In summary it can be said that all three methods were in substantial agreement. They were within one or two percent, except in two localized regions of the case in which E_1 is less than E_2, where the deflections become small in passing from positive to negative values. Even in these regions the results obtained from the empirical equation are regarded as satisfactory.

Table 1

Computed Deflections (mils) for 1,000 pound load

				W_1 (r = 10.0 inches)			W_2 (r = 15.6 inches)			W_3 (r = 26.0 inches)			W_4 (r = 37.4 inches)			W_5 (r = 49.0 inches)		
E_1 (psi)	E_2 (psi)	$\frac{E_1}{E_2}$	h (in)	ELAST. MOD.	BISTRO	NEW	ELAST. MOD.	BISTRO	NEW	ELAST. MOD.	BISTRO	NEW	ELAST. MOD.	BISTRO	NEW	ELAST. MOD.	BISTRO	NEW
10^7	10^4	10^3	5	0.99	0.99	1.00	0.93	0.93	0.95	0.81	0.81	0.84	0.68	0.68	0.69	0.57	0.57	0.56
			10	0.52	0.52	0.51	0.51	0.51	0.51	0.48	0.48	0.49	0.45	0.45	0.46	0.41	0.41	0.43
			20	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.26	0.25	0.25	0.25	0.24	0.24	0.25
			40	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13	0.13
10^6	10^4	10^2	5	1.86	1.85	1.91	1.55	1.55	1.59	1.09	1.09	1.07	0.75	0.75	0.72	0.54	0.54	0.52
			10	1.07	1.07	1.08	0.99	0.99	1.01	0.84	0.84	0.87	0.70	0.70	0.71	0.57	0.57	0.56
			20	0.57	0.57	0.57	0.55	0.55	0.55	0.51	0.51	0.52	0.47	0.47	0.49	0.43	0.43	0.44
			40	0.30	0.30	0.30	0.30	0.30	0.29	0.28	0.28	0.28	0.27	0.27	0.27	0.26	0.26	0.26
10^5	10^4	10	5	2.65	2.65	2.67	1.77	1.77	1.74	0.98	0.98	0.97	0.65	0.65	0.65	0.49	0.49	0.49
			10	1.94	1.93	1.98	1.56	1.56	1.60	1.06	1.06	1.05	0.73	0.73	0.71	0.53	0.53	0.52
			20	1.20	1.20	1.21	1.06	1.06	1.08	0.86	0.86	0.88	0.70	0.70	0.71	0.56	0.56	0.56
			40	0.74	0.74	0.74	0.64	0.64	0.65	0.56	0.56	0.57	0.50	0.50	0.51	0.44	0.44	0.46
10^4	10^4	1	5	2.39	2.39	2.39	1.53	1.53	1.53	0.92	0.92	0.92	0.64	0.64	0.64	0.49	0.49	0.49
			10	2.39	2.39	2.39	1.53	1.53	1.53	0.92	0.92	0.92	0.64	0.64	0.64	0.49	0.49	0.49
			20	2.39	2.39	2.39	1.53	1.53	1.53	0.92	0.92	0.92	0.64	0.64	0.64	0.49	0.49	0.49
			40	2.39	2.39	2.39	1.53	1.53	1.53	0.92	0.92	0.92	0.64	0.64	0.64	0.49	0.49	0.49
10^3	10^4	0.1	5	-0.01	-0.04	-0.42	0.85	0.86	0.80	0.80	0.80	0.82	0.62	0.62	0.62	0.48	0.48	0.48
			10	-0.15	-0.06	+0.21	-0.58	-0.57	-0.69	0.35	0.35	0.24	0.44	0.44	0.45	0.41	0.41	0.43
			20	7.45	7.52	7.85	1.30	1.32	1.59	-0.42	-0.42	-0.39	-0.08	-0.08	-0.17	0.15	0.15	0.08
			40	14.9	14.9	15.1	6.68	6.69	6.89	1.60	1.60	1.78	0.09	0.09	0.20	-0.21	-0.21	-0.17

Note: Values listed under the heading "ELAST. MOD." were computed using equation 5 of TTI Research Report 123-6 (reference 3)

Values listed under the heading "BISTRO" were computed using the program of reference 2.

Values listed under the heading "NEW" were computed using equation 1 of this report.

4. CONCLUSIONS

1. An empirical equation has been derived for computing the surface deflections of a two-layer elastic structure.
2. This equation is much simpler and less costly to use than the previously available equations which involve integration of expressions containing Bessel functions.
3. The present equation gives results which are in substantial agreement with those of the previously available methods.

5. REFERENCES

(1) Michelow, J. "Analysis of Stresses and Displacements in an N-Layered Elastic System Under a Load Uniformly Distributed over a Circular Area." California Research Corporation, Richmond, California, September, 1973, and Computer Program "CHEVRON" based thereon, Chevron Oil Company, California.

(2) Computer program "BRISTO", Koninklijke/Shell-Laboratorium, Amsterdam.

(3) Scrivner, F. H.; Michalak, C. H.; and Moore, W. M. "Calculation of the Elastic Moduli of a Two Layer Pavement System from Measured Surface Deflections." Research Report Number 123-6, Texas Transportation Institute, Texas A&M University, College Station, Texas, March 1971, page 9, Equation (5), and computer program "ELASTIC MODULUS" modified as mentioned on page 13.

(4) Burmister, D. M. "The Theory of Stresses and Displacements in Layered Systems and Applications to the Design of Airport Runways." Proceedings, Highway Research Board, Vol. 23, 1943, page 130, Equation (n).