A FINITE-ELEMENT METHOD FOR TRANSVERSE VIBRATIONS OF BEAMS AND PLATES

by

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Development of Methods for Computer Simulation
of Beam-Columns and Grid-Beam and Slab Systems

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PREFACE

This report presents the results of an analytical study which was undertaken to develop an implicit numerical method for determining the transient and steady-state vibrations of elastic beams and plates. The study consists of (1) a theoretical analysis of the stability of difference equations used, (2) the formulation of the difference equations for the general solution of the beam and plate, and (3) a demonstration of the method by computer solutions of example problems. A supplemental report will describe the use of the associated computer programs for the beam and plate and will further illustrate the application of these programs to highway engineering problems.

Report 56-1 in the List of Reports provides an explanation of the basic procedures which are used in these programs. Although the programs are written in FORTRAN-63 for the CDC 1604 computer, minor changes would make these programs compatible with an IBM 7090 system. Copies of the programs and data cards for the example problems in this report may be obtained from the Center for Highway Research at The University of Texas.

Support for this project was provided by the Texas Highway Department, under Research Project 3-5-63-56 (HPR-1-4), in cooperation with the U. S. Department of Transportation, Bureau of Public Roads. Some related support was provided by the National Science Foundation. The computer time was contributed by the Computation Center of The University of Texas. The authors are grateful to these organizations and to the many individuals who helped in this study.

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LIST OF REPORTS

Report No. 56-1, "A Finite-Element Method of Solution for Linearly Elastic Beam-Columns" by Hudson Matlock and T. Allan Haliburton, presents a finite-element solution for beam-columns that is a basic tool in subsequent reports.

Report No. 56-2, "A Computer Program to Analyze Bending of Bent Caps" by Hudson Matlock and Wayne B. Ingram, describes the application of the beam-column solution to the particular problem of bent caps.

Report No. 56-3, "A Finite-Element Method of Solution for Structural Frames" by Hudson Matlock and Berry Ray Grubbs, describes a solution for frames with no sway.

Report No. 56-4, "A Computer Program to Analyze Beam-Columns under Movable Loads" by Hudson Matlock and Thomas P. Taylor, describes the application of the beam-column solution to problems with any configuration of movable non-dynamic loads.

Report No. 56-5, "A Finite-Element Method for Bending Analysis of Layered Structural Systems" by Wayne B. Ingram and Hudson Matlock, describes an alternating-direction iteration method for solving two-dimensional systems of layered grids-over-beams and plates-over-beams.

Report No. 56-6, "Discontinuous Orthotropic Plates and Pavement Slabs" by W. Ronald Hudson and Hudson Matlock, describes an alternating-direction iteration method for solving complex two-dimensional plate and slab problems with emphasis on pavement slabs.

Report No. 56-7, "A Finite-Element Analysis of Structural Frames" by T. Allan Haliburton and Hudson Matlock, describes a method of analysis for rectangular plane frames with three degrees of freedom at each joint.

Report No. 56-8, "A Finite-Element Method for Transverse Vibrations of Beams and Plates" by Harold Salani and Hudson Matlock, describes an implicit procedure for determining the transient and steady-state vibrations of beams and plates, including pavement slabs.

Report No. 56-9, "A Direct Computer Solution for Plates and Pavement Slabs" by C. Fred Stelzer, Jr., and W. Ronald Hudson, describes a direct method for solving complex two-dimensional plate and slab problems.

Report No. 56-10, "A Finite-Element Method of Analysis for Composite Beams" by Thomas P. Taylor and Hudson Matlock, describes a method of analysis for composite beams with any degree of horizontal shear interaction.

ABSTRACT

A finite-element method is developed to determine the transverse linear deflections of a vibrating beam or plate. The method can be used to obtain numerical solutions to varied beam and plate vibration problems which can not be readily solved by other known methods. The solutions for the beam and plate are separate formulations which have been programmed for a digital computer. Both solutions permit arbitrary variations in bending stiffness, mass density and dynamic loading. The static equations have been included in the development so that the initial deflections can be conveniently established. In the beam, the difference equations are solved by a recursive procedure. For the plate, the same procedure is combined with an alternating-direction technique to obtain an iterated solution. The numerical results demonstrate that the method is applicable to a wide range of vibration problems which are relevant to a beam or plate.

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NOMENCLATURE

Symbol	Typical Units	Definition
A	in.	Constant
C_1	in.	Constant
C_2	in.	Constant
D	lb-in^2/in	Flexural stiffness of plate
d	lb-sec/in^2	Distributed damping coefficient
E	lb/in^2	Modulus of elasticity
e	-	Base of natural logarithms
F	lb-in^2	Bending stiffness = EI
g^2	-	$\frac{EI\ h_t^2}{\rho\ h_x^4}$
h_p	in.	Length of plate increment
h_t	sec	Length of time increment
h_x	in.	Length of beam or plate increment
h_y	in.	Length of plate increment
I	in^4	Moment of inertia of the cross section
i	-	Index for plate axis
j	-	Index for plate or beam axis
k	-	Index for time axis
L	in.	Length of beam or plate
M	-	Number of beam or plate increments
M_b	in-lb	Bending moment
m	-	Index
N	-	Number of plate increments
n	-	Index

Symbol	Typical Units	Definition
P	lb	Axial load
Q	lb/sta or lb/mesh point	Concentrated transverse load on a beam or concentrated transverse load on a plate
q	lb/in or lb/in^2	Transverse load per unit length of beam or transverse load per unit area of plate
R	in-lb/sta per rad	Concentrated rotational restraint
r	in-lb/in per rad	Rotational restraint per unit length
S	lb/in per sta or lb/in per mesh point	Concentrated stiffness of elastic foundation for a beam or concentrated stiffness of elastic foundation for a plate
s	lb/in^2 or lb/in^3	Stiffness of elastic foundation per unit length of beam or stiffness of elastic foundation per unit area of plate
t	sec	Time
T_c	in-lb/sta	Concentrated applied couple
t_c	in-lb/in	Applied couple per unit length
u^2	-	$\frac{Dh_t^2}{\rho h_p^4}$
v	in/sec	Velocity
w	in.	Transverse deflection for a beam or plate
x	in.	Distance along axis of a beam or plate
y	in.	Distance along axis of a plate
α_m	radians	Angle
β_n	radians	Angle
λ	lb/in^3	Closure parameter
ν	-	Poisson's ratio
ρ	$lb\ sec^2/in^2$ or $lb\ sec^2/in^3$	Mass density per unit length of beam or mass density per unit area of plate
ϕ	-	Exponent

CHAPTER 1. INTRODUCTION

Advances in science and technology have brought about an increasing need for solutions to structural problems in which dynamic behavior is an important factor. Classical solutions are available for a limited class of problems in this category. The development of the high-speed digital computer has made it feasible to obtain approximate numerical solutions for a vast number of heretofore unsolved problems.

The primary purpose of this investigation is to develop a finite-element method for determining the transverse time-dependent linear deflections of a beam or plate. The method is based on an implicit formula which was introduced by Crank and Nicolson (Ref 5)* to solve the second order heat flow problem.

Essentially, the beam or plate is replaced by an arbitrary number of finite elements and the time dimension is divided into discrete intervals. This representation readily permits the flexural stiffness, elastic restraints and the loading to be discontinuous. The governing partial linear differential equation is approximated by a difference equation and a numerical solution is obtained at specified intervals of time. The difference equation for the unknown deflection may be formulated explicitly or implicitly. In an explicit formula, there is only one unknown deflection in each difference equation, whereas, in an implicit formula, there are several unknown deflections in each equation. Thus the resulting set of difference equations must be solved simultaneously to obtain the unknown deflections.

Finite difference solutions for initial value problems are subject to

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*See References on p 51.

instability. This can be illustrated by considering the following equation for an undamped transversely vibrating beam:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho \frac{\partial^2 w}{\partial t^2} = 0 \qquad (1.1)$$

In the foregoing, E is the modulus of elasticity, I is the moment of inertia, ρ is the mass density per unit length, w is the deflection, x is the distance along the beam and t is the time. For suitable boundary conditions and a given initial displacement, the beam will vibrate periodically. If the deflections are calculated from a solution of the partial differential equation, the contribution from the higher characteristic frequencies is usually negligible. However, in a finite difference solution, it is possible for the higher frequencies to cause the calculated deflections to become unbounded as time approaches infinity. In his book on difference methods, Richtmyer (Ref 11) discusses the equivalence of stability and convergence. For properly defined problems, stability insures convergence. Crandall (Ref 4) and other investigators have discussed the stability of finite difference approximations for Eq 1.1.

The stability criteria and pictorial representations of the explicit and implicit formulas for a beam and plate will be presented in the subsequent discussion. Both formulations have been programmed for a digital computer. However, the development of the equations and the numerical results will pertain to the implicit solution. As a convenience in establishing the initially deflected shape of a beam or plate, the equations of statics have been included in this development. All difference equations are based on the assumptions of linear elasticity and elementary beam and thin plate theories. The symbols adopted for use in this paper are defined where they first appear and are listed in the Nomenclature.

CHAPTER 2. STABILITY OF THE BEAM EQUATION

From a theoretical standpoint, the use of difference equations for the solution of a linear transient problem is complicated by stability requirements. In this discussion, a finite difference solution is stable if the solution is bounded as time approaches infinity. To facilitate a difference representation of the terms in the vibrating beam equation, it is convenient to establish a rectangular grid in an x,t plane. The coordinate axes for the grid are the beam and the time axes, and the lines in the grid intersect at mesh points. Any mesh point may be located by station numbers which are identified by the indices j and k with respect to the beam and time axes. The distances between the grid lines in the coordinate directions are fixed by the lengths of the beam increment h_x and the time increment h_t. This grid is illustrated in Fig 1.

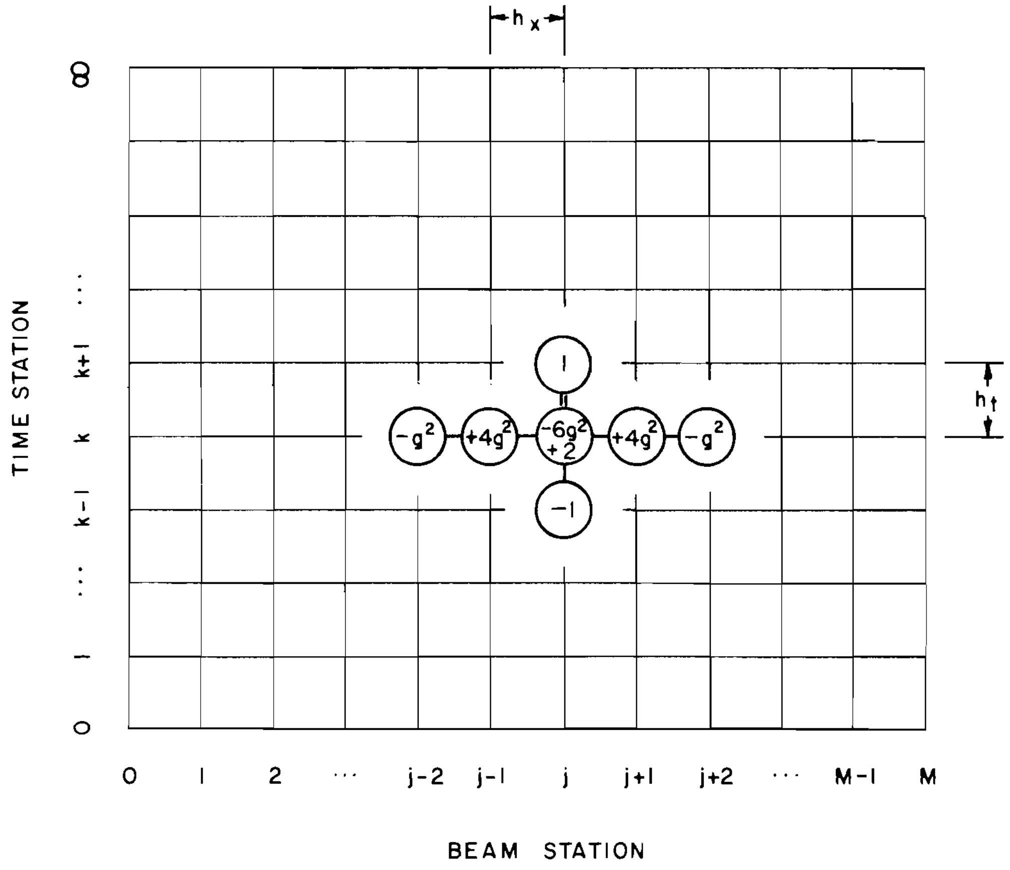
Explicit Formula

An examination of the explicit formula for a uniform beam will demonstrate the stability criterion which was first established by Collatz (Ref 1). The explicit difference approximation for Eq 1.1 is

$$g^2 \left[w_{j-2,k} - 4w_{j-1,k} + 6w_{j,k} - 4w_{j+1,k} + w_{j+2,k} \right] + w_{j,k-1} - 2w_{j,k} + w_{j,k+1} = 0 \qquad (2.1)$$

wherein

$$g^2 = \frac{EI}{\rho} \frac{h_t^2}{h_x^4}$$



$$w_{j,k+1} = -g^2 w_{j-2,k} + 4g^2 w_{j-1,k} + (-6g^2+2)w_{j,k} + 4g^2 w_{j+1,k} - g^2 w_{j+2,k} - w_{j,k-1}$$

where $g^2 = \frac{EI}{\rho} \frac{h_t^2}{h_x^4}$

Fig 1. Explicit operator for the transverse deflections of a uniform beam.

At $k = 0$, the initial deflections and velocities are specified. Therefore, the value of $w_{j,k+1}$ is the only unknown in the equation. In Fig 1, the operator associated with Eq 2.1 is superimposed on the rectangular grid. To solve for each unknown deflection at $k = 1$, the operator is applied successively at $j = 1, 2, \ldots, M-1$. The boundary conditions are introduced to establish the deflections at the ends of the beam. In a similar manner, the unknown deflections are calculated for $k = 2, 3, \ldots, \infty$.

For a beam with hinged ends and M segments or increments, a solution to Eq 2.1 is assumed to be

$$w_{j,k} = A \sin (j\beta_n)\, e^{k\phi} \tag{2.2}$$

in which A is a constant, $j = 0, 1, 2, \ldots, M$, and $k = 2, 3, 4, \ldots, \infty$.

Equation 2.2 is substituted into Eq 2.1 to establish

$$\begin{aligned} g^2 e^{k\phi} \Big[&\sin (j-2)\, \beta_n - 4 \sin (j-1)\, \beta_n + 6 \sin (j\beta_n) \\ &- 4 \sin (j+1)\, \beta_n + \sin (j+2)\, \beta_n \Big] \\ &+ \sin j\beta_n \left[e^{(k-1)\phi} - 2e^{k\phi} + e^{(k+1)\phi} \right] = 0 \end{aligned} \tag{2.3}$$

The following trigonometric identities are used to simplify Eq 2.3:

$$\sin (\theta \pm \gamma) = \sin \theta \cos \gamma \pm \cos \theta \sin \gamma$$

and

$$\cos 2\theta = 2 \cos^2 \theta - 1$$

Hence, Eq 2.3 becomes

$$\frac{1}{e^{k\phi}} \left[e^{(k-1)\phi} - 2e^{k\phi} + e^{(k+1)\phi} \right] = -4g^2 (1 - \cos \beta_n)^2 \tag{2.4}$$

This may be reduced to

$$e^{2\phi} + e^{\phi}\left[4g^2 \ (1 - \cos \beta_n)^2 - 2\right] + 1 = 0 \qquad (2.5)$$

On the boundaries, independent of k, the deflections and moments are zero. Thus,

$$w_{0,k} = w_{M,k} = \frac{\partial^2 w_{0,k}}{\partial x^2} = \frac{\partial^2 w_{M,k}}{\partial x^2} = 0 \qquad (2.6)$$

From Eq 2.2,

$$\sin (M\beta_n) = 0$$

Hence,

$$M\beta_n = \pi, 2\pi, \ldots, n\pi$$

or

$$\beta_n = \frac{n\pi}{M} \qquad (2.7)$$

where

$$n = 1, 2, 3, \ldots, M-1$$

Therefore, Eq 2.2 becomes

$$w_{j,k} = \sum_{n=1}^{M-1} A_n \sin (j \frac{n\pi}{M}) \ e^{k\phi} \qquad (2.8)$$

The roots of the quadratic Eq 2.5 are substituted into Eq 2.8 so that

$$w_{j,k} = \sum_{n=1}^{M-1} A_n \sin (j \frac{n\pi}{M}) \left[C_1 \ (e^{\phi_1})^k + C_2 \ (e^{\phi_2})^k\right] \qquad (2.9)$$

where C_1 and C_2 are constants. In Eq 2.9, for $w_{j,k}$ to be bounded for all values of k, the roots of the quadratic, e^{ϕ_1} and e^{ϕ_2}, must satisfy the condition that

$$|e^{\phi_1}|, \ |e^{\phi_2}| \leq 1 \qquad (2.10)$$

This condition may be satisfied by defining g^2 in Eq 2.5. Thus, the limiting value of g^2 occurs when the discriminant

$$(16\ g^2 - 2)^2 - 4 \leq 0 \qquad (2.11)$$

for $\beta_n = \pi$

Expanding Eq 2.11 discloses that

$$4\ g^2 - 1 \leq 0$$

and

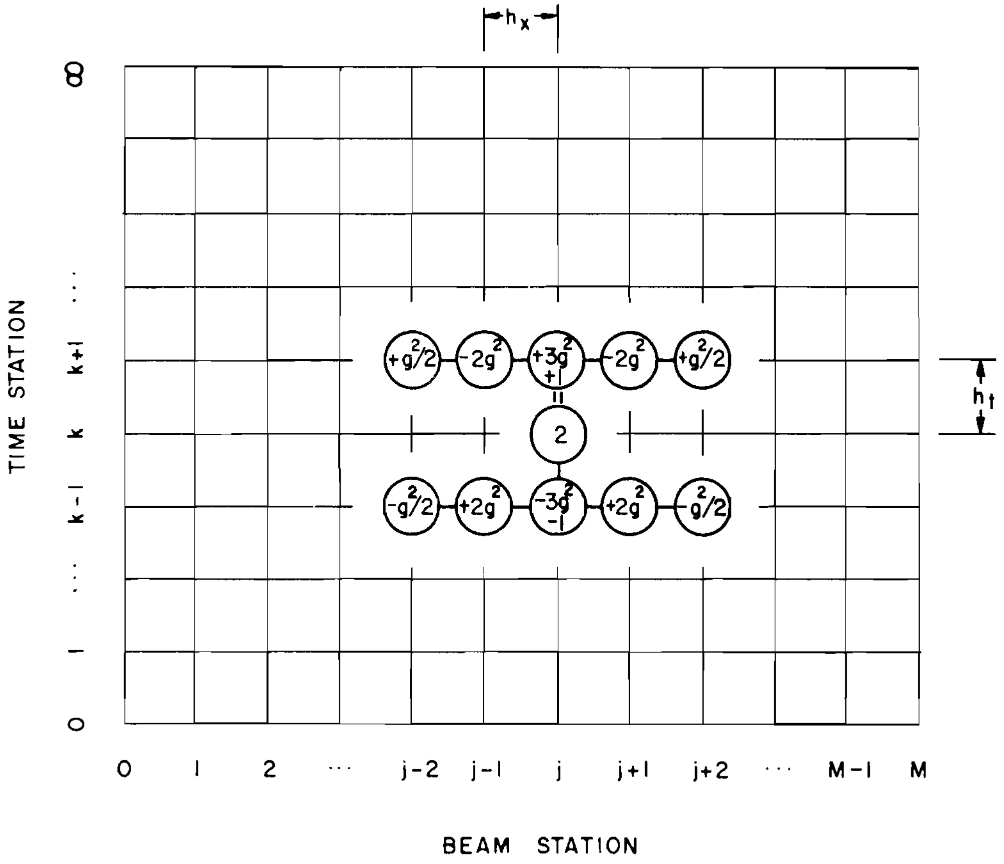
$$g^2 \leq \frac{1}{4} \qquad (2.12)$$

The preceding analysis is based on a uniform beam with hinged supports. For a stable solution, the maximum value of g^2, or $\frac{EI}{\rho}\frac{h_t^2}{h_x^4}$, is prescribed by Eq 2.12. Because of this limitation, the explicit formula will not be used in the subsequent development of the dynamic beam equation.

Implicit Formula

In Fig 2, an implicit operator of the Crank-Nicolson (Ref 5) form is shown for Eq 1.1. All deflections at Station k+1 are unknown. The fourth derivative term that was previously at the k^{th} station has been divided equally between the stations at k-1 and k+1. For any Station j, this implies that the deflection at Station k is an average of the sum of the deflections at Stations k-1 and k+1. At k=0, the initial deflections and velocities are specified. To solve for the unknown deflections at k=1, the operator is applied systematically at j = 1, 2, ..., M-1. This procedure establishes a set of simultaneous equations wherein each equation includes five unknown deflections. These equations may be solved by any convenient method. In a similar fashion, the unknown deflections are determined for k = 2, 3, 4, ..., ∞.

The admissibility of the implicit formula can be established by a procedure suggested by Young (Ref 16). Let L(w) be the differential equation and G(w) be a Taylor series expansion of the terms in the implicit formula



$$(g^2/2)w_{j-2,k+1} - 2g^2 w_{j-1,k+1} + (3g^2+1)w_{j,k+1}$$

$$-2g^2 w_{j+1,k+1} + (g^2/2)w_{j+2,k+1} = 2w_{j,k} - (g^2/2)w_{j-2,k-1}$$

$$+ 2g^2 w_{j-1,k-1} + (-3g^2 - 1)w_{j,k-1} + 2g^2 w_{j+1,k-1} - (g^2/2)w_{j+2,k-1}$$

$$\text{where} \quad g^2 = \frac{EI}{\rho} \, \frac{h_t^2}{h_x^4}$$

Fig 2. Implicit operator for the transverse deflections of a uniform beam.

about the point j,k . When $G(w)$ is subtracted from $L(w)$, the remainder, or truncation error, is of the order $(h_x)^2$ and $(h_t)^2$. Furthermore, h_t is a given function of h_x .

Thus the

$$\lim_{h_x \to 0} \left[L(w) - G(w) \right] = 0 \tag{2.13}$$

and the admissibility of the implicit formula is established.

The implicit difference approximation to Eq 1.1 is

$$\begin{aligned} \frac{g^2}{2} \Big[& w_{j-2,k+1} - 4w_{j-1,k+1} + 6w_{j,k+1} - 4w_{j+1,k+1} + w_{j+2,k+1} \\ & + w_{j-2,k-1} - 4w_{j-1,k-1} + 6w_{j,k-1} - 4w_{j+1,k-1} + w_{j+2,k-1} \Big] \\ & + w_{j,k-1} - 2w_{j,k} + w_{j,k+1} = 0 \end{aligned} \tag{2.14}$$

To establish the stability criterion, Eq 2.2 is substituted into Eq 2.14 to yield

$$\begin{aligned} \frac{g^2}{2} \Big\{ & e^{(k+1)\phi} \Big[\sin (j-2) \beta_n - 4 \sin (j-1) \beta_n + 6 \sin (j\beta_n) \\ & - 4 \sin (j+1) \beta_n + \sin (j+2) \beta_n \Big] + e^{(k-1)\phi} \Big[\sin (j-2) \beta_n \\ & - 4 \sin (j-1) \beta_n + 6 \sin (j\beta_n) - 4 \sin (j+1) \beta_n \\ & + \sin (j+2) \beta_n \Big] \Big\} + \sin (j\beta_n) \left[e^{(k-1)\phi} - 2e^{k\phi} + e^{(k+1)\phi} \right] \\ & = 0 \end{aligned} \tag{2.15}$$

The above equation reduces to

$$\begin{aligned} & \frac{1}{e^{(k+1)\phi} + e^{(k-1)\phi}} \left[e^{(k-1)\phi} - 2e^{k\phi} + e^{(k+1)\phi} \right] \\ & = 2g^2 (1 - \cos \beta_n)^2 \end{aligned} \tag{2.16}$$

and the quadratic equation becomes

$$e^{2\phi} - e^{\phi}\left[\frac{2}{1 + 2g^2\ (1 - \cos\beta_n)^2}\right] + 1 = 0 \tag{2.17}$$

The value of β_n is given in Eq 2.7. The roots of the quadratic satisfy Eq 2.10 for all $g^2 > 0$. Therefore, the implicit formula is stable for all positive values of EI , ρ , h_t , and h_x .

The preceding discussion of stability has been based on free vibration of a uniform beam and well defined boundary conditions. Analytical proofs for more complicated cases are not feasible. For example, if the same beam has uniform rotational restraints r , foundation springs s , and an axial tension P , the quadratic form becomes

$$e^{2\phi} - e^{\phi}\left[\frac{2\rho}{\rho + h_t^{\,2}\left(2\,\frac{EI}{h_x^{\,4}}\,(1 - \cos\beta_n)^2 + \frac{s}{2} + \frac{r+P}{h_x^{\,2}}\,(1 - \cos\beta_n)\right)}\right] + 1 = 0 \tag{2.18}$$

An evaluation of stability from Eq 2.18 is not practicable. However, stable numerical solutions have been obtained for complex problems.

Crandall (Ref 4) has shown that the optimum implicit formula for a uniform beam has a truncation error of the order $(h_t)^3$. In a recent paper, Tucker (Ref 15) used an implicit formula which has a truncation error of the order (h_t) . In this study, the general development of the beam and plate equations will be based on the Crank-Nicolson (Ref 5) implicit form which has a truncation error of the order $(h_t)^2$.

CHAPTER 3. DEVELOPMENT OF THE BEAM EQUATIONS

The finite-element beam solution consists of the static equation, the dynamic equation related to the initial velocities and the dynamic equation. The static equation is due to Matlock (Ref 9) and is discussed briefly herein. Central differences (Ref 3) are used in all derivations except where otherwise noted. The coordinate system which was described in the preceding chapter is applicable in the following development.

Static Equation

The beam segment in Fig 3 illustrates the static loads and elastic restraints which may be imposed on the beam to establish its initially deflected shape. A finite-element model of this segment has been developed by Matlock (Ref 9). Equation 3.1 is obtained by summing moments and forces on the beam segment in Fig 3.

$$\frac{d^2M_b}{dx^2} = q - sw + \frac{d}{dx}\left[t_c + (r + P)\frac{dw}{dx}\right] \tag{3.1}$$

In the foregoing, M_b is the bending moment, q is the transverse load per unit length, s is the elastic stiffness of the foundation per unit length, t_c is an applied couple per unit length, r is a rotational restraint per unit length and P is an axial load. Combining Eq 3.1 with the differential equation for a beam

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = \frac{d^2M_b}{dx^2} \tag{3.2}$$

establishes Eq 3.3

$$\frac{d^2}{dx^2}\left[EI\frac{d^2w}{dx^2}\right] = q - sw + \frac{d}{dx}\left[t_c + (r + P)\frac{dw}{dx}\right] \tag{3.3}$$

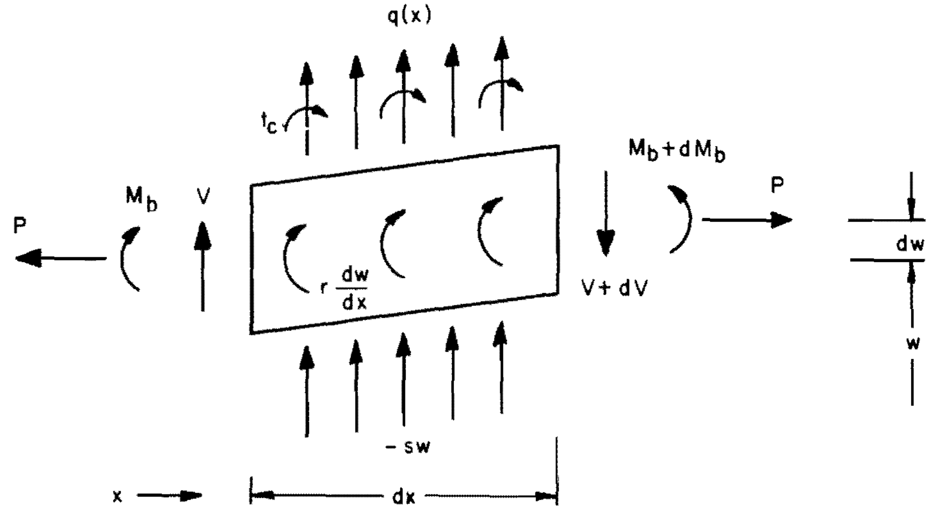


Fig 3. Beam segment with static loads and elastic restraints.

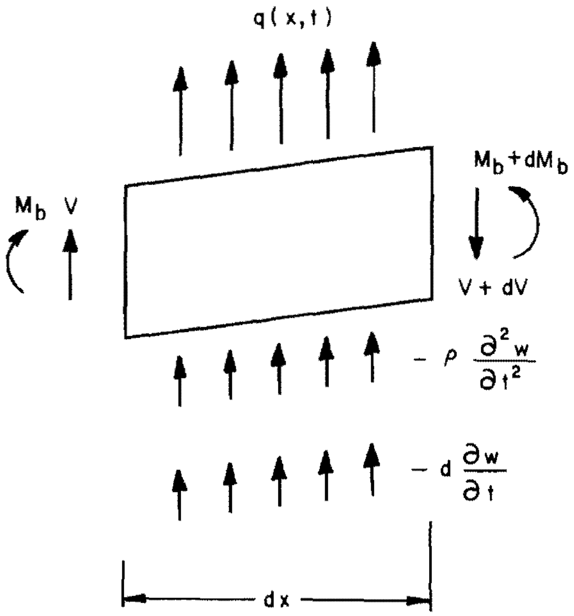


Fig 4. Beam segment with transient loads.

In a difference equation, the distributed quantities q , r , t_c and s are lumped as corresponding concentrated quantities Q , R , T_c and S at each incremental point along the beam. Equation 3.3 involves the derivative of a product of two variables. In transforming this differential equation to a difference equation, the left side of the equation is expanded from the outside to the inside in the following manner:

$$\frac{d^2}{dx}\left[F\,\frac{d^2w}{dx^2}\right] = \frac{1}{h_x^{\,2}}\left\{\left(F\,\frac{d^2w}{dx^2}\right)_{j-1} - 2\left(F\,\frac{d^2w}{dx^2}\right)_j + \left(F\,\frac{d^2w}{dx^2}\right)_{j+1}\right\}$$

$$= \frac{1}{h_x^{\,4}}\left\{F_{j-1}\,(w_{j-2} - 2w_{j-1} + w_j) - 2F_j\,(w_{j-1} - 2w_j + w_{j+1})\right.$$

$$\left. + F_{j+1}\,(w_j - 2w_{j+1} + w_{j+2})\right\} \tag{3.4}$$

In Eq 3.4, F represents the bending stiffness and h_x is the length of a beam increment. Similarly, Eq 3.3 is converted to the difference equation

$$\left[F_{j-1} - 0.25\,h_x\,(R_{j-1} + h_x\,P_{j-1})\right] w_{j-2} - \left[2\,(F_{j-1} + F_j)\right] w_{j-1}$$

$$+ \left[F_{j-1} + 4F_j + F_{j+1} + h_x^{\,3}\,S_j + 0.25\,h_x\,(R_{j-1} + h_x\,P_{j-1})\right.$$

$$\left. + 0.25\,h_x\,(R_{j+1} + h_x\,P_{j+1})\right] w_j - \left[2\,(F_j + F_{j+1})\right] w_{j+1}$$

$$+ \left[F_{j+1} - 0.25\,h_x\,(R_{j+1} + h_x\,P_{j+1})\right] w_{j+2}$$

$$= h_x^{\,3}\,Q_j - 0.5\,h_x^{\,2}\,(T_{c_{j-1}} - T_{c_{j+1}}) \tag{3.5}$$

The application of this equation at each incremental point results in a set of simultaneous equations which is solved by a recursive procedure. This procedure and the boundary conditions will be discussed subsequently.

Dynamic Equation

The partial differential equation for the transverse vibrations of a beam can be derived from d'Alembert's principle. The concept of reversed effective forces, or inertial forces, in d'Alembert's principle is quite easily visualized. Imagine that the inertial and viscous drag forces and an externally applied force q(x,t) are superimposed on the beam segment which is shown in Fig 4. Thus the differential equation for a vibrating beam is

$$\frac{\partial^2}{\partial x^2}\left[F\,\frac{\partial^2 w}{\partial x^2}\right] = -\,\rho\,\frac{\partial^2 w}{\partial t^2} - d\,\frac{\partial w}{\partial t} + q\,(x,t) \qquad (3.6)$$

where d is the coefficient of viscous damping and the other symbols have the same meaning as before. The quantities r , s and P , which affect the stiffness of a beam at any instant of time, are added to Eq 3.6 and this yields

$$\frac{\partial^2}{\partial x^2}\left[F\,\frac{\partial^2 w}{\partial x^2}\right] + sw - \frac{\partial}{\partial x}\left[(r + P)\,\frac{\partial w}{\partial x}\right] + \rho\,\frac{\partial^2 w}{\partial t^2} + d\,\frac{\partial w}{\partial t} = q\,(x,t)\,. \qquad (3.7)$$

The implicit representation of Eq 3.7* is

$$\begin{aligned}
& Y_a w_{j-2,k+1} + Y_b w_{j-1,k+1} + \left[Y_c + \frac{h_x^{\,4}}{h_t^{\,2}}\,\rho_j + \frac{h_x^{\,4}}{h_t}\,d_j\right] w_{j,k+1} \\
& \quad + Y_d w_{j+1,k+1} + Y_e w_{j+2,k+1} = h_x^{\,3}\,Q_{j,k} + \left[2\,\frac{h_x^{\,4}}{h_t^{\,2}}\,\rho_j\right] w_{j,k} \\
& \quad - \left[\frac{h_x^{\,4}}{h_t^{\,2}}\,\rho_j\right] w_{j,k-1} + \left[\frac{h_x^{\,4}}{h_t}\,d_j\right] w_{j,k} - Y_a w_{j-2,k-1} \\
& \quad - Y_b w_{j-1,k-1} - Y_c w_{j,k-1} - Y_d w_{j+1,k-1} - Y_e w_{j+2,k-1} \qquad (3.8)
\end{aligned}$$

in which

$$Y_a = \frac{1}{2}\left[F_{j-1} - 0.25\,h_x\,(R_{j-1} + h_x P_{j-1})\right]$$

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* A derivation of the implicit formula for Eq 3.7 is given in Appendix 1.

$$Y_b = -\left[F_{j-1} + F_j\right]$$

$$Y_c = \frac{1}{2}\left[F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j + 0.25\, h_x (R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1})\right]$$

$$Y_d = -\left[F_j + F_{j+1}\right]$$

$$Y_e = \frac{1}{2}\left[F_{j+1} - 0.25\, h_x (R_{j+1} + h_x P_{j+1})\right] \tag{3.9}$$

$$k = 1, 2, 3, \ldots, \infty$$

In the foregoing, h_t is the length of time increment. The remaining symbols have been previously defined. In Eq 3.8, the unknown deflections at $k+1$ appear on the left side of the equation, and the known deflections at k and $k-1$ appear on the right side of the equation.

At the outset, the deflections and velocities at $k=0$ are given. With these initial conditions, the unknown deflections at $k=1$ are then calculated to begin the transient solution. This is accomplished by rewriting Eq 3.8 so that the generic indices $k+1$, k and $k-1$ become 1 , $\frac{1}{2}$ and 0 respectively. Furthermore, the initial deflections and velocities are introduced in the computational procedure in accordance with the following equations:

$$\frac{\partial w}{\partial t}\bigg|_{j,0} = \frac{-w_{j,0} + w_{j,\frac{1}{2}}}{h_t/2} \tag{3 10}$$

and

$$\rho \frac{\partial^2 w}{\partial t^2}\bigg|_{j,\frac{1}{2}} = \rho_j \frac{w_{j,0} - 2w_{j,\frac{1}{2}} + w_{j,1}}{(h_t/2)^2} \tag{3.11}$$

The unknown deflections at $k=\frac{1}{2}$ are eliminated by combining Eqs 3.10 and 3.11.

Consequently, the deflections at k=1 are calculated. Commencing at k=2 and thereafter, the solution progresses with time in accordance with Eq 3.8. This is demonstrated in Fig 5.

The effects of rotatory inertia and shear deformation have been omitted in the derivation of the dynamic equation. A discussion of these effects is given in Ref 12.

Method of Solution for the Difference Equations

There are several systematic procedures available to solve simultaneous equations. For an efficient machine procedure, it is convenient to use a method of elimination described by Matlock (Ref 9).

The difference equation, whether static or dynamic, may be written in the form

$$\bar{a}_j w_{j-2,k} + \bar{b}_j w_{j-1,k} + \bar{c}_j w_{j,k} + \bar{d}_j w_{j+1,k} + \bar{e}_j w_{j+2,k} = \bar{f}_{j,k} \qquad (3.12)$$

$$k = 0,\ 1,\ 2,\ 3,\ \ldots,\ \infty \ .$$

The terms $\bar{a}_j$, $\bar{b}_j$, $\bar{c}_j$, $\bar{d}_j$, $\bar{e}_j$ and $\bar{f}_{j,k}$ may be recognized by comparing the foregoing equation with either the static Eq 3.5 or the dynamic Eq 3.8. For instance, in Eq 3.8,

$$\bar{a}_j = Y_a$$

$$\bar{b}_j = Y_b$$

$$\bar{c}_j = Y_c + \frac{h_x^4}{h_t^2} \rho_j + \frac{h_x^4}{h_t} d_j$$

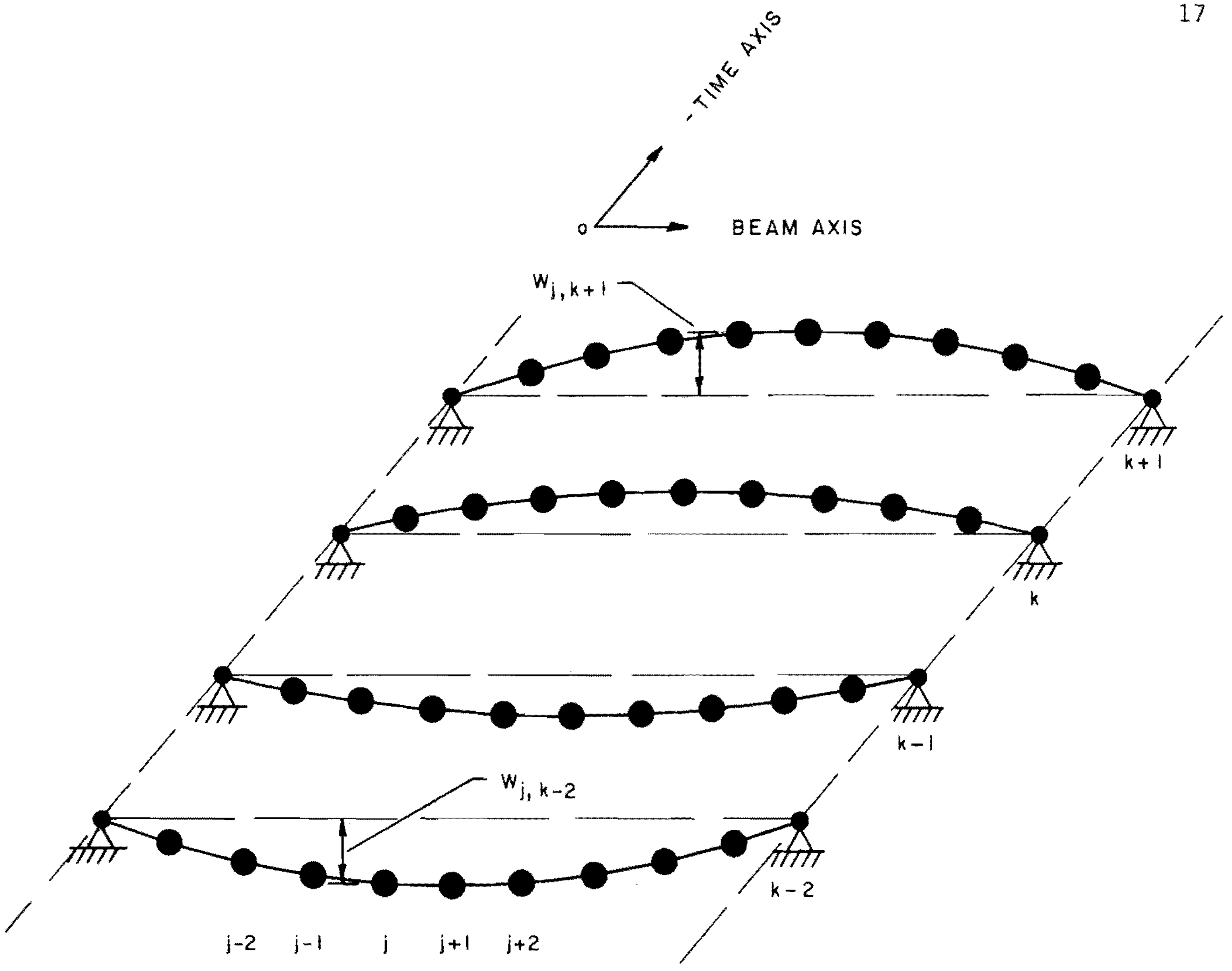
$$\bar{d}_j = Y_d$$

$$\bar{e}_j = Y_e$$

and

$$\bar{f}_{j,k} = h_x^3 Q_{j,k-1} + \left[2 \frac{h_x^4}{h_t^2} \rho_j\right] w_{j,k-1} - \left[\frac{h_x^4}{h_t^2} \rho_j\right] w_{j,k-2}$$

(equation continued)



BEAM: 0, 1, 2, . . . , M

PRESCRIBED BOUNDARIES AT 0, M (Illustrated above as a hinge)

TIME: 0, 1, 2, . . . , ∞

DEFLS ARE KNOWN AT k-2, k-1, k

DEFLS ARE UNKNOWN AT k+1

Fig 5. Propagation of solution for unknown beam deflections.

$$+\left[\frac{h_x^4}{h_t} d_j\right] w_{j,k-1} - Y_a w_{j-2,k-2} - Y_b w_{j-1,k-2} - Y_c w_{j,k-2}$$

$$- Y_d w_{j+1,k-2} - Y_e w_{j+2,k-2}$$

The solution to Eq 3.12 is assumed to be

$$w_{j,k} = A_j + B_j w_{j+1,k} + C_j w_{j+2,k} \tag{3.13}$$

in which

$$A_j = D_j (E_j A_{j-1} + \bar{a}_j A_{j-2} - \bar{f}_{j,k}) \tag{3.14}$$

$$B_j = D_j \ (E_j C_{j-1} + \bar{d}_j) \tag{3.15}$$

$$C_j = D_j \ (\bar{e}_j) \tag{3.16}$$

$$D_j = - 1 \ / \ (E_j B_{j-1} + \bar{a}_j C_{j-2} + \bar{c}_j) \tag{3.17}$$

$$E_j = \bar{a}_j B_{j-2} + \bar{b}_j \tag{3.18}$$

Proceeding from either end of the beam in what is called a forward direction, Equations 3.14 through 3.18 are applied at every station, including one fictitious station beyond each end of the beam. On the reverse pass, the unknown deflections are calculated from Eq 3.13.

Boundaries and Specified Conditions

Although the equations have not been established in a matrix array, it is convenient to consider the coefficients $\bar{a}_j$, ..., $\bar{e}_j$ as terms in a quintuple-diagonal coefficient matrix and the unknown deflections and known loads as column matrices. The first and last equations represent the moment at the free edge of a beam, and the second and next-to-the-last equations represent the shear one-half increment inside the free edge. For a uniform beam with an unloaded free boundary, the first and second equations are

$$w_{-1,k} - 2w_{0,k} + w_{1,k} = 0 \tag{3.19}$$

and

$$-w_{-1,k} + 3w_{0,k} - 3w_{1,k} + w_{2,k} = 0 \tag{3.20}$$

Thus an approximation of the natural boundary conditions for zero moment and shear are automatically created by zero stiffness values beyond the ends of the beam.

Specified deflections are established by equating A_j to the desired deflection and setting B_j and C_j equal to zero in Eq 3.13. To specify a slope at the jth station, the coefficients A_j , B_j and C_j at Stations j-1 and j+1 are recalculated on the basis of the reaction couple that must be developed about the jth station (Ref 9).

CHAPTER 4. NUMERICAL RESULTS - BEAM

The static and dynamic equations that were developed in the preceding chapter have been programmed in FORTRAN for the Control Data Corporation 1604 computer. A listing of this program, DCB1, a guide for data input, and a summary flow diagram are in Appendix 4.

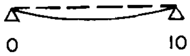
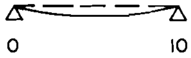
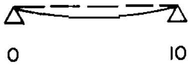
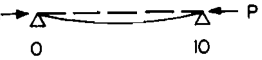
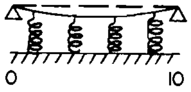
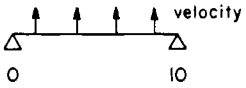
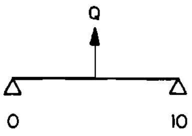
Verification of the Method

Table 1 illustrates the problems which have been selected to verify the method. The theoretical angular frequency of vibration for each problem is given in Timoshenko (Ref 12). The period of vibration corresponding to the lowest angular frequency was divided into an arbitrary number of time increments. For all problems, the number of beam increments is 10, the increment length is 12 in., the stiffness is 1.08×10^9 lb-in^2, and the mass density is 9.04×10^{-3} lb-sec^2/in^2. Each beam has hinged support.

In Problems 1, 2 and 3, the time increments are 2.653×10^{-4} sec, 5.306×10^{-4} sec and 2.565×10^{-3} sec. The initially deflected shape of each beam is established as one-half cycle of a sine wave. This is the fundamental mode of vibration of the beam. At $k=0$, the beam is released and the deflections are noted during the ensuing vibrations. The deflected shape of the beam at the conclusion of the first period is similar to its initial shape. This is illustrated in Table 1 by the recorded values of the initial deflections and the subsequent deflections at the end of the first period. These three problems demonstrate that a small time increment is desirable.

Problems 4 and 5 are similar to Problem 1 with the following alterations. In Problem 4, the axial load is -3.70×10^5 lb and the time increment is

TABLE 1. A SUMMARY OF THE NUMERICAL RESULTS

		VALUES AT CENTER OF SPAN		
BEAM AT INITIAL CONDITIONS	NUMBER OF TIME INCREMENTS PER FUNDAMENTAL PERIOD OF VIBRATION BASED ON A THEORETICAL SOLUTION	INITIAL DEFLECTION (Inches)	SUBSEQUENT DEFLECTION (Inches)	
			TIME STATION	w
(1) 	100	-2.004	99	-1.987
			100	-1.999
			101	-2.004
			102	-2.001
(2) 	50	-2.004	49	-1.955
			50	-1.995
			51	-2.005
			52	-1.983
(3) 	10	-2.004	9	-0.7231
			10	-1.637
			11	-2.018
			12	-1.743
(4) 	100	-3.952	99	-3.937
			100	-3.951
			101	-3.950
			102	-3.933
(5) 	100	-6.672	99	-6.643
			100	-6.669
			101	-6.670
			102	-6.644
(6) 	100	0.0	25	1.619×10^{-1}
			99	-1.393×10^{-2}
			100	-6.512×10^{-3} *
			101	7.156×10^{-4}
(7) 	100	0.0	50	6.690
			99	7.954×10^{-2}
			100	3.407×10^{-2} *
			101	7.937×10^{-3}

* DEFLECTION IS ZERO IN THEORETICAL SOLUTION GIVEN BY TIMOSHENKO (REF. 12)

3.752×10^{-4} sec. In Problem 5, the uniform foundation spring is 12.0×10^{3} lb/in/sta and the time increment is 1.540×10^{-4} sec.

The beam in Problem 6 has zero initial deflections and a uniform initial velocity of 30 in/sec everywhere except at the supports. The time increment is 2.653×10^{-4} sec. Theoretically, the deflections at the end of the first period are zero.

In Problem 7, a concentrated load of 1.0×10^{5} lb is applied suddenly at the middle of the span and is removed at the end of the first period. The time increment is 2.653×10^{-4} sec. At the conclusion of the first period and thereafter, the deflections are zero.

Excluding Problem 3, the maximum error in the numerical results based on the theoretical solutions is about 4%. Furthermore, these results confirm that the finite-element method described herein can be used to solve vibrating beam problems.

Example Problems

Two example problems have been selected to illustrate the versatility of the finite-element method. The partially embedded beam, which is described in Fig 6, is subjected to an axial load and a transient pulse. In addition to the hinged supports, there is a rotational restraint at the upper boundary. The soil modulus has been converted at each station to an equivalent elastic spring. A damping factor of 10.0 lb-sec/in^2 has been assumed arbitrarily. Figure 6b shows the deflected shape of the beam at the conclusion of the pulse, or $k = 18$, and at a subsequent time. Figure 6c illustrates the response of a typical station on the beam.

The second example, which is sketched in Fig 7, is a three-span beam with a constant load moving along the beam at a uniform velocity of one beam increment per time increment. Figure 7b illustrates the response of the beam

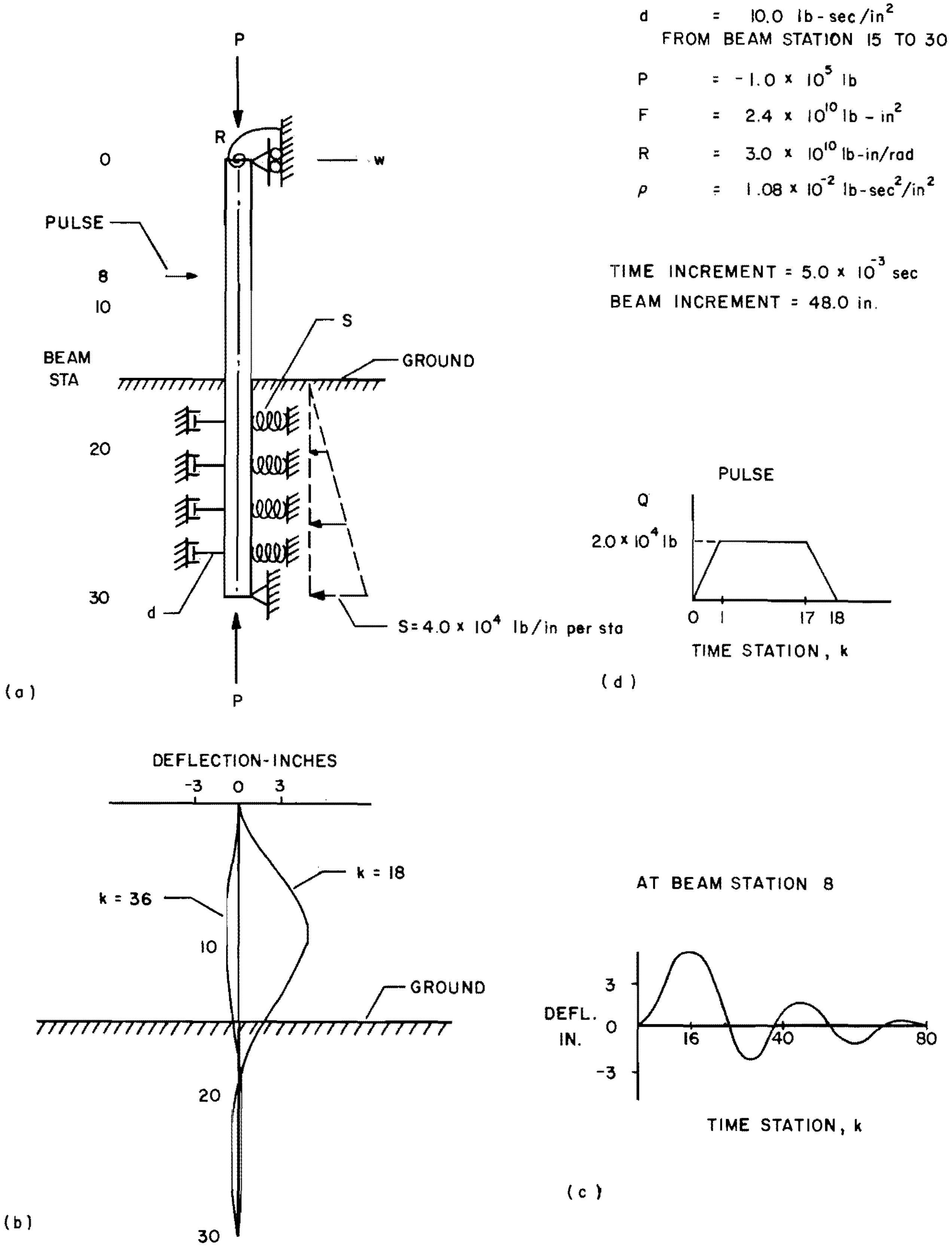
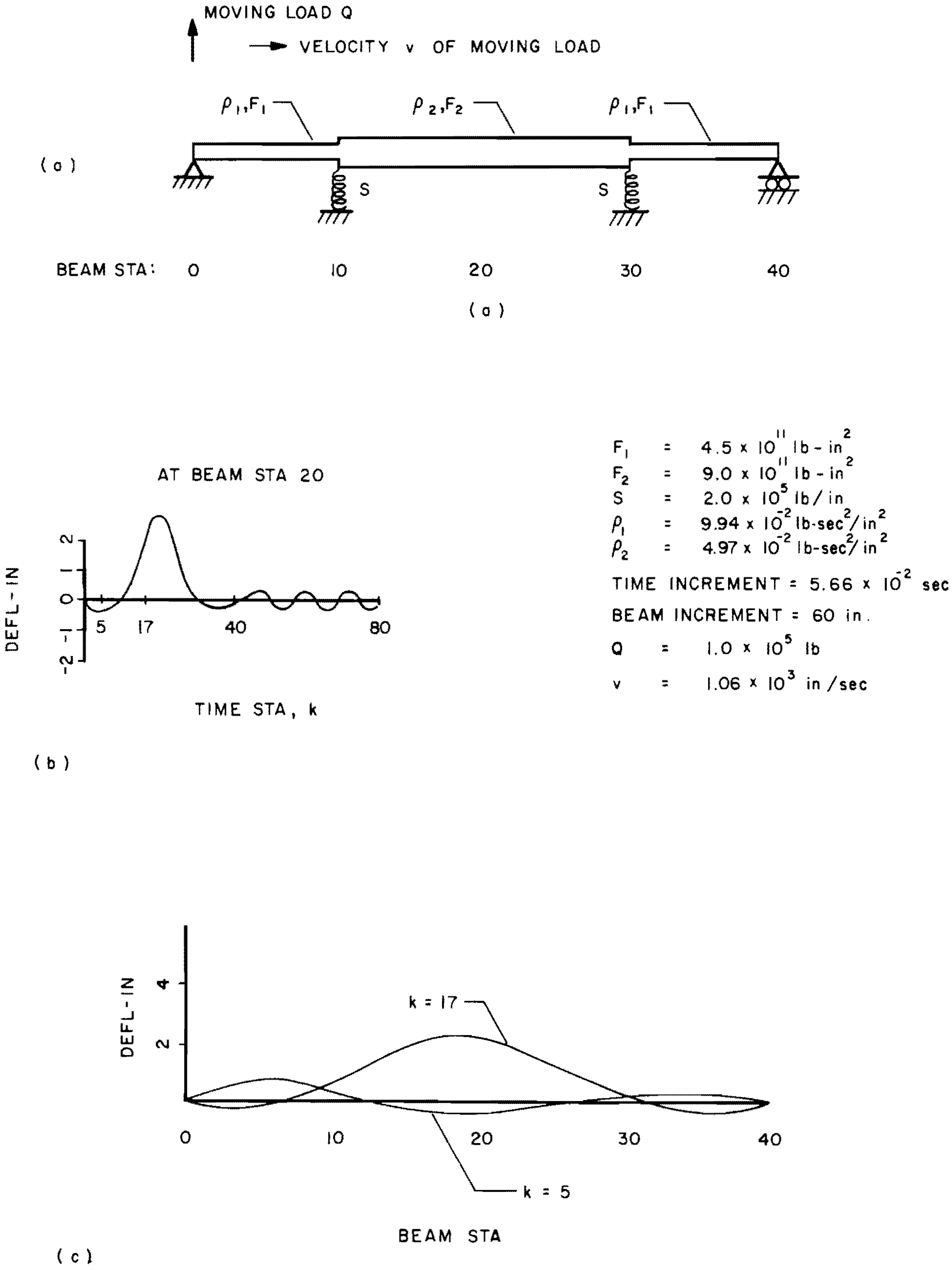


Fig 6. Partially embedded beam subjected to a load pulse.



F_1 = 4.5×10^{11} lb - in^2

F_2 = 9.0×10^{11} lb - in^2

S = 2.0×10^5 lb/in

ρ_1 = 9.94×10^{-2} lb-sec^2/in^2

ρ_2 = 4.97×10^{-2} lb-sec^2/in^2

TIME INCREMENT = 5.66×10^{-2} sec

BEAM INCREMENT = 60 in.

Q = 1.0×10^5 lb

v = 1.06×10^3 in/sec

Fig 7. Moving load on a three-span beam.

at Station 20. Figure 7c is a plot of the beam deflections at the two indicated times.

For the Control Data Corporation 1604 computer, the execution time required for each solution is approximately 45 seconds.

CHAPTER 5. STABILITY OF THE PLATE EQUATION

A difference solution for the vibrating plate equation must meet the requirements of stability. The restrictions that have been established for the beam equation are not applicable to a plate, but the same procedures are involved. Therefore, the following development will parallel the previous work.

The equation for the transverse deflections of a vibrating plate is

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + \rho\frac{\partial^2 w}{\partial t^2} = 0 \tag{5.1}$$

where w is the deflection, D is the uniform flexural stiffness, x and y are the rectangular coordinate axes, t is time and ρ is the mass per unit area of the plate. The independent variables in Eq 5.1 are x, y and t. Therefore, a difference representation of the terms in the above equation requires a three-dimensional coordinate system in which x, y and t are the three coordinate axes. A rectangular grid, whose lines are parallel to the x and y axes, is established at each interval of time. The intersections of these grid lines are known as mesh points. Any mesh point may be located by station numbers which are defined by the indices i, j and k with respect to the coordinate axes. In the x or y-direction, the distance between adjacent grid lines is fixed by the length of the plate increment h_p.

Explicit Formula

Explicitly, the finite difference formula for Eq 5.1 is

$$u^2 \left\{ w_{i-2,j,k} + w_{i+2,j,k} + w_{i,j-2,k} + w_{i,j+2,k} + 20w_{i,j,k} \right.$$

$$- 8\left[w_{i-1,j,k} + w_{i+1,j,k} + w_{i,j-1,k} + w_{i,j+1,k}\right]$$

(equation continued)

$$+ 2 \left[w_{i-1,j+1,k} + w_{i+1,j+1,k} + w_{i-1,j-1,k} + w_{i+1,j-1,k} \right] \Big\}$$

$$+ w_{i,j,k-1} - 2w_{i,j,k} + w_{i,j,k+1} = 0 \qquad (5.2)$$

wherein

$$u^2 = \frac{D}{\rho} \frac{h_t^2}{h_p^4}$$

Two initial conditions and eight boundary conditions are prescribed. At $k = 1$ and thereafter, the only unknown is $w_{i,j,k+1}$. The operator corresponding to Eq 5.2 is shown in Fig 8. To solve explicitly for each unknown deflection at any time station, the operator is used successively at every mesh point in the x,y plane. The boundary conditions are introduced to establish the deflections along the edges of the plate. In this manner, the solution marches forward with time.

For a rectangular plate with M by N increments and hinged supports along the edges, a solution is assumed to be of the form

$$w_{i,j,k} = Ae^{k\phi} \sin (i\alpha_m) \sin (j\beta_n) \qquad (5.3)$$

where

$$i = 0, 1, 2, \ldots, M$$

$$j = 0, 1, 2, \ldots, N$$

and

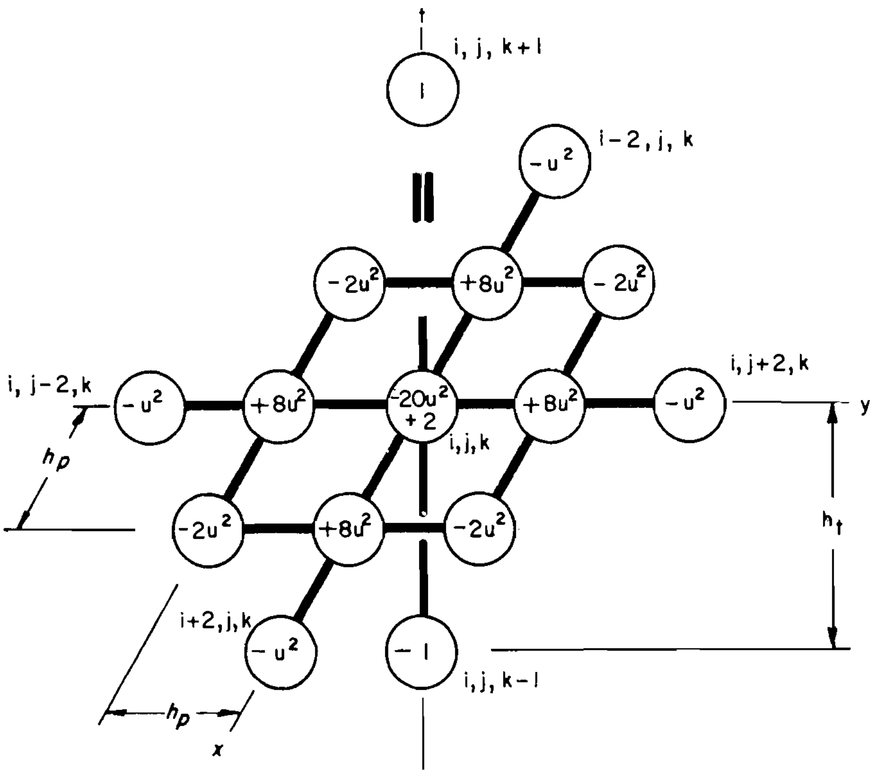
$$k = 2, 3, \ldots, \infty \; .$$

A substitution of Eq 5.3 into Eq 5.2 establishes that

$$u^2 e^{k\phi} \Big\{ \sin (j\beta_n) \Big[\sin (i-2) \alpha_m - 8 \sin (i-1) \alpha_m + 20 \sin (i\alpha_m)$$

$$- 8 \sin (i+1) \alpha_m + \sin (i+2) \alpha_m \Big]$$

(equation continued)



$$
\begin{aligned}
w_{i,j,k+1} = & -u^2\left[w_{i-2,j,k} + w_{i+2,j,k} + w_{i,j-2,k} + w_{i,j+2,k}\right] \\
& -2u^2\left[w_{i-1,j-1,k} + w_{i+1,j-1,k} + w_{i-1,j+1,k} + w_{i+1,j+1,k}\right] \\
& +8u^2\left[w_{i-1,j,k} + w_{i,j-1,k} + w_{i+1,j,k} + w_{i,j+1,k}\right] \\
& +(-20u^2 + 2)\, w_{i,j,k} - w_{i,j,k-1}
\end{aligned}
$$

$$
\text{where} \quad u^2 = \frac{D}{\rho}\;\frac{h_t^2}{h_p^4}
$$

Fig 8. Explicit operator for the transverse deflections of a uniform plate.

$$\begin{aligned}&+ \sin (i\alpha_m) \Big[\sin (j-2)\ \beta_n - 8 \sin (j-1)\ \beta_n - 8 \sin (j+1)\ \beta_n \\ &+ \sin (j+2)\ \beta_n\Big] + 2 \Big[\sin (i-1)\ \alpha_m \sin (j-1)\ \beta_n \\ &+ \sin (i-1)\ \alpha_m \sin (j+1)\ \beta_n + \sin (i+1)\ \alpha_m \sin (j-1)\ \beta_n \\ &+ \sin (i+1)\ \alpha_m \sin (j+1)\ \beta_n\Big]\Big\} + \sin (i\alpha_m) \sin (j\beta_n) \Big[e^{(k-1)\phi} \\ &- 2e^{k\phi} + e^{(k+1)\phi}\Big] = 0\end{aligned} \tag{5.4}$$

A simplification of Eq 5.4 yields

$$\begin{aligned}e^{-\phi} - 2 + e^{\phi} = -4u^2 \Big\{ &\Big[\cos \alpha_m - 2\Big]^2 + \Big[\cos \beta_n - 2\Big]^2 - 4 \\ &+ 2 \cos \alpha_m \cos \beta_n\Big\}\end{aligned} \tag{5.5}$$

Equation 5.5 reduces to

$$\begin{aligned}e^{2\phi} + e^{\phi} \Big\{4u^2 &\Big[(\cos \alpha_m - 2)^2 + (\cos \beta_n - 2)^2 - 4 \\ &+ 2 \cos \alpha_m \cos \beta_n\Big] - 2\Big\} + 1 = 0\end{aligned} \tag{5.6}$$

On the boundaries, independent of k, the deflections must satisfy the following equations:

$$w_{i,0,k} = w_{i,N,k} = 0 \tag{5.7}$$

$$w_{0,j,k} = w_{M,j,k} = 0 \tag{5.8}$$

$$- w_{i,-1,k} = w_{i,1,k} \tag{5.9}$$

$$- w_{i,N-1,k} = w_{i,N+1,k} \tag{5.10}$$

$$- w_{-1,j,k} = w_{1,j,k} \tag{5.11}$$

and

$$- w_{M-1,j,k} = w_{M+1,j,k} \tag{5.12}$$

The boundary conditions are satisfied for

$$\alpha_m = \frac{m}{M}\pi \qquad m = 1, 2, \ldots, M-1 \tag{5.13}$$

and

$$\beta_n = \frac{n}{N}\pi \qquad n = 1, 2, \ldots, N-1 \tag{5.14}$$

Thus, Eq 5.3 becomes

$$w_{i,j,k} = \sum_{m=1}^{M-1} \sum_{n=1}^{N-1} A_m A_n \sin (i \frac{m\pi}{M}) \sin (j \frac{n\pi}{N}) \left[C_1 (e^{\phi_1})^k + C_2 (e^{\phi_2})^k\right] \tag{5.15}$$

in which C_1 and C_2 are constants.

For stability,

$$|e^{\phi_1}|, |e^{\phi_2}| \leq 1 \tag{5.16}$$

An examination of Eq 5.6 shows that Eq 5.16 is satisfied if the discriminant

$$\left\{4u^2 \left[(\cos \alpha_m - 2)^2 + (\cos \beta_n - 2)^2 - 4 + 2 \cos \alpha_m \cos \beta_n\right] - 2\right\}^2 - 4 \leq 0 \tag{5.17}$$

For $\alpha_m = \beta_n = \pi$, Eq 5.17 reveals that

$$u^2 \leq \frac{1}{16} \tag{5.18}$$

For a stable explicit solution, the maximum value of $u^2 = \frac{Dh_t^2}{\rho h_p^4}$ is predicted by Eq 5.18. For this reason, the explicit formula will not be used in the development of the dynamic plate equation.

To verify this stability criterion and to gain some insight of the behavior of an unstable solution, a numerical experiment was performed with an explicit plate program. The experiment consisted of five problems in which a square plate with hinged supports about the edges was divided into a 4 x 4 grid. For each problem, D , ρ and h_p were constants and the time increment h_t was calculated on the basis of a prescribed value for the ratio $\frac{Dh_t^2}{\rho h p^4}$. The values for this ratio were 0.04, 0.05, 0.06, 0.08 and 0.1. On the basis of Eq 5.18, instability could be predicted for a ratio of 0.0625. At $k=0$, the initial deflections were specified. An examination of the computed deflections revealed a divergent oscillatory solution for the largest ratio. The deflections became increasingly larger at each successive time interval. At ratios of 0.05, 0.06 and 0.08, irregularities were noted in the computed deflections.

Implicit Formula

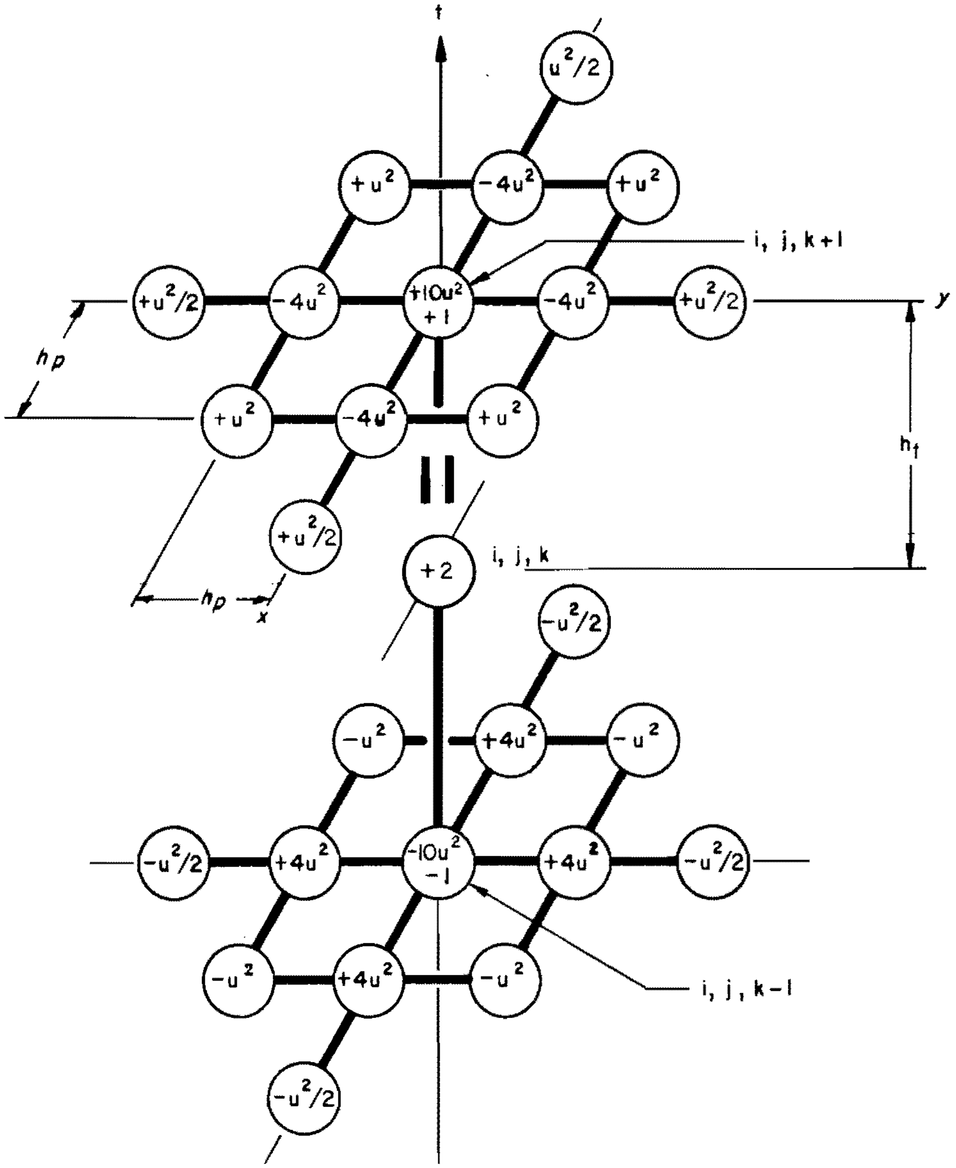
Figure 9 illustrates the implicit formula and operator for Eq 5.1. The fourth derivative terms that were previously at the k^{th} station have been divided equally between the stations at $k-1$ and $k+1$. This assumes that the deflections at the k^{th} station are an average of the sum of the corresponding deflections at Stations $k-1$ and $k+1$. All deflections at $k+1$ are unknown, whereas those at k and $k-1$ are known from previous solutions. Thus, for an implicit solution, a set of simultaneous equations must be solved.

The stability criterion for the implicit plate formula may be established by the same procedure that was employed for the explicit formula. Accordingly, Eq 5.3 is substituted into the equation that is shown in Fig 9. A separation of variables yields

$$e^{2\phi} - e^{\phi}\left\{\frac{2}{1+2u^2\left[(\cos\alpha_m-2)^2+(\cos\beta_n-2)^2+2\cos\alpha_m\cos\beta_n-4\right]}\right\} + 1 = 0 \qquad (5.19)$$

The roots of the preceding quadratic equation satisfy Eq 5.16 for all

$$u^2 > 0 \qquad (5.20)$$



$$u^2/2\left[w_{i-2,j,k+1} + w_{i,j-2,k+1} + w_{i+2,j,k+1} + w_{i,j+2,k+1}\right] + u^2\left[w_{i-1,j-1,k+1} + w_{i+1,j-1,k+1}\right.$$

$$\left. + w_{i+1,j+1,k+1} + w_{i-1,j+1,k+1}\right] - 4u^2\left[w_{i-1,j,k+1} + w_{i,j-1,k+1} + w_{i+1,j,k+1} + w_{i,j+1,k+1}\right]$$

$$+ (10u^2 + 1)\, w_{i,j,k+1} = 2w_{i,j,k}$$

$$-u^2/2\left[w_{i-2,j,k-1} + w_{i,j-2,k-1} + w_{i+2,j,k-1} + w_{i,j+2,k-1}\right] - u^2\left[w_{i-1,j-1,k-1} + w_{i+1,j-1,k-1}\right.$$

$$\left. + w_{i+1,j+1,k-1} + w_{i-1,j+1,k-1}\right] + 4u^2\left[w_{i-1,j,k-1} + w_{i,j-1,k-1} + w_{i+1,j,k-1} + w_{i,j+1,k-1}\right]$$

$$+(-10u^2 - 1)\, w_{i,j,k-1}$$

where $u^2 = \frac{D}{\rho}\,\frac{h_t^2}{h_p^4}$

Fig 9. Implicit operator for the transverse deflections of a uniform plate.

Hence, the implicit formula is stable for any choice of positive values for D , ρ , h_p and h_t . In a subsequent chapter, the implicit formula will be employed to solve for the deflections of a nonuniform plate. Analytical proofs for other boundary conditions are not readily attainable. Nonetheless, stability is indicated by the fact that numerical solutions have been obtained for problems with other well defined boundaries.

CHAPTER 6. DEVELOPMENT OF THE PLATE EQUATIONS

The finite-element plate solution includes the static equation, the dynamic equation related to the initial velocities and the dynamic equation. Shear deformations, linear damping and the effects of rotatory inertia have been omitted.

Static Equation

Consideration of static equilibrium and the moment-curvature relationship (Ref 13) yields

$$\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right]$$

$$+ 2\frac{\partial^2}{\partial x\partial y}\left[D\,(1-\nu)\,\frac{\partial^2 w}{\partial x\partial y}\right] = q - sw \qquad (6.1)$$

where

$$D = \frac{Eh^3}{12\,(1-\nu^2)}$$

In the foregoing, h is the plate thickness, ν is Poisson's ratio, s is the foundation modulus and q is the transverse static load. The coordinate system which was described in the preceding chapter is applicable in the following development.

In the finite-element solution, it is assumed that the increment length h_x in the x-direction does not necessarily equal the increment length h_y in the y-direction. Furthermore, the stiffness D and the lumped quantities S and Q may vary from one mesh point to another. The variation in D accounts for a changing plate thickness, but the plate properties are isotropic. The partial derivatives in Eq 6.1 are expanded in the same manner that was used for the beam equation. This establishes the difference equation

$$\begin{aligned} & X_1 w_{i-2,j} + \left[X_2 + Y_7 + 2Z_2\right] w_{i-1,j} + \left[X_3 + Y_3 + 2Z_5\right] w_{i,j} \\ & + \left[X_4 + Y_{10} + 2Z_8\right] w_{i+1,j} + X_5 w_{i+2,j} + Y_1 w_{i,j-2} \\ & + \left[Y_2 + X_{10} + 2Z_4\right] w_{i,j-1} + \left[Y_4 + X_7 + 2Z_6\right] w_{i,j+1} \\ & + Y_5 w_{i,j+2} + \left[X_9 + Y_6 + 2Z_1\right] w_{i-1,j-1} \\ & + \left[X_6 + Y_8 + 2Z_3\right] w_{i-1,j+1} + \left[X_{11} + Y_9 + 2Z_7\right] w_{i+1,j-1} \\ & + \left[X_8 + Y_{11} + 2Z_9\right] w_{i+1,j+1} = \left[Q_{i,j} - S_{i,j} w_{i,j}\right] \frac{1}{h_x h_y} \end{aligned} \qquad (6.2)$$

In the above equation, the coefficients $X_1, \ldots, X_{11}$, $Y_1, \ldots, Y_{11}$ and $Z_1, \ldots, Z_9$ are defined in Appendix 2. A finite-element model of the plate has been developed by Hudson (Ref 6).

Dynamic Equation

The partial differential equation of motion for forced lateral vibration of a plate is

$$\begin{aligned} & \frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)\right] + \frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right)\right] \\ & + 2 \frac{\partial^2}{\partial x \partial y}\left[D\,(1-\nu)\,\frac{\partial^2 w}{\partial x \partial y}\right] + sw + \rho \frac{\partial^2 w}{\partial t^2} = q\,(x,y,t) \end{aligned} \qquad (6.3)$$

where $q\,(x,y,t)$ is the imposed lateral force. The implied difference equation for Eq 6.3* is

$$\begin{aligned} & \frac{1}{2}\,(X_1) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i-2,j} + \frac{1}{2}\,(X_2 + Y_7 + 2Z_2) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i-1,j} \\ & + \left\{\frac{1}{2}\,\left(X_3 + Y_3 + 2Z_5 + \frac{S_{i,j}}{h_x h_y}\right) + \frac{\rho_{i,j}}{h_t^2}\right\} \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i,j} \end{aligned}$$

(equation continued)

---

* A derivation of the implicit formula for Eq 6.3 is given in Appendix 3.

$$+ \frac{1}{2} (X_4 + Y_{10} + 2Z_8) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i+1,j} + \frac{1}{2} (X_5) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i+2,j}$$

$$+ \frac{1}{2} (Y_1) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i,j-2} + \frac{1}{2} (Y_2 + X_{10} + 2Z_4) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i,j-1}$$

$$+ \frac{1}{2} (Y_4 + X_7 + 2Z_6) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i,j+1} + \frac{1}{2} (Y_5) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i,j+2}$$

$$+ \frac{1}{2} (X_9 + Y_6 + 2Z_1) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i-1,j-1}$$

$$+ \frac{1}{2} (X_6 + Y_8 + 2Z_3) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i-1,j+1}$$

$$+ \frac{1}{2} (X_{11} + Y_9 + 2Z_7) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i+1,j-1}$$

$$+ \frac{1}{2} (X_8 + Y_{11} + 2Z_9) \begin{bmatrix} w_{k+1} \\ w_{k-1} \end{bmatrix}_{i+1,j+1} - \frac{2\rho_{i,j}}{h_t^2} w_{i,j,k}$$

$$= \frac{Q_{i,j,k}}{h_x h_y} \tag{6.4}$$

The compact notation in brackets in Eq 6.4 implies a multiplication of the coefficients by the deflections at $k+1$ and $k-1$. In Eq 6.4, the solution for the unknown deflections at $k+1$ is dependent on the known deflections at k and $k-1$.

To begin the transient solution at $k = 0$, Eq 6.4 is modified so that the generic indices $k+1$, k and $k-1$ become 1, $\frac{1}{2}$ and 0, respectively. In addition, the initial velocities and deflections are introduced in the computational procedure in accordance with the following equations:

$$\left.\frac{\partial w}{\partial t}\right|_{i,j,0} = \frac{-w_{i,j,0} + w_{i,j,\frac{1}{2}}}{h_t/2} \tag{6.5}$$

and

$$\rho \frac{\partial^2 w}{\partial t^2}\Big|_{i,j,\frac{1}{2}} = \rho_{i,j} \frac{w_{i,j,0} - 2w_{i,j,\frac{1}{2}} + w_{i,j,1}}{(h_t/2)^2} \tag{6.6}$$

The unknown deflections at $k = \frac{1}{2}$ are eliminated by combining Eqs 6.5 and 6.6. Thus the deflections at $k = 1$ are calculated. Beginning at $k = 2$, the plate deflections are determined from Eq 6.4 for each time interval as the solution marches forward.

Method of Solution for the Difference Equations

To obtain a solution for the unknown deflections, either static or dynamic, the appropriate equation is applied at each mesh point within the interior of the plate, along all boundaries, and at one mesh point outside of these boundaries. For a square plate which has been divided into M intervals in both directions, this procedure will introduce $(M+3)^2 - 4$ unknowns in $(M+3)^2 - 4$ equations. In matrix form this becomes

$$\left[B\right]\left[w\right] = \left[C\right] \tag{6.7}$$

$\left[B\right]$ is a square matrix with a predominant number of zero terms, but the nonzero terms are not banded about the main diagonal. These equations may be solved conveniently by an iterative procedure which is known as an alternating-direction-implicit, or ADI, method. In a comparison with other iterative methods, Young (Ref 17) has shown for second order difference equations that the ADI method has the most rapid rate of convergence. Conte and Dames (Ref 2) were among the first to utilize the ADI method to solve for the static deflections of a plate. Tucker (Ref 14) used this method successfully to solve the static grid-beam problem.

The ADI method is comparable to line relaxation in the x and y-directions. Basically, for an ADI solution, Eq 6.4 is solved for the deflections

$\left[\overline{wx}\right]$ in an x system and the deflections $\left[\overline{wy}\right]$ in a y system at alternate iterations. Equation 6.8 shows the iterative procedure employed to solve Eq 6.4 for the x system at iteration $n+\frac{1}{2}$.

$$\begin{aligned}
&\frac{1}{2}(X_1)\left[\overline{wx}_{i-2,j,k+1}\right]^{n+\frac{1}{2}} + \frac{1}{2}(X_2 + Z_2)\left[\overline{wx}_{i-1,j,k+1}\right]^{n+\frac{1}{2}} \\
&+ \left\{\frac{1}{2}(X_3 + Z_5 + S_{i,j}) + \frac{\rho_{i,j}}{h_t^2} + \lambda_m\right\}\left[\overline{wx}_{i,j,k+1}\right]^{n+\frac{1}{2}} \\
&+ \frac{1}{2}(X_4 + Z_8)\left[\overline{wx}_{i+1,j,k+1}\right]^{n+\frac{1}{2}} \\
&+ \frac{1}{2}(X_5)\left[\overline{wx}_{i+2,j,k+1}\right]^{n+\frac{1}{2}} \\
&= \frac{Q_{i,j,k}}{h_x h_y} + \lambda_m\left[\overline{wy}_{i,j,k+1}\right]^{n} - \sum\left[X,Y,Z,\frac{\rho}{h_t^2}\right]\left[w\right] \qquad (6.8)
\end{aligned}$$

In the foregoing, $\left[\overline{wx}\right]^{n+\frac{1}{2}}$ are the unknown deflections for the x system at iteration $n+\frac{1}{2}$, and $\left[\overline{wx}\right]^{n}$ and $\left[\overline{wy}\right]^{n}$ are the known deflections from the n^{th} iteration for the x and y systems, respectively. The summation term on the right hand side of Eq 6.8 implies a multiplication of the remaining X , Y , Z and $\frac{\rho}{h_t^2}$ terms in Eq 6.4 with their respective deflections at iteration $n+\frac{1}{2}$ or n , or at a previous time interval. The closure parameter λ_m will be discussed subsequently. Equation 6.8 involves M+3 unknowns in M+3 equations along a single line of mesh points in the x-direction. An equation similar to Eq 6.8 can be written for the y system. One iteration consists of solving 2M+2 lines in the x and y-directions. The total number of equations solved in each iteration is (2M+2)(M+3) .

Each equation has five non-zero terms banded about the main diagonal in the coefficient matrix. This quintuple-diagonal system of equations is solved by the same method which was described previously for the beam equations. The solution is reached when $|\overline{wx} - \overline{wy}|$ is less than a specified closure tolerance.

Boundaries and Specified Conditions

For an unloaded free edge at $x = a$, the following difference approximations for moment and shear are automatically satisfied in the plate solution by zero stiffness beyond the edge of the plate.

$$w_{a-1,j} - 2\ (1+\nu)\ w_{a,j} + w_{a+1,j} + \nu\ (w_{a,j-1} + w_{a,j+1}) = 0 \qquad (6.9)$$

and

$$- (2-\nu)\ w_{a-1,j-1} + (2-\nu)\ w_{a,j-1} - w_{a-2,j} + \left(3 + 2\ (2-\nu)\right) w_{a-1,j}$$
$$- \left(3 + 2\ (2-\nu)\right) w_{a,j} + w_{a+1,j} - (2-\nu)\ w_{a-1,j+1}$$
$$+ (2-\nu)\ w_{a,j+1} = 0 \qquad (6.10)$$

Equations 6.9 and 6.10 are equivalent to the Kirchhoff boundary conditions (Ref 13) which are

$$\left(\frac{\partial^2 w}{\partial x^2} + \nu\ \frac{\partial^2 w}{\partial y^2}\right) = 0 \qquad (6.11)$$

and

$$\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\ \frac{\partial^3 w}{\partial x \partial y^2}\right) = 0 \qquad (6.12)$$

In the numerical solution, a zero deflection is conveniently established by inserting very stiff elastic foundation springs at the desired mesh points. No provision has been made to prescribe the slope at any boundary. However, this could be accomplished by the same procedure that was used for a beam.

Closure Parameters

The scalars λ_1 , $\lambda_2, \ldots, \lambda_m$ in Eq 6.8 are closure parameters that accelerate the convergence of the iterative procedure. In fact, these parameters are the key to an efficient solution. For a symmetric problem (Ref 6), these parameters have been related to the eigenvalues of the difference equations along any line in either the x or y system.

The parameters for the static equation as it is formulated in this development may be determined from

$$\frac{D}{h_x^4}\left[w_{i-2,j} - 6w_{i-1,j} + 10w_{i,j} - 6w_{i+1,j} + w_{i+2,j}\right]$$

$$= \lambda_m w_{i,j} \tag{6.13}$$

In the above equation, the plate stiffness D is a constant and the increment lengths h_x and h_y are equal. For hinged boundaries and M intervals, a solution is assumed to be

$$w_{i,j} = \sin (i\alpha_m) \tag{6.14}$$

where

$$\alpha_m = \frac{m\pi}{M}$$

This yields

$$\lambda_m = \frac{D}{h_x^4} 4 \left(1 - \cos \frac{m\pi}{M}\right) \left(2 - \cos \frac{m\pi}{M}\right) \tag{6.15}$$

$$m = 1, 2, \ldots, M-1$$

There are M-1 parameters which are used in cyclic order in the static and dynamic equations. If the problem has mixed boundary conditions and non-uniform stiffness, the closure parameters may be estimated from Eq 6.15.

The closure parameters for each system are inversely proportional to h_x^4 and h_y^4 . For an efficient solution, the iterative procedure must account for this variation in closure parameters. Ingram (Ref 7) has demonstrated that optimum closure is obtained if the calculated parameters for the x system are used in the solution of the y system and vice versa. This scheme has been included in the plate solution.

CHAPTER 7. NUMERICAL RESULTS - PLATE

The development of the plate equations in Chapter 6 has been assembled in a FORTRAN program for the Control Data Corporation 1604 computer. A listing of this program, DPI1, a guide for data input, and a summary flow diagram are in Appendix 5. Four problems are used to interpret the computed results of the plate program. Problems 1, 2, and 3 are intended to illustrate the effect of variations in number of plate increments and length of time increment on the accuracy of the solution and on the amount of computation time required to propagate the solution through a given number of time increments. If the plate in initially deflected in the shape of its fundamental mode of vibration and then released, theoretically this deflected shape will be repeated at the end of each integer multiple of the fundamental period of vibration. The program was modified to permit specification of initial deflections, but since this is of little practical use it was not made a permanent part of the final version.

Problem 1: 4 × 4 Grid

A plate with hinged supports along the edges is divided into a 4 × 4 grid. The increment lengths h_x and h_y are 12 in., the uniform stiffness is 2.5×10^6 lb-in., Poisson's ratio is 0.25, the mass density is 7.5×10^{-4} lb-sec^2/in^3, the increment of time h_t is 4.233×10^{-4} sec and the closure parameters are 1.83×10^2, 9.62×10^2 and 2.24×10^3 lb/in^3. The theoretical period of vibration for the lowest angular frequency (Ref 12) is $30\ h_t$. At $k=0$, the initial deflections of the plate are

$$w_{i,j,0} = \sin\left(\frac{i\pi h_x}{L}\right) \sin\left(\frac{j\pi h_y}{L}\right) \qquad (7.1)$$

in which L is 48.0 in. This shape corresponds to a normal mode of vibration. The plate is then released. At the conclusion of the first period, or $30\ h_t$, the shape of the plate is similar to its initial shape. In Table 2, this similarity is shown for selected mesh points. The maximum variation between the initial deflections and the deflections at the conclusion of the first period is about 9 percent. For a closure tolerance of 1.0×10^{-6} in., four iterations are required to solve for the unknown deflections for each time increment. The computer execution time is 1.2 minutes for 30 increments of time.

Problem 2: 8 × 8 Grid

For this problem, the plate is divided into an 8 × 8 grid. Thus, the increment lengths h_x and h_y are 6 in. and the summation in Eq 7.1 is changed accordingly. The remaining dimensions are the same as those in the preceding problem. The closure parameters are 2.93×10^3, 4.0×10^3, 5.0×10^3, 7.0×10^3, 1.0×10^4, 1.54×10^4, and 3.58×10^5 lb/in^3. Seven iterations are required for each time increment. The similarity between the initial deflections and the deflections at the end of the first period is illustrated in Table 2. The variation in the deflections is about 2 percent. The computer execution time is 6.3 minutes for 30 time increments.

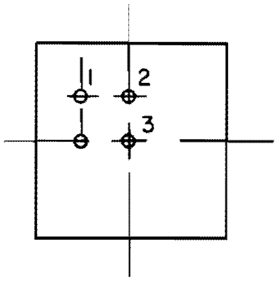
Problem 3: 4 × 4 Grid and Reduced Time Increment

This problem is identical to Problem 1 with the exception that the time increment h_t is 2.117×10^{-4} sec, which is one-half of the value used in Problem 1. The deflections are shown in Table 2. Three iterations are required for each increment of time and the computer execution time is 1.7 minutes for 60 increments of time. The variation in the deflections for this problem is about 7 percent.

Problem 4: Moving Load on a Rectangular Plate

Three different solutions have been obtained for the uniform plate which

TABLE 2. A COMPARISON OF THE NUMERICAL RESULTS



τ = FUNDAMENTAL PERIOD OF VIBRATION OF THEORETICAL PLATE

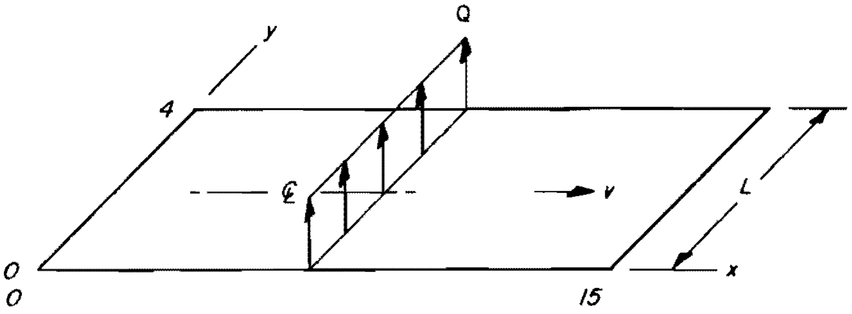
MESH POINT	PROBLEM 1 4 x 4 GRID $\tau = 30\ h_t$		PROBLEM 2 8 x 8 GRID $\tau = 30\ h_t$	
	DEFL	TIME	DEFL	TIME
1	0.5000 in.	(0)	0.5000 in.	(0)
	0.4580 in.	(τ)	0.4910 in.	(τ)
2	0.7071 in.	(0)	0.7071 in.	(0)
	0.6478 in.	(τ)	0.6944 in.	(τ)
3	1.0000 in.	(0)	1.0000 in.	(0)
	0.9161 in.	(τ)	0.9821 in.	(τ)

	PROBLEM 3 4 x 4 GRID $\tau = 60\ h_t$	
	DEFL	TIME
1	0.5000 in.	(0)
	0.4697 in.	(τ)
2	0.7071 in.	(0)
	*	(τ)
3	1.0000 in.	(0)
	0.9393 in.	(τ)

* NOT INCLUDED IN OUTPUT

is described in Fig 10. First, the static load is applied at $i = 7$ and the resulting static deflections are noted. For the two dynamic solutions, the initial velocities and deflections are zero and the moving load is applied successively at $i = 0, 1, 2, \ldots, 15$. In one solution, the velocity of the moving load is 9.45×10^2 in/sec. For the other solution, the velocity of the moving load is 3.78×10^3 in/sec. The deflections are noted when the load is at $i = 7$. Figure 10b illustrates the deflected shape of the plate for the three solutions. Figure 11 shows the contours of the deflections for the same solutions. The closure tolerance is 1.0×10^{-6} in. and the closure parameters are 0.7, 1.0, 4.0, 6.0, and 11.0 lb/in^3. The static solution requires 50 iterations. The dynamic solutions require 16 iterations for each time increment when h_t is 5.08×10^{-2} sec and 5 iterations when h_t is 1.25×10^{-2} sec.

This problem was selected to demonstrate the effect that the velocity of a moving load has on the response of a plate. For $v = 9.45 \times 10^2$ in/sec, the dynamic deflection at $i = 7$ is greater than the static deflection. However, for $v = 3.78 \times 10^3$ in/sec, the dynamic deflection at $i = 7$ is less than the static deflection and the traveling wave lags behind the moving load. This phenomenon was discussed by Reismann (Ref 10) in his theoretical solution for a long rectangular plate.



(a)

Hinged supports along all edges
Corners are held down

15 x 4 GRID

$D = 2.5 \times 10^6$ lb - in

$h_x = h_y = 4.8$ in.

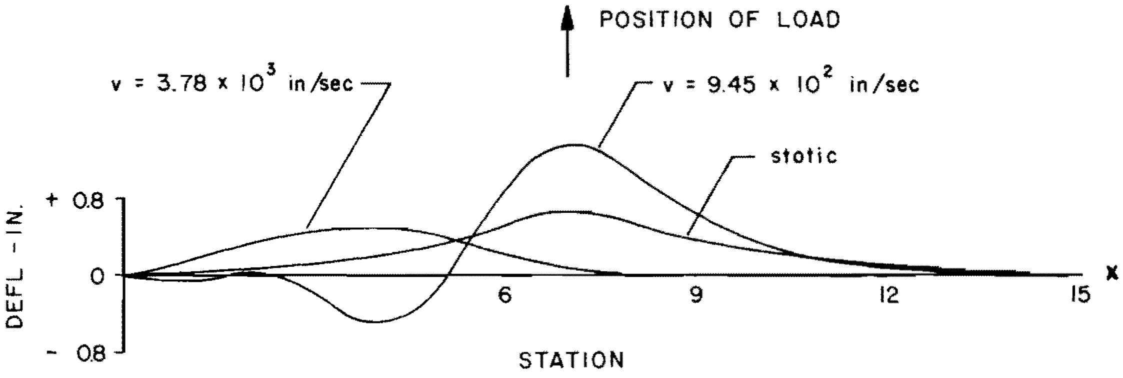
$L = 192$ in.

$\nu = 0.25$

$Q = 1.0 \times 10^3$ lb/sta

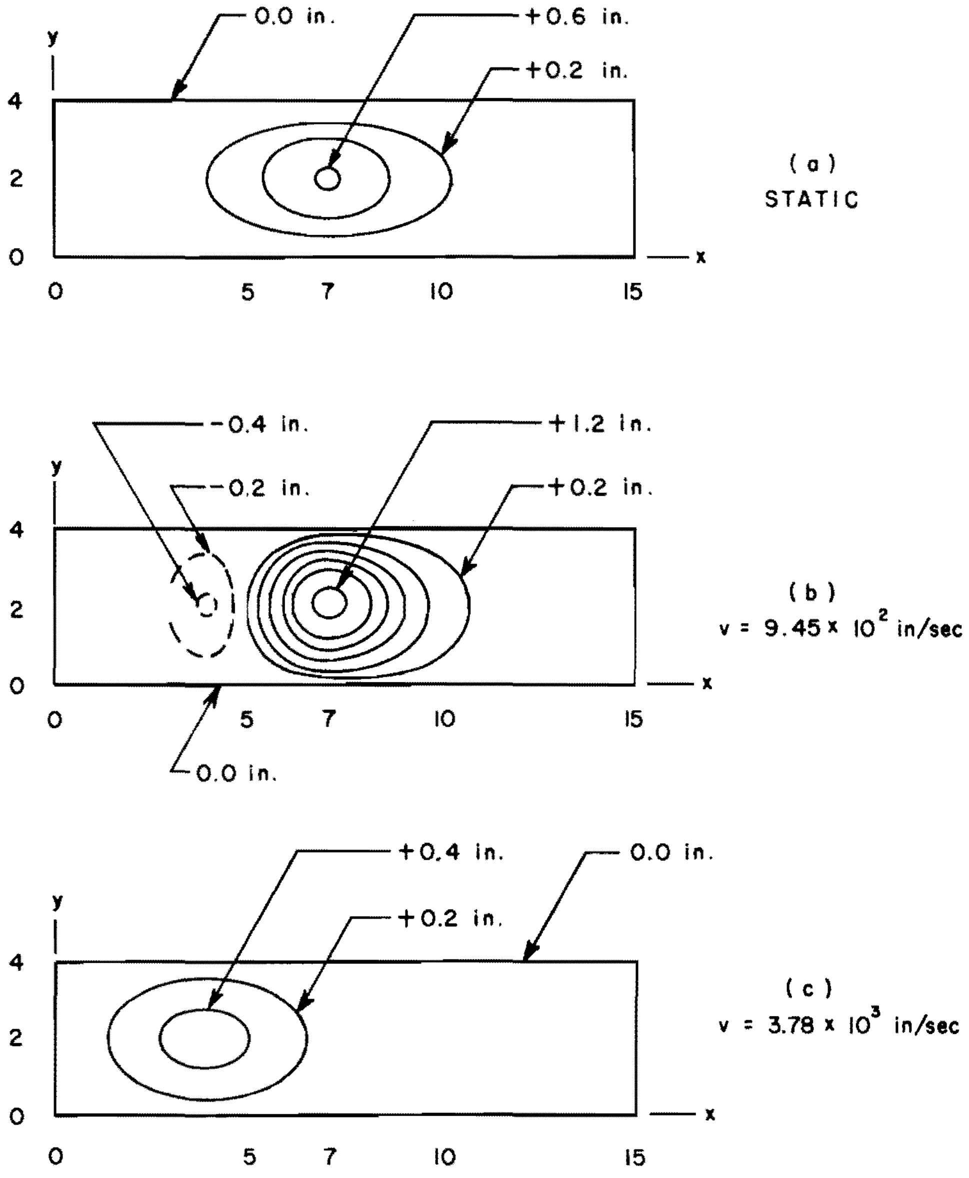
FOR $v = 9.45 \times 10^2$ in/sec
$h_t = 5.08 \times 10^{-2}$ sec

FOR $v = 3.78 \times 10^3$ in/sec
$h_t = 1.27 \times 10^{-2}$ sec



(b) PROFILE OF THE TRANSVERSE DEFLECTIONS ALONG THE CENTERLINE

Fig 10. Moving load on a rectangular plate.



CONTOUR INTERVAL = 0.2 in.

LOAD IS AT STATION i = 7

i refers to sta along the x axis

Fig 11. Contours of transverse deflections for a rectangular plate.

CHAPTER 8. CONCLUSIONS

A finite-element method has been presented to determine the response of a vibrating beam or plate. The method is based on an implicit difference formula of the Crank-Nicolson form. An examination of the difference equations for a uniform beam and plate disclosed that the implicit formula is not subject to instability. Therefore, this formula has been used in the development of the beam and plate equations. Although several investigators have used difference equations to solve the equation of motion for a uniform beam, the general development described herein is applicable to nonuniform beams and plates.

For the beam equation, the development includes externally applied dynamic loading, rotational restraints, elastic foundation supports, axial loads and viscous damping. For the plate equation, the development is arbitrarily restricted to externally applied dynamic loading and elastic foundation supports. Separate computer programs have been written in FORTRAN-63 for the solutions of the beam and plate equations. Both programs permit the flexural stiffnesses, elastic restraints, mass densities and loads to be discontinuous. Numerical examples demonstrate that the programs will be useful in solving many diverse problems whose solutions are not easily attainable by other known methods.

The present beam program effectively uses about 60 percent of the core storage of the Control Data Corporation 1604 computer. In contrast, the plate program utilizes the entire core storage of the computer and is restricted to problems whose maximum grid dimensions are 15 X 15. This limitation can be alleviated by storing a portion of the program on auxiliary tape.

A future extension of the preceding development will incorporate nonlinear flexural stiffness, foundation supports and damping. In addition, coupling between response of the beam, or slab, and response of a moving mass must be considered. The fundamental ideas and procedures described herein may have a potential application in shell dynamics and in other initial-value problems in engineering.

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APPENDIX 1

DYNAMIC BEAM EQUATION

APPENDIX 1. DYNAMIC BEAM EQUATION

The partial differential equation for the vibrating beam has been shown to be

$$\frac{\partial^2}{\partial x^2}\left[F\,\frac{\partial^2 w}{\partial x^2}\right] + sw - \frac{\partial}{\partial x}\left[(r + P)\,\frac{\partial w}{\partial x}\right] + \rho\,\frac{\partial^2 w}{\partial t^2} + d\,\frac{\partial w}{\partial t} = q(x,t) \tag{A1.1}$$

A finite difference form of the above equation is derived in the following manner. All symbols have been previously defined. Expansion of Eq A1.1 establishes

$$\begin{aligned}
&\frac{1}{h_x^2}\left[\left(F\,\frac{\partial^2 w}{\partial x^2}\right)_{j-1,k} - 2\left(F\,\frac{\partial^2 w}{\partial x^2}\right)_{j,k} + \left(F\,\frac{\partial^2 w}{\partial x^2}\right)_{j+1,k}\right] \\
&\quad + (sw)_{j,k} - \frac{1}{2h_x}\left[-\left((r + P)\,\frac{\partial w}{\partial x}\right)_{j-1,k}\right. \\
&\quad \left. + \left((r + P)\,\frac{\partial w}{\partial x}\right)_{j+1,k}\right] + \frac{\rho_j}{h_t^2}\left[w_{j,k-1} - 2w_{j,k} + w_{j,k+1}\right] \\
&\quad + \frac{d_j}{h_t}\left[-w_{j,k} + w_{j,k+1}\right] = q_{j,k}
\end{aligned} \tag{A1.2}$$

and

$$\begin{aligned}
&\frac{1}{h_x^4}\left[F_{j-1}\,(w_{j-2,k} - 2\,w_{j-1,k} + w_{j,k})\right. \\
&\quad - 2\,F_j\,(w_{j-1,k} - 2\,w_{j,k} + w_{j+1,k}) \\
&\quad \left. + F_{j+1}\,(w_{j,k} - 2\,w_{j+1,k} + w_{j+2,k})\right] + s_j w_{j,k}
\end{aligned}$$

(equation continued)

$$- \frac{1}{4h_x^2} \Big[- (r + P)_{j-1} \; (-w_{j-2,k} + w_{j,k} \;)$$

$$+ \; (r \; + P)_{j+1} \; (-w_{j,k} + w_{j+2,k} \;) \Big]$$

$$+ \frac{\rho_j}{h_t^2} \Big[\; w_{j,k-1} - 2 \, w_{j,k} + w_{j,k+1} \; \Big]$$

$$+ \frac{d_j}{h_t} \Big[- w_{j,k} + w_{j,k+1} \; \Big] \; = \; q_{j,k} \qquad \text{(A1.3)}$$

Furthermore, let

$$Q_{j,k} \; = \; h_x q_{j,k} \qquad \text{(A1.4)}$$

$$S_j \; = \; h_x s_j \qquad \text{(A1.5)}$$

and

$$R_j \; = \; h_x r_j \qquad \text{(A1.6)}$$

Equations A1.3, A1.4, A1.5 and A1.6 are combined to yield

$$\Big[\; F_{j-1} - 0.25 \; h_x \; (R_{j-1} + h_x P_{j-1}) \Big] \, w_{j-2,k}$$

$$- \; 2 \Big[F_{j-1} + F_j \; \Big] \, w_{j-1,k} + \Big[\; F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j$$

$$+ \; 0.25 \; h_x \; (R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1} \;) \; \Big] \, w_{j,k}$$

(equation continued)

$$
\begin{aligned}
& - 2 \Big[F_j + F_{j+1} \Big] w_{j+1,k} + \Big[F_{j+1} - 0.25\, h_x \,(R_{j+1} \\
& + h_x P_{j+1}) \Big] w_{j+2,k} + \frac{h_x^4}{h_t^2} \rho_j \Big[w_{j,k-1} - 2\, w_{j,k} + w_{j,k+1} \Big] \\
& + d_j \frac{h_x^4}{h_t} \Big[- w_{j,k} + w_{j,k+1} \Big] = h_x^3 Q_{j,k} \qquad \text{(A1.7)}
\end{aligned}
$$

For an implicit formula, the preceding equation becomes

$$
\begin{aligned}
& 0.5 \Big[F_{j-1} - 0.25\, h_x \,(R_{j-1} + h_x P_{j-1}) \Big] w_{j-2,k+1} \\
& \quad - \Big[F_{j-1} + F_j \Big] w_{j-1,k+1} + \Big\{ 0.5 \Big[F_{j-1} + 4F_j + F_{j+1} + h_x^3 S_j \\
& \quad + 0.25\, h_x \,(R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1}) \Big] + \frac{h_x^4}{h_t^2} \rho_j \\
& \quad + \frac{h_x^4}{h_t} d_j \,) \Big\} w_{j,k+1} - \Big[F_j + F_{j+1} \Big] w_{j+1,k+1} \\
& \quad + 0.5 \Big[F_{j+1} - 0.25\, h_x \,(R_{j+1} + h_x P_{j+1}) \Big] w_{j+2,k+1} \\
& \quad = h_x^3 Q_{j,k} + 2 \frac{h_x^4}{h_t^2} \rho_j w_{j,k} - \frac{h_x^4}{h_t^2} \rho_j w_{j,k-1} + \frac{h_x^4}{h_t} d_j w_{j,k} \\
& \quad - 0.5 \Big[F_{j-1} - 0.25\, h_x \,(R_{j-1} + h_x P_{j-1}) \Big] w_{j-2,k-1} \\
& \quad + \Big[F_{j-1} + F_j \Big] w_{j-1,k-1} - 0.5 \Big[F_{j-1} + 4\, F_j + F_{j+1} \\
& \quad + h_x^3 S_j + 0.25\, h_x \,(R_{j-1} + h_x P_{j-1} + R_{j+1} + h_x P_{j+1}) \Big] w_{j,k-1}
\end{aligned}
$$

(equation continued)

$$+ \left[F_j + F_{j+1} \right] w_{j+1,k-1} - 0.5 \left[F_{j+1} - 0.25\, h_x \,(R_{j+1} + h_x P_{j+1}) \right] w_{j+2,k-1} \qquad (A1.8)$$

The foregoing equation corresponds to Eq 3.8 in the text.

APPENDIX 2

STATIC PLATE COEFFICIENTS

APPENDIX 2. STATIC PLATE COEFFICIENTS

$$X_1 = \frac{1}{h_x^4} D_{i-1,j}$$

$$X_2 = -\frac{2}{h_x^4}\left[D_{i-1,j} + D_{i,j}\right] - \frac{2\nu}{h_x^2\,h_y^2} D_{i-1,j}$$

$$X_3 = \frac{1}{h_x^4}\left[D_{i-1,j} + 4D_{i,j} + D_{i+1,j}\right] + \frac{4\nu}{h_x^2\,h_y^2} D_{i,j}$$

$$X_4 = -\frac{2}{h_x^4}\left[D_{i,j} + D_{i+1,j}\right] - \frac{2\nu}{h_x^2\,h_y^2} D_{i+1,j}$$

$$X_5 = \frac{1}{h_x^4} D_{i+1,j}$$

$$X_6 = \frac{\nu}{h_x^2\,h_y^2} D_{i-1,j}$$

$$X_7 = -\frac{2\nu}{h_x^2\,h_y^2} D_{i,j}$$

$$X_8 = \frac{\nu}{h_x^2\,h_y^2} D_{i+1,j}$$

$$X_9 = X_6$$

$$X_{10} = X_7$$

$$X_{11} = X_8$$

$$Y_1 = \frac{1}{h_y^4} D_{i,j-1}$$

$$Y_2 = -\frac{2}{h_y^4}\left[D_{i,j-1} + D_{i,j}\right] - \frac{2\nu}{h_x^2\, h_y^2} D_{i,j-1}$$

$$Y_3 = \frac{1}{h_y^4}\left[D_{i,j-1} + 4D_{i,j} + D_{i,j+1}\right] + \frac{4\nu}{h_x^2\, h_y^2} D_{i,j}$$

$$Y_4 = -\frac{2}{h_y^4}\left[D_{i,j} + D_{i,j+1}\right] - \frac{2\nu}{h_x^2\, h_y^2} D_{i,j+1}$$

$$Y_5 = \frac{1}{h_y^4} D_{i,j+1}$$

$$Y_6 = \frac{\nu}{h_x^2\, h_y^2} D_{i,j-1}$$

$$Y_7 = -\frac{2\nu}{h_x^2\, h_y^2} D_{i,j}$$

$$Y_8 = \frac{\nu}{h_x^2\, h_y^2} D_{i,j+1}$$

$$Y_9 = Y_6$$

$$Y_{10} = Y_7$$

$$Y_{11} = Y_8$$

Let

$$T_{i,j} = (1-\nu)\, D_{i-1/2,\ j-1/2}$$

$$T_{i,j+1} = (1-\nu)\, D_{i-1/2,\ j+1/2}$$

$$T_{i+1,j} = (1-\nu)\, D_{i+1/2,\ j-1/2}$$

$$T_{i+1,j+1} = (1-\nu)\, D_{i+\frac{1}{2},j+\frac{1}{2}}$$

Then

$$Z_1 = \frac{1}{h_x^2\, h_y^2}\, T_{i,j}$$

$$Z_2 = -\frac{1}{h_x^2\, h_y^2}\left[T_{i,j} + T_{i,j+1}\right]$$

$$Z_3 = \frac{1}{h_x^2\, h_y^2}\, T_{i,j+1}$$

$$Z_4 = -\frac{1}{h_x^2\, h_y^2}\left[T_{i,j} + T_{i+1,j}\right]$$

$$Z_5 = \frac{1}{h_x^2\, h_y^2}\left[T_{i,j} + T_{i,j+1} + T_{i+1,j} + T_{i+1,j+1}\right]$$

$$Z_6 = -\frac{1}{h_x^2\, h_y^2}\left[T_{i,j+1} + T_{i+1,j+1}\right]$$

$$Z_7 = \frac{1}{h_x^2\, h_y^2}\, T_{i+1,j}$$

$$Z_8 = -\frac{1}{h_x^2\, h_y^2}\left[T_{i+1,j} + T_{i+1,j+1}\right]$$

$$Z_9 = \frac{1}{h_x^2\, h_y^2}\, T_{i+1,j+1}$$

APPENDIX 3

DYNAMIC PLATE EQUATION

APPENDIX 3. DYNAMIC PLATE EQUATION

The partial differential equation for a transversely vibrating plate is

$$\frac{\partial^2}{\partial x^2}\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]+\frac{\partial^2}{\partial y^2}\left[D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right]$$

$$+\,2\frac{\partial^2}{\partial x\partial y}\left[D\left(1-\nu\right)\ \frac{\partial^2 w}{\partial x\partial y}\ \right]+sw+\rho\frac{\partial^2 w}{\partial t^2}\ =\ q\ (x,y,t) \qquad \text{(A3.1)}$$

The finite difference form of Eq A3.1 is derived in the following manner. An expansion of Eq A3.1 establishes

$$\frac{1}{h_x^2}\left\{\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]_{i-1,j,k}-2\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]_{i,j,k}\right.$$

$$\left.+\left[D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)\right]_{i+1,j,k}\right\}+\frac{1}{h_y^2}\left\{\left[D\left(\frac{\partial^2 w}{\partial y^2}\right.\right.\right.$$

$$\left.\left.+\,\nu\frac{\partial^2 w}{\partial x^2}\right)\right]_{i,j-1,k}-2\left[D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right]_{i,j,k}$$

$$\left.+\left[D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)\right]_{i,j+1,k}\right\}$$

$$+\,\frac{2(1-\nu)}{h_x h_y}\left\{\left(D\frac{\partial^2 w}{\partial x\partial y}\right)_{i-1/2,j-1/2,k}-\left(D\frac{\partial^2 w}{\partial x\partial y}\right)_{i-1/2,j+1/2,k}\right.$$

$$\left.-\left(D\frac{\partial^2 w}{\partial x\partial y}\right)_{i+1/2,j-1/2,k}+\left(D\frac{\partial^2 w}{\partial x\partial y}\right)_{i+1/2,j+1/2,k}\right\}+(sw)_{i,j,k}$$

$$+\,\frac{\rho_{i,j}}{h_t^2}\left[w_{i,j,k-1}-2w_{i,j,k}+w_{i,j,k+1}\right]\ =\ q_{i,j,k} \qquad \text{(A3.2)}$$

and

$$
\begin{aligned}
&\frac{D_{i-1,j}}{h_x^2} \Big\{ \frac{1}{h_x^2} \Big[w_{i-2,j,k} - 2w_{i-1,j,k} + w_{i,j,k} \Big] \\
&\quad + \frac{\nu}{h_y^2} \Big[w_{i-1,j-1,k} - 2w_{i-1,j,k} + w_{i-1,j+1,k} \Big] \Big\} \\
&\quad - 2 \frac{D_{i,j}}{h_x^2} \Big\{ \frac{1}{h_x^2} \Big[w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k} \Big] \\
&\quad + \frac{\nu}{h_y^2} \Big[w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k} \Big] \Big\} \\
&\quad + \frac{D_{i+1,j}}{h_x^2} \Big\{ \frac{1}{h_x^2} \Big[w_{i,j,k} - 2w_{i+1,j,k} + w_{i+2,j,k} \Big] \\
&\quad + \frac{\nu}{h_y^2} \Big[w_{i+1,j-1,k} - 2\, w_{i+1,j,k} + w_{i+1,j+1,k} \Big] \Big\} \\
&\quad + \frac{D_{i,j-1}}{h_y^2} \Big\{ \frac{1}{h_y^2} \Big[w_{i,j-2,k} - 2w_{i,j-1,k} + w_{i,j,k} \Big] \\
&\quad + \frac{\nu}{h_x^2} \Big[w_{i-1,j-1,k} - 2w_{i,j-1,k} + w_{i+1,j-1,k} \Big] \Big\} \\
&\quad - 2 \frac{D_{i,j}}{h_y^2} \Big\{ \frac{1}{h_y^2} \Big[w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k} \Big] \\
&\quad + \frac{\nu}{h_x^2} \Big[w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k} \Big] \Big\}
\end{aligned}
$$

(equation continued)

$$
\begin{aligned}
&+ \frac{D_{i,j+1}}{h_y^2} \Big\{ \frac{1}{h_y^2} \Big[w_{i,j,k} - 2w_{i,j+1,k} + w_{i,j+2,k} \Big] \\
&+ \frac{\nu}{h_x^2} \Big[w_{i-1,j+1,k} - 2w_{i,j+1,k} + w_{i+1,j+1,k} \Big] \Big] \Big\} \\
&+ \frac{2(1-\nu)}{h_x h_y} \Big\{ \frac{D_{i-1/2,j-1/2}}{h_x h_y} \Big[w_{i-1,j-1,k} - w_{i-1,j,k} - w_{i,j-1,k} \\
&+ w_{i,j,k} \Big] - \frac{D_{i-1/2,j+1/2}}{h_x h_y} \Big[w_{i-1,j,k} - w_{i-1,j+1,k} - w_{i,j,k} \\
&+ w_{i,j+1,k} \Big] - \frac{D_{i+1/2,j-1/2}}{h_x h_y} \Big[w_{i,j-1,k} - w_{i,j,k} - w_{i+1,j-1,k} \\
&+ w_{i+1,j,k} \Big] + \frac{D_{i+1/2,j+1/2}}{h_x h_y} \Big[w_{i,j,k} - w_{i,j+1,k} - w_{i+1,j,k} \\
&+ w_{i+1,j+1,k} \Big] \Big\} + s_{i,j} w_{i,j,k} \\
&+ \frac{\rho_{i,j}}{h_t^2} \Big[w_{i,j,k-1} - 2w_{i,j,k} + w_{i,j,k+1} \Big] = q_{i,j,k}
\end{aligned}
\tag{A3.3}
$$

Let

$$S_{i,j} = h_x h_y \, s_{i,j} \tag{A3.4}$$

$$Q_{i,j,k} = h_x h_y \, q_{i,j,k} \tag{A3.5}$$

$$C_{i,j} = (1-\nu) \, D_{i-1/2, \, j-1/2} \tag{A3,6}$$

$$C_{i,j+1} = (1-\nu)\, D_{i-1/2,\,j+1/2} \tag{A3.7}$$

$$C_{i+1,j} = (1-\nu)\, D_{i+1/2,\,j-1/2} \tag{A3.8}$$

and

$$C_{i+1,j+1} = (1-\nu)\, D_{i+1/2,\,j+1/2} \tag{3.9}$$

Equations A3.3 through A3.9 are combined to yield

$$\frac{D_{i-1,j}}{h_x^2}\left\{\frac{1}{h_x^2}\left[w_{i-2,j,k} - 2w_{i-1,j,k} + w_{i,j,k}\right]\right.$$

$$\left. + \frac{\nu}{h_y^2}\left[w_{i-1,j-1,k} - 2w_{i-1,j,k} + w_{i-1,j+1,k}\right]\right\}$$

$$- 2\,\frac{D_{i,j}}{h_x^2}\left\{\frac{1}{h_x^2}\left[w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k}\right]\right.$$

$$\left. + \frac{\nu}{h_y^2}\left[w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k}\right]\right\}$$

$$+ \frac{D_{i+1,j}}{h_x^2}\left\{\frac{1}{h_x^2}\left[w_{i,j,k} - 2w_{i+1,j,k} + w_{i+2,j,k}\right]\right.$$

$$\left. + \frac{\nu}{h_y^2}\left[w_{i+1,j-1,k} - 2w_{i+1,j,k} + w_{i+1,j+1,k}\right]\right\}$$

$$+ \frac{D_{i,j-1}}{h_y^2}\left\{\frac{1}{h_y^2}\left[w_{i,j-2,k} - 2w_{i,j-1,k} + w_{i,j,k}\right]\right.$$

(equation continued)

$$+ \frac{\nu}{h_x^2} \Big[w_{i-1,j-1,k} - 2w_{i,j-1,k} + w_{i+1,j-1,k} \Big] \Big] \Big\}$$

$$- \frac{2D_{i,j}}{h_y^2} \Big\{ \frac{1}{h_y^2} \Big[w_{i,j-1,k} - 2w_{i,j,k} + w_{i,j+1,k} \Big]$$

$$+ \frac{\nu}{h_x^2} \Big[w_{i-1,j,k} - 2w_{i,j,k} + w_{i+1,j,k} \Big] \Big] \Big\}$$

$$+ \frac{D_{i,j+1}}{h_y^2} \Big\{ \frac{1}{h_y^2} \Big[w_{i,j,k} - 2w_{i,j+1,k} + w_{i,j+2,k} \Big]$$

$$+ \frac{\nu}{h_x^2} \Big[w_{i-1,j+1,k} - 2w_{i,j+1,k} + w_{i+1,j+1,k} \Big] \Big] \Big\}$$

$$+ \frac{2}{h_x^2 h_y^2} \Big\{ C_{i,j} \Big[w_{i-1,j-1,k} - w_{i-1,j,k} - w_{i,j-1,k} + w_{i,j,k} \Big]$$

$$- C_{i,j+1} \Big[w_{i-1,j,k} - w_{i-1,j+1,k} - w_{i,j,k} + w_{i,j+1,k} \Big]$$

$$- C_{i+1,j} \Big[w_{i,j-1,k} - w_{i,j,k} - w_{i+1,j-1,k} + w_{i+1,j,k} \Big]$$

$$+ C_{i+1,j+1} \Big[w_{i,j,k} - w_{i,j+1,k} - w_{i+1,j,k} + w_{i+1,j+1,k} \Big] \Big\}$$

$$+ \frac{S_{i,j}}{h_x h_y} w_{i,j,k} + \frac{\rho_{i,j}}{h_t^2} \Big[w_{i,j,k-1} - 2w_{i,j,k} + w_{i,j,k+1} \Big]$$

$$= \frac{Q_{i,j,k}}{h_x h_y} \qquad \text{(A3.10)}$$

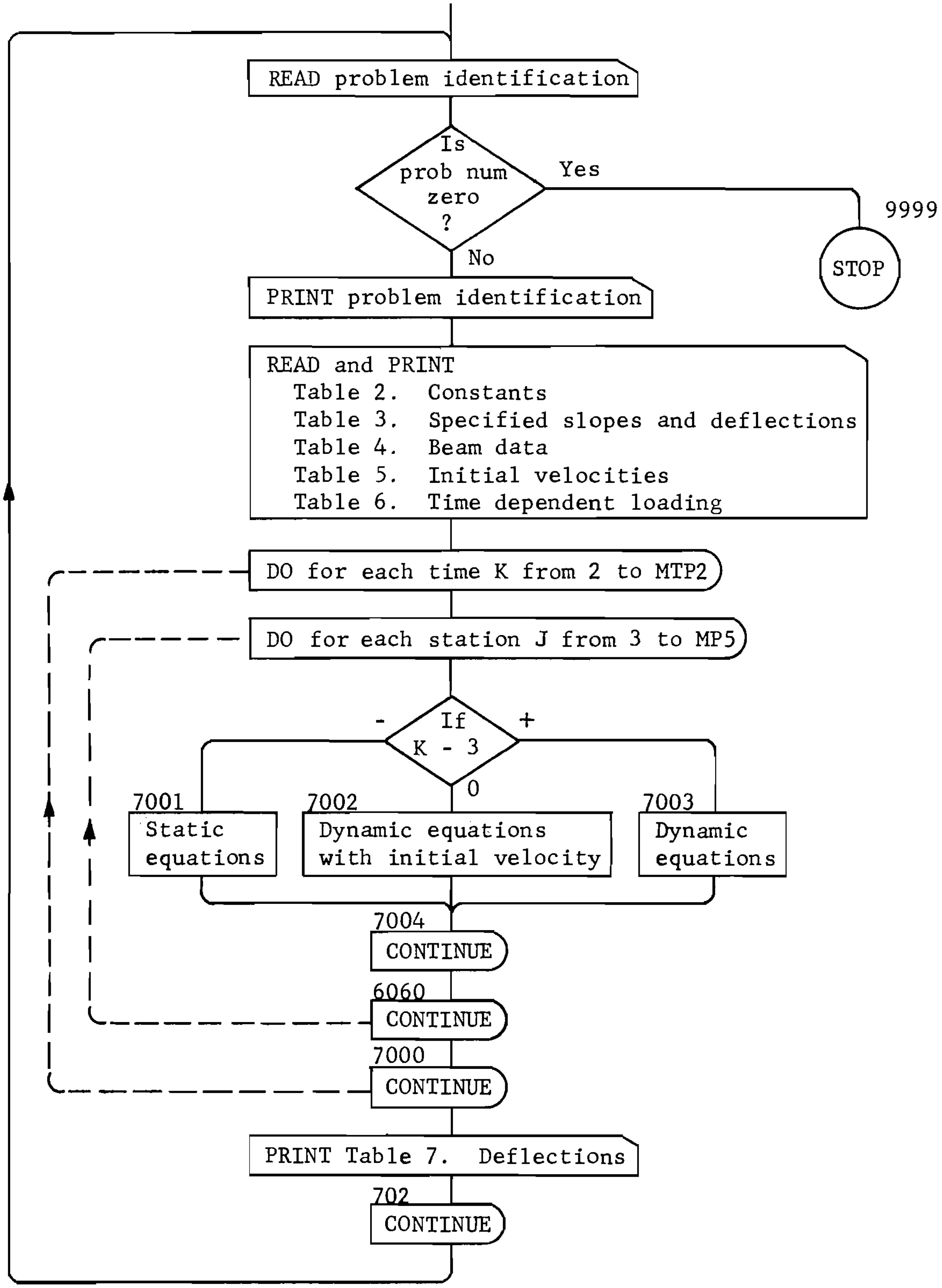
The static plate coefficients, which are defined in Appendix 2, are substituted into Eq A3.10. For an implicit formula, Eq A3.10 becomes

$$
\begin{aligned}
&\frac{1}{2}\,(X_1)\,\Big[w_{i-2,j,k-1} + w_{i-2,j,k+1}\Big] + \frac{1}{2}\,(X_2 + Y_7 + 2Z_2)\,\Big[w_{i-1,j,k-1} \\
&\quad + w_{i-1,j,k+1}\Big] + \Big[\frac{1}{2}\,(X_3 + Y_3 + 2Z_5 + S_{i,j}) + \rho_{i,j}\Big]\Big[w_{i,j,k-1} \\
&\quad + w_{i,j,k+1}\Big] + \frac{1}{2}\,(X_4 + Y_{10} + 2Z_8)\,\Big[w_{i+1,j,k-1} + w_{i+1,j,k+1}\Big] \\
&\quad + \frac{1}{2}\,(X_5)\,\Big[w_{i+2,j,k-1} + w_{i+2,j,k+1}\Big] + \frac{1}{2}\,(Y_1)\,\Big[w_{i,j-2,k-1} \\
&\quad + w_{i,j-2,k+1}\Big] + \frac{1}{2}\,(Y_2 + X_{10} + 2Z_4)\,\Big[w_{i,j-1,k-1} + w_{i,j-1,k+1}\Big] \\
&\quad + \frac{1}{2}\,(Y_4 + X_7 + 2Z_6)\,\Big[w_{i,j+1,k-1} + w_{i,j+1,k+1}\Big] \\
&\quad + \frac{1}{2}\,(Y_5)\,\Big[w_{i,j+2,k-1} + w_{i,j+2,k+1}\Big] + \frac{1}{2}\,(X_9 + Y_6 \\
&\quad + 2Z_1)\,\Big[w_{i-1,j-1,k-1} + w_{i-1,j-1,k+1}\Big] + \frac{1}{2}\,(X_6 + Y_8 \\
&\quad + 2Z_3)\,\Big[w_{i-1,j+1,k-1} + w_{i-1,j+1,k+1}\Big] + \frac{1}{2}\,(X_{11} + Y_9 \\
&\quad + 2Z_7)\,\Big[w_{i+1,j-1,k-1} + w_{i+1,j-1,k+1}\Big] + \frac{1}{2}\,(X_8 + Y_{11} \\
&\quad + 2Z_9)\,\Big[w_{i+1,j+1,k-1} + w_{i+1,j+1,k+1}\Big] - \frac{2\rho_{i,j}}{h_t^2}\,w_{i,j,k} \\
&\quad = \frac{Q_{i,j,k}}{h_x h_y}\ . \qquad \text{(A3.11)}
\end{aligned}
$$

APPENDIX 4

SUMMARY FLOW DIAGRAM, GUIDE FOR DATA INPUT, AND LISTING FOR PROGRAM DBC1

SUMMARY FLOW DIAGRAM - DBCI



GUIDE FOR DATA INPUT FOR PROGRAM DBC1 (BEAM)

with Supplementary Notes

DBC1 GUIDE FOR DATA INPUT --- Card forms

IDENTIFICATION OF PROGRAM AND RUN (2 alphanumeric cards per run)

1 80

1 80

IDENTIFICATION OF PROBLEM (one card each problem)

PROB	Description of problem (alphanumeric)
1 5	11 80

TABLE 2. CONSTANTS (one card)

				NUM CARDS IN TABLE			
NUM BEAM INCRS	BEAM INCR LENGTH	NUM TIME INCRS	TIME INCR LENGTH	3	4	5	6
6 10	20	26 30	40	46 50	56 60	66 70	76 80

TABLE 3. SPECIFIED DEFLECTIONS AND SLOPES (number of cards according to TABLE 2)

BEAM STATION	CASE	DEFLECTION	SLOPE
6 10	16 20	30	40

CASE = 1 for deflection only, 2 for slope only, 3 for both

TABLE 4 BEAM DATA AND STATIC LOADING (number of cards according to TABLE 2). Data <u>added</u> to storage as lumped quatities per increment length, linearly interpolated between values input at indicated end stations, with 1/2-values at each end station. Concentrated effects are established as full values at single stations by setting final station = initial station. (2 cards per set of data required)

<u>FROM</u> BEAM STA	<u>TO</u> BEAM STA	ENTER 1 IF CONT'D ON NEXT SET OF CARDS	F BENDING STIFFNESS	RHO MASS DENSITY	DF DAMPING COEF

6 10 15 20 30 40 50

Q TRANSVERSE LOAD	S SPRING SUPPORT	P AXIAL LOAD	T TRANSVERSE COUPLE	R ROTATIONAL RESTRAINT

6 15 25 35 45 55

TABLE 5 INITIAL VELOCITIES (number of cards according to TABLE 2) Full values of velocity occur at each station and the input is not cumulative.

<u>FROM</u> BEAM STA	<u>TO</u> BEAM STA	ENTER 1 IF CONT'D ON NEXT CARD	WV INITIAL VELOCITY

6 10 15 20 30

TABLE 6 TIME DEPENDENT LOADING (number of cards according to TABLE 2) Full values of load occur at each station and the input is not cumulative.

<u>FROM</u> BEAM STA	<u>TO</u> BEAM STA	<u>FROM</u> TIME STA	<u>TO</u> TIME STA	ENTER 1 IF CONT'D ON NEXT CARD	QT TIME DEPENDENT LOADING

6 10 15 20 25 30 40

	STOP CARD (one blank card at end of run)

1 5

GENERAL PROGRAM NOTES

The data cards must be stacked in proper order for the program to run.

A consistent system of units should be used for all input data; for example: pounds, inches, and seconds.

All 5-space words are understood to be integers . . . . $\boxed{-4321}$

All 10-space words are floating-point decimal numbers in an E format . . . . $\boxed{-4.321E+03}$

All integer data words must be right justified in the field provided.

The calculated deflections for all beam stations are printed in tabular form for each station.

The program will adjust the number of time stations so that this value will be a multiple of five. Thus, the number of time stations input will be increased by the computer by one to four to accommodate the output format.

TABLE 2. CONSTANTS

Typical units for the beam and time increment lengths are inches and seconds.

The maximum number of beam increments into which the beam-column may be divided is 100.

There is no maximum number of time increments, except that dynamic loading may be specified for only the first 110 time increments.

TABLE 3. SPECIFIED DEFLECTIONS AND SLOPES

The maximum number of stations at which deflections and slopes may be specified is 20.

Cards must be arranged in order of station numbers.

A slope may not be specified closer than 3 increments from another specified slope.

A deflection may not be specified closer than 2 increments from a specified slope, except that both a deflection and a slope may be specified at the same station.

TABLE 4. BEAM DATA AND STATIC LOADING

Typical units:

variables:	F	RHO	DF	Q	S	P	T	R
values per station:	lb-in^2	lb-sec^2/in^2	lb-sec/in	lb	lb/in	lb	lb-in	lb-in-rad

Axial tension or compression values P must be stated at each station in the same manner as any other distributed data; there is no provision in the program to automatically distribute the internal effects of an externally applied axial force.

For the interpolation and distribution process, there are four variations in the station numbering and in referencing for continuation to succeeding cards. These variations are explained and illustrated on the following page.

There are no restrictions on the order of cards in Table 4, except that within a distribution sequence the stations must be in regular order.

TABLE 5. INITIAL VELOCITIES

Typical units:

variable:	WV
values per station:	in/sec

A linear variation in initial velocities may be specified for any interval of beam stations, including the two end stations. The sequential order of the stations must be observed.

Initial velocities are input in the same manner as distributed quantities in Table 4, except that full values occur at every beam station and the input is not cumulative.

TABLE 6. TIME DEPENDENT LOADING

Typical units:

variable:	QT
values per station:	lb

The time dependent loading may be specified for any beam station and for a maximum of 110 time stations.

The program permits any continuous linear variation in loading with time; however, if the loading is input for an interval of beam stations, the timewise variation in loading must be the same for every station within the interval.

The sequential order of both beam and time stations must be observed.

Full values of load occur at each station and the input is not cumulative.

STATION NUMBERING AND REFERENCING FOR TABLE 4.

Fixed-position Data

Individual-card Input

	FROM BEAM STA	TO BEAM STA	CONT'D TO NEXT CARD?	F	Q	
Case a.1. Data concentrated at one sta	7	7	0 = NO		3.0	●
Case a.2. Data uniformly distributed	5	15	0 = NO	2.0		◒
	15	20	0 = NO	4.0	1.0	◓
	10	20	0 = NO		2.0	○
Multiple-card Sequence						
Case b. First-of-sequence	25		1 = YES	0.0	2.0	⊘
Case c. Interior-of-sequence		30	1 = YES	4.0	2.0	⦸
		35	1 = YES	2.0	0.0	⊖
Case d. End-of-sequence		40	0 = NO	2.0		

Resulting Distributions of Data

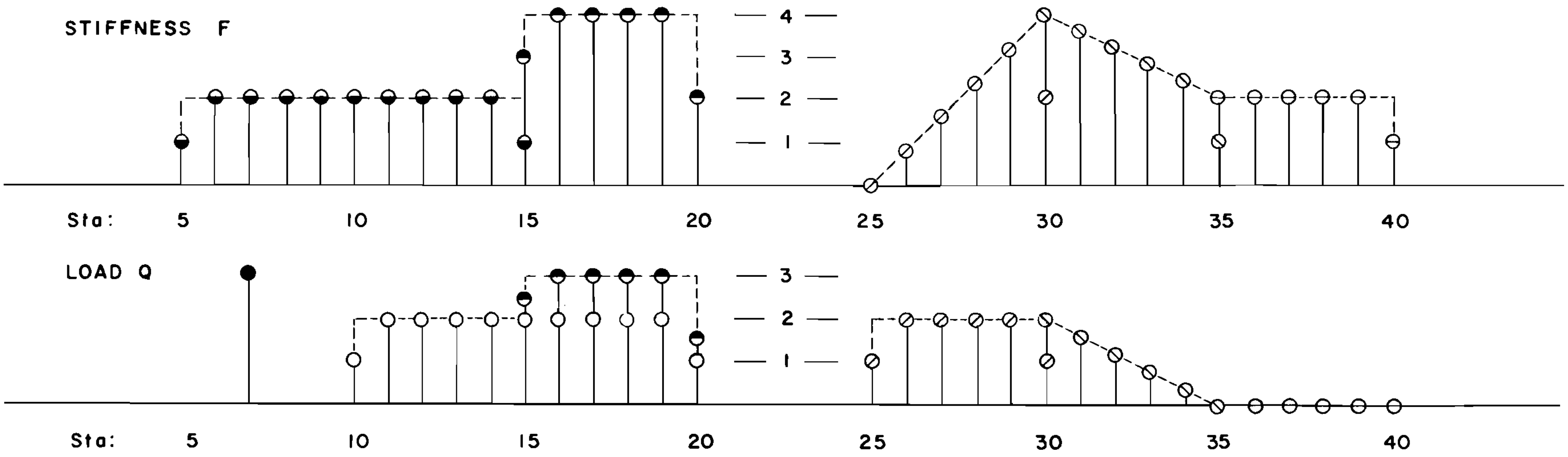
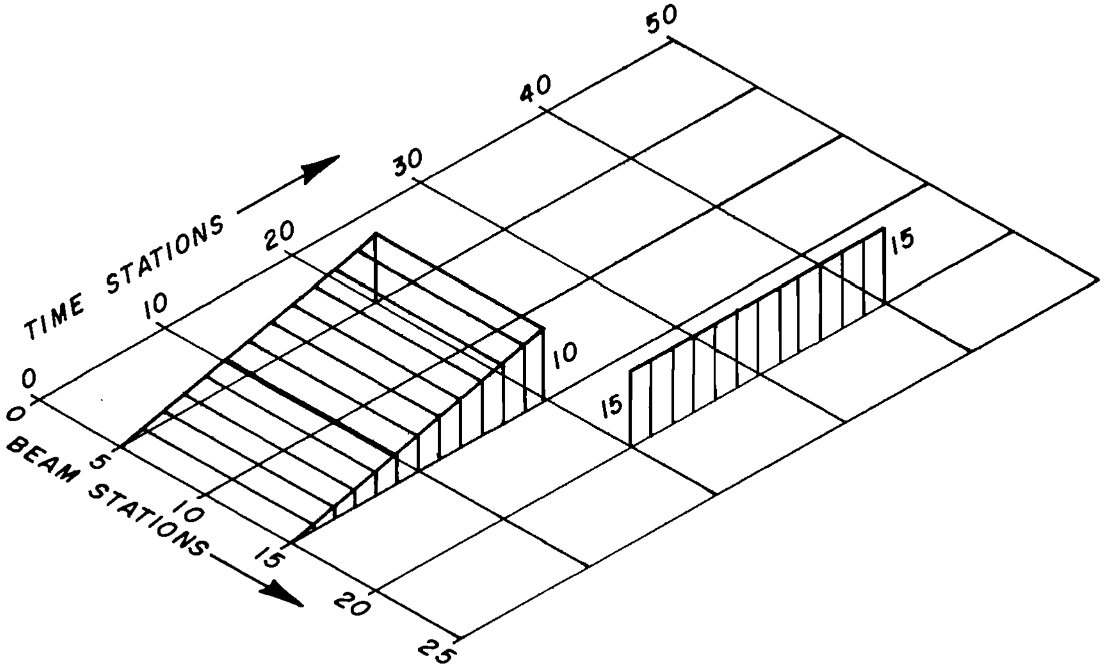


TABLE 6. TIME DEPENDENT LOADING (continued)

The variable QT is input at any beam station and time station by specifying j and k in the FROM and TO columns.

EXAMPLES OF PERMISSIBLE INPUT OF THE VARIABLE QT ARE SHOWN BELOW

BEAM STATIONS			TIME STATIONS			CONT'D TO NEXT CARD ?	$Q_{j,k}$
FROM	TO		FROM	TO			
5	15		0	20		1 = YES	0
5	15		0	20		0 = NO	10
20	20		20	40		0 = NO	15



```
-COOP,CE051118,MATLOCK,S/2S.           DBC1    DECK 1
-FTN,E,R,N.
      PROGRAM DBC1
    1 FORMAT (5X,52HPROGRAM  DBF1 - DECK 5       -  HJ SALANI, H MATLOCK22JL5 ID
     1            28H  REVISION DATE = 12 JUN 66 )
C-----SOLVES FOR THE DYNAMIC RESPONSE OF A BEAM BY AN IMPLICIT METHOD   01JL5
C-----NOTATION  FOR DBC  1                                              01JL5
C         AN1( ), AN2( ),ETC IDENTIFICATION AND REMARKS (ALPHA-NUM)     12JE3
C         DF(J)              DAMPING COEF                               01JL5
C         DWS( )             VALUE OF SPECIFIED SLOPE DW/DX             04JE3
C         ESM                MULTIPLIER FOR HALF VALUES AT END STAS     07JE3
C         FN1,FN2,F(J)       FLEXURAL STIFFNESS (EI) (INPUT AND TOTAL)  12JE3
C         H                  BEAM INCREMENT                             09JL5
C         HT                  TIME INCREMENT                            01JL5
C         ITEST               BLANK FIELD FOR ALPHANUMERIC ZERO         22JL5
C         J                   BEAM STATION                              09JL5
C         J1, J2              INITIAL AND FINAL STATIONS IN SEQUENCE    05JE3
C         JS                  STA OF SPECIFIED DEFLECTION OR SLOPE      05JE3
C         K                   TIME STATION                              09JL5
C         KASE                CASE NUM FOR SPECIFIED CONDITIONS         07JE3
C         KASE                1=DEFL, 2=SLOPE, 3=BOTH                   01JL5
C         M                   TOTAL NUMBER OF INCREMENTS OF BMCOL       12JE3
C         M                   MAX NUM = 50                              01JL5
C         MT                  NUMBER TIME INCREMENTS                    01JL5
C         MT                  MAX NUM NOT SPECIFIED                     09JL5
C         NCT3,4,5 AND 6      NUM CARDS IN TABLES 3,4,5 AND 6           07JN6
C         NPROB               PROBLEM NUMBER (PROG STOPS IF ZERO)       25MY3
C         NS                  INDEX NUM FOR SPECIFIED CONDITIONS        05JE3
C         PN1, PN2, P(J)      AXIAL TENSION OR COMPRESSION(INPUT, TOTAL)12JE3
C         QN1, QN2, Q(J)      TRANSVERSE FORCE (INPUT AND TOTAL)        23MR4
C         QT(J,K)             TIME DEPENDENT TRANSVERSE LOADING         01JL5
C         QT(J,K)             MAX NUM (50,110)                          09JL5
C         RHO(J)              MASS DENSITY OF BEAM                      01JL5
C         RN1, RN2, R(J)      ROTATIONAL RESTRAINT ( INPUT, TOTAL )     12JE3
C         SN1, SN2, S(J)      SPRING SUPPORT STIFFNESS (INPUT AND TOTAL)23MR4
C         TN1, TN2, T(J)      TRANSVERSE TORQUE ( INPUT, TOTAL )        12JE3
C         W(J,K)              LATERAL DEFL OF BEAM AT  J,K              09JL5
C         WS(JS)              SPECIFIED VALUE OF DEFL AT STA JS         12JE3
C         WV(J)               INITIAL VELOCITY                          09JL5
C         XF,XB               MULTIPLIER                                01JL5
      DIMENSION     AN1(32), AN2(14), F(107), Q(107), S(107), T(107),   07JN6
     1              R(107), P(107), A(107), B(107), C(107), W(107,8),   07JN6
     2              KEY(107), WS(20), DWS(20), QT(107,110), RHO(107),   07JN6
     3              WV(107), DF(107)                                    07JN6
   10 FORMAT (    5H     , 80X, 10HI-----TRIM )                         27FE4 ID
   11 FORMAT (    5H1    , 80X, 10HI-----TRIM )                         27FE4 ID
   12 FORMAT (  16A5  )                                                 04MY3 ID
   13 FORMAT (        5X, 16A5 )                                        27FE4 ID
   14 FORMAT ( A5, 5X, 14A5 )                                           18FE5 ID
   15 FORMAT (///10H     PROB , /5X, A5, 5X, 14A5 )                     18FE5 ID
   16 FORMAT (///17H     PROB (CONTD), /5X, A5, 5X, 14A5 )              18FE5 ID
   19 FORMAT (///48H     RETURN THIS PAGE TO TIME RECORD FILE -- HM )   12MR5 ID
   21 FORMAT ( 2( 5X, I5, E10.3 ), 4( 5X, I5) )                         07JN6
   31 FORMAT ( 2(5X, I5), 2E10.3 )                                      23MR4
   41 FORMAT ( 5X, 3I5, 3E10.3 )                                        07JN6
```

```
201 FORMAT (///25H     TABLE 2.  CONSTANTS   ,                          01JL5
   1      / 5X, 25H     NUM BEAM INCRE       , 20X, I10,                01JL5
   2      / 5X, 25H     BEAM INCRE LENGTH    ,20X, E10.3,               01JL5
   3      / 5X, 25H     NUM TIME INCRE       ,20X, I10,                 01JL5
   4      / 5X, 25H     TIME INCRE LENGTH    , 20X, E10.3,              01JL5
   5      / 5X, 25H     NUM CARDS TABLE 3    ,20X, I10,                 01JL5
   6      / 5X, 25H     NUM CARDS TABLE 4    ,20X, I10,
   7      / 5X, 25H     NUM CARDS TABLE 5    , 20X, I10,                07JN6
   8      / 5X, 25H     NUM CARDS TABLE 6    , 20X, I10 )               07JN6
300 FORMAT (///47H     TABLE 3 - SPECIFIED DEFLECTIONS AND SLOPES       20JA4
   1   /    5X, 48H     STA      CASE       DEFLECTION        SLOPE    )01JL5
311 FORMAT (       10X, I3, 7X, I2, 8X, E10.3, 9X, 4HNONE )             23MR4
312 FORMAT (       10X, I3, 7X, I2, 11X, 4HNONE, 8X, E10.3 )            23MR4
313 FORMAT (       10X, I3, 7X, I2, 3X, 2(5X, E10.3) )                  23MR4
400 FORMAT (///45H     TABLE 4.  BEAM DATA AND STATIC LOADING       )  01JL5
411 FORMAT (5X,30H     FROM     TO     CONTD     ,/,10X, 3( I4, 4X ),  07JN6
   1      //5X, 45H        F             RHO             DF      ,    07JN6
   2             10H        Q ,/, 5X,  4( 5X, E10.3 ) ,                07JN6
   3      //5X, 45H        S              P              T      ,     07JN6
   4             10H        R ,/, 5X,  4( 5X, E10.3 ) , // )           07JN6
412 FORMAT (5X,30H     FROM     TO     CONTD     ,/, 10X, I4, 12X, I4, 01JL5
   1      //5X, 45H        F             RHO             DF      ,    07JN6
   2             10H        Q ,/, 5X,  4( 5X, E10.3 ) ,                07JN6
   3      //5X, 45H        S              P              T      ,     07JN6
   4             10H        R ,/, 5X,  4( 5X, E10.3 ) , // )           07JN6
413 FORMAT (5X,30H     FROM     TO     CONTD     ,/, 18X, I4, 4X, I4,  01JL5
   1      //5X, 45H        F             RHO             DF      ,    07JN6
   2             10H        Q ,/, 5X,  4( 5X, E10.3 ) ,                07JN6
   3      //5X, 45H        S              P              T      ,     07JN6
   4             10H        R ,/, 5X,  4( 5X, E10.3 ) , // )           07JN6
500 FORMAT (///37H     TABLE 7.  D E F L E C T I O N S   ,/,           01JL5
   1             35H         J=BEAM AXIS, K=TIME AXIS         )        01JL5
511 FORMAT (       5X, I4, 2X, 6E12.3 )                                 23MR4
602 FORMAT   ( 5X, 5(5X, E10.3) )                                       01JL5
604 FORMAT   ( 5(5X, E10.3) )                                           01JL5
605 FORMAT (///35H     TABLE 5. INITIAL VELOCITIES    )                 07JN6
606 FORMAT ( 5X, 5E10.3 )                                               07JN6
607 FORMAT (///40H     TABLE 6.  TIME DEPENDENT LOADING    ,/,          01JL5
   1      5X,    30H     BEAM STA       TIME STA  ,/,                   07JN6
   2      5X,    50H     FROM  TO       FROM  TO     CONTD       QT     ) 07JN6
608 FORMAT ( 10X, 2I4, 7X, 2I4, 5X, I4, 2X, E10.3 )                     07JN6
609 FORMAT ( 5X, 5I5, E10.3 )                                           07JN6
610 FORMAT   ( 10X, I3, 3X, I3, 6X, I3, 3X, I3, 5X, E10.3)              01JL5
611 FORMAT ( 10X, 2I4, 7X, I4, 9X, I4, 2X, E10.3 )                      07JN6
612 FORMAT ( 5X, 3I5, E10.3 )                                           07JN6
613 FORMAT ( 5X, 25H      FROM      TO      WV    ,/, 10X, 2( I4, 3X ),  07JN6
   1      E10.3 )                                                       07JN6
614 FORMAT  ( 13X, 4HR(J), 11X, 4HP(J), 10X, 5HDF(J), 9X, 6HRHO(J))     01JL5
615 FORMAT ( 5X, 34H      FROM      TO      CONTD      WV     ,/, 10X, I4,  07JN6
   1      12X, I4, 4X, E10.3 )                                          07JN6
616 FORMAT ( 5X, 34H      FROM      TO      CONTD      WV     ,/, 17X, I4,  07JN6
   1      5X, I4, 4X, E10.3 )                                           07JN6
617 FORMAT   (/, 18X, 2HK=, I3, 10X, 2HK=, I3, 10X, 2HK=, I3, 10X,      01JL5
   1              2HK=, I3, 10X, 2HK=, I3 )                             01JL5
618 FORMAT (     7H     J= , I3,  5( 5X, E10.3 ))                       01JL5
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  619 FORMAT ( 30X, I4, 5X, I4, 2X, E10.3 )                             07JN6
  904 FORMAT (// 40H      TOO MUCH DATA FOR AVAILABLE STORAGE // )      04FE4
  907 FORMAT ( //40H      ERROR STOP -- STATIONS NOT IN ORDER )         03FE4
C-----START EXECUTION OF PROGRAM - SEE GENERAL FLOW CHART               23MR4
               ITEST = 5H                                               19MR5 ID
 1000 PRINT 10                                                          12JL3 ID
      CALL TIME                                                         18FE5 ID
C-----PROGRAM AND PROBLEM IDENTIFICATION                                04MY3 ID
      READ  12, ( AN1(N), N = 1, 32 )                                   18FE5 ID
 1010 READ  14,  NPROB, ( AN2(N), N = 1, 14 )                           28AG3 ID
          IF ( NPROB - ITEST ) 1020, 9990, 1020                         26FE5 ID
 1020 PRINT 11                                                          26AG3 ID
      PRINT 1                                                           18FE5 ID
      PRINT 13, ( AN1(N), N = 1, 32 )                                   18FE5 ID
      PRINT 15, NPROB, ( AN2(N), N = 1, 14 )                            26AG3 ID
C-----INPUT TABLE 2, CONSTANTS                                          01JL5
 1210 READ     21, M, H, MT, HT, NCT3, NCT4, NCT5, NCT6                 07JN6
      PRINT    201, M, H, MT, HT, NCT3, NCT4, NCT5, NCT6                07JN6
C-----COMPUTE CONSTANTS AND INDEXES                                     10JE3
 1240          HT2 = H + H                                              03JE3
               HTE2 = HT * HT                                           01JL5
               HE2 = H * H                                              30MY3
               HE3 = H * HE2                                            30MY3
               HE4 = H * HE3                                            01JL5
               MP1 = M + 1                                              01JL5
               MP4 = M + 4                                              30MY3
               MP5 = M + 5                                              30MY3
               MP6 = M + 6                                              10JE3
               MP7 = M + 7                                              30MY3
               MTP2 = MT + 2                                            01JL5
               MTP9 = MT + 9                                            01JL5
               H41T = HE4 / HT                                          01JL5
               H4T2 = HE4 / HTE2                                        01JL5
               XF= 0.5                                                  01JL5
               XB= 0.5                                                  01JL5
C-----INPUT TABLE 3, SPECIFIED SLOPES AND DEFLECTIONS                   01JL5
 1300 PRINT 300                                                         03JE3
 1310     DO 1315  J = 3, MP5                                           23MR4
               KEY(J) = 1                                               03JE3
 1315     CONTINUE                                                      03JE3
 1325          IF ( NCT3 - 20 ) 1327, 1327, 1326                        01JL5
 1326 PRINT 904                                                         04JE3
          GO TO 1010                                                    09JL5
 1327          JS = 3                                                   03FE4
          DO 1350 N = 1, NCT3                                           01JL5
      READ 31, IN1, KASE, WS(N), DWS(N)                                 03FE4
          IF ( IN1 + 4 - JS ) 1328, 1328, 1329                          03FE4
 1328 PRINT 907                                                         03FE4
          GO TO 9999                                                    03FE4
 1329          JS = IN1 + 4                                             03FE4
C-----SET INDEXES FOR FUTURE CONTROL OF SPECIFIED CONDITION ROUTINES     10JE3
          GO TO ( 1330, 1335, 1340 ), KASE                              05JE3
 1330          KEY(JS) = 2                                              05JE3
      PRINT 311, IN1, KASE, WS(N)                                       03FE4
          GO TO 1350                                                    03JE3
```

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 1335          KEY(JS-1) = 3                                            05JE3
               KEY(JS+1) = 5                                            05JE3
      PRINT 312, IN1, KASE, DWS(N)                                      03FE4
          GO TO 1350                                                    03JE3
 1340          KEY(JS-1) = 3                                            05JE3
               KEY(JS) = 4                                              05JE3
               KEY(JS+1) = 5                                            05JE3
      PRINT 313, IN1, KASE, WS(N), DWS(N)                               03FE4
 1350     CONTINUE                                                      03JE3
 1399     CONTINUE                                                      04JE3
C     CLEAR STORAGE                                                     01JL5
          DO 1402 J=1,MP7                                               01JL5
               F(J) = 0.0                                               30MY3
               Q(J) = 0.0                                               19MR4
               S(J) = 0.0                                               19MR4
               T(J) = 0.0                                               30MY3
               R(J) = 0.0                                               30MY3
               P(J) = 0.0                                               30MY3
               RHO(J) = 0.0                                             01JL5
               DF(J) = 0.0                                              01JL5
               WV(J) = 0.0                                              01JL5
          DO 1403 K= 1, 110                                             01JL5
               QT(J,K) = 0.0                                            01JL5
 1403     CONTINUE                                                      01JL5
          DO 1402 KD= 1, 8                                              01JL5
               W(J,KD) = 0.0                                            01JL5
 1402     CONTINUE                                                      04JE3
C-----INPUT TABLE 4, BEAM DATA                                           01JL5
               NCH4 = NCT4 / 2                                          01JL5
 1400 PRINT 400                                                         04JE3
 1406          KR2 = 0                                                  04JE3
          DO 1480 N=1, NCH4                                             01JL5
               KR1 = KR2                                                28MY3
      READ     41, IN1, IN2, KR2, FN2, RHON2, DFN2                      07JN6
      READ     606, QN2, SN2, PN2, TN2, RN2                             07JN6
               JN = IN1 + 4                                             28MY3
               J2 = IN2 + 4                                             28MY3
               KSW = 1 + KR2 + 2 * KR1                                  28MY3
          GO TO ( 1407, 1410, 1415, 1415 ), KSW                         04JE3
 1407 PRINT    411, IN1, IN2, KR2, FN2, RHON2, DFN2, QN2, SN2, PN2, TN2,07JN6
     1              RN2                                                 07JN6
          GO TO 1420                                                    04EJ3
 1410 PRINT    412, IN1, KR2, FN2, RHON2, DFN2, QN2, SN2, PN2, TN2, RN2 07JN6
          GO TO 1420                                                    04JE3
 1415 PRINT    413, IN2, KR2, FN2, RHON2, DFN2, QN2, SN2, PN2, TN2, RN2 07JN6
          GO TO 1435                                                    04JE3
 1420          J1 = JN                                                  04JE3
 1425          FN1 = FN2                                                04JE3
               QN1 = QN2                                                28MY3
               SN1 = SN2                                                28MY3
               TN1 = TN2                                                28MY3
               RN1 = RN2                                                28MY3
               PN1 = PN2                                                28MY3
               DFN1 = DFN2                                              01JL5
               RHON1 = RHON2                                            01JL5
```

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          GO TO ( 1435, 1480, 9999, 1480 ), KSW                         22JA4
C-----SEE FLOW CHART, TABLE INTERPOL AND DISTRIB                        23MR4
 1435          JINCR = 1                                                07JE3
               ESM = 1.0                                                07JE3
          IF ( J2 - J1 ) 1437, 1450, 1440                               03FE4
 1437 PRINT 907                                                         03FE4
          GO TO 1010                                                    01JL5
 1440          DENOM = J2 - J1                                          07JE3
               ISW = 1                                                  07JE3
          GO TO 1455                                                    07JE3
 1450          DENOM = 1.0                                              07JE3
               ISW = 0                                                  07JE3
 1455     DO 1460  J = J1, J2, JINCR                                    04JE3
               DIFF = J - J1                                            28MY3
               PART = DIFF / DENOM                                      28MY3
               F(J) = F(J) + ( FN1 + PART * ( FN2 - FN1 ) ) * ESM       28MY3
               Q(J) = Q(J) + ( QN1 + PART * ( QN2 - QN1 ) ) * ESM       19MR4
               S(J) = S(J) + ( SN1 + PART * ( SN2 - SN1 ) ) * ESM       19MR4
               T(J) = T(J) + ( TN1 + PART * ( TN2 - TN1 ) ) * ESM       28MY3
               R(J) = R(J) + ( RN1 + PART * ( RN2 - RN1 ) ) * ESM       28MY3
               P(J) = P(J) + ( PN1 + PART * ( PN2 - PN1 ) ) * ESM       28MY3
               DF(J) = DF(J) + ( DFN1 + PART * (DFN2 - DFN1) ) * ESM    01JL5
               RHO(J) = RHO(J) + ( RHON1 + PART * (RHON2 - RHON1) ) *ESM01JL5
 1460     CONTINUE                                                      04JE0
          IF ( ISW ) 9999, 1470, 1465                                   03FE4
 1465          JINCR = J2 - J1                                          07JE3
               ESM = - 0.5                                              07JE3
               ISW = 0                                                  28MY3
          GO TO 1455                                                    04JE3
 1470     GO TO ( 1480, 9999, 1480, 1475 ), KSW                         23JA4
 1475          J1 = J2                                                  04JE3
          GO TO 1425                                                    04JE3
 1480     CONTINUE                                                      04JE3
C-----INPUT TABLE 5, INITIAL VELOCITIES                                 01JL5
      PRINT    605                                                      07JN6
               KR2 = 0                                                  07JN6
          DO 1493 N=1, NCT5                                             07JN6
               KR1 = KR2                                                07JN6
      READ     612, IN1, IN2, KR2, WV2                                  07JN6
               JN = IN1 + 4                                             07JN6
               J2 = IN2 + 4                                             07JN6
               KSW = 1 + KR2 + 2 * KR1                                  07JN6
          GO TO ( 1481, 1482, 1483, 1483 ), KSW                         07JN6
 1481 PRINT    613, IN1, IN2, WV2                                       07JN6
          GO TO 1484                                                    07JN6
 1482 PRINT    615, IN1, KR2, WV2                                       07JN6
          GO TO 1484                                                    07JN6
 1483 PRINT    616, IN2, KR2, WV2                                       07JN6
          GO TO 1486                                                    07JN6
 1484          J1 = JN                                                  07JN6
 1485          WV1 = WV2                                                07JN6
          GO TO ( 1486, 1493, 9999, 1493 ), KSW                         07JN6
 1486     IF ( J2 - J1 ) 1487, 1489, 1488                               07JN6
 1487 PRINT    907                                                      07JN6
          GO TO 1010                                                    07JN6
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 1488          DENOM = J2 - J1                                          07JN6
          GO TO 1490                                                    07JN6
 1489          DENOM = 1.0                                              07JN6
 1490     DO 1491 J = J1, J2                                            07JN6
               DIFF = J - J1                                            07JN6
               PART = DIFF / DENOM                                      07JN6
               WV(J) = WV1 + PART * ( WV2 - WV1)                        07JN6
 1491     CONTINUE                                                      07JN6
          GO TO ( 1493, 9999, 1493, 1492 ), KSW                         07JN6
 1492          J1 = J2                                                  07JN6
          GO TO 1485                                                    07JN6
 1493     CONTINUE                                                      07JN6
C-----INPUT TABLE 6, TIME DEPENDENT LOADING                             01JL5
      PRINT 607                                                         01JL5
               KR2 = 0                                                  07JN6
          DO   635 N = 1, NCT6                                          07JN6
               KR1 = KR2                                                07JN6
      READ     609, IN1, IN2, KN1, KN2, KR2, QTN                        07JN6
               J1 = IN1 + 4                                             07JN6
               J2 = IN2 + 4                                             07JN6
               KN = KN1 + 2                                             07JN6
               K2 = KN2 + 2                                             07JN6
               KSW = 1 + KR2 + 2 * KR1                                  07JN6
          GO TO ( 620, 621, 622, 622 ), KSW                             07JN6
  620 PRINT    608, IN1, IN2, KN1, KN2, KR2, QTN                        07JN6
          GO TO 623                                                     07JN6
  621 PRINT    611, IN1, IN2, KN1, KR2, QTN                             07JN6
          GO TO 623                                                     07JN6
  622 PRINT    619, KN2, KR2, QTN                                       07JN6
          GO TO 625                                                     07JN6
  623          K1 = KN                                                  07JN6
  624          QN1 = QTN                                                07JN6
          GO TO ( 625, 635, 9999, 635 ), KSW                            07JN6
  625     IF ( J2 - J1 ) 626, 627, 627                                  07JN6
  626 PRINT    907                                                      07JN6
          GO TO 9999                                                    07JN6
  627     IF ( K2 - K1 ) 628, 629, 630                                  07JN6
  628 PRINT    907                                                      07JN6
          GO TO 9999                                                    07JN6
  629          DENOM = 1.0                                              07JN6
          GO TO 631                                                     07JN6
  630          DENOM = K2 - K1                                          07JN6
  631     DO 633 J = J1, J2                                             07JN6
          DO 632 K = K1, K2                                             07JN6
               DIFF = K - K1                                            07JN6
               PART = DIFF / DENOM                                      07JN6
               QT(J,K) = QN1 + PART * ( QTN - QN1 )                     07JN6
  632     CONTINUE                                                      07JN6
  633     CONTINUE                                                      07JN6
          GO TO ( 635, 635, 635, 634 ), KSW                             07JN6
  634          K1 = K2                                                  07JN6
          GO TO 624                                                     07JN6
  635     CONTINUE                                                      07JN6
C-----START OF BEAM-COLUMN SOLUTION                                     10JE3
      PRINT 11                                                          08MY3 ID
```

```
      PRINT 1                                                           18FE5 ID
      PRINT 13, ( AN1(N), N = 1, 32 )                                   18FE5 ID
      PRINT 16, NPROB, ( AN2(N), N = 1, 14 )                            28AG3 ID
      PRINT 500                                                         23MR4
               K = 1                                                    01JL5
          DO 7009 NOT= 8, MTP9, 5                                       01JL5
          IF ( NOT - MT - 4 ) 7009, 7005, 7005                          01JL5
 7005          MTP  = NOT                                               01JL5
          GO TO 7006                                                    01JL5
 7009     CONTINUE                                                      01JL5
 7006     DO 7000 KD = 2, MTP                                           01JL5
               K = K + 1                                                01JL5
 6000          NS = 1                                                   04JE3
          DO 6060 J = 3, MP5                                            04JE3
          IF   (110-KD) 704, 705, 705                                   01JL5
  704          QTP = 0.0                                                01JL5
          GO TO 706                                                     01JL5
  705          QTP = QT(J, KD-1)                                        01JL5
  706     CONTINUE                                                      01JL5
C-----COMPUTE MATRIX COEFFS AT EACH STA J                                10JE3
               YA = F(J-1) - 0.25 * H * ( R(J-1) + H * P(J-1) )         01JL5
               YB = - 2.0 * ( F(J-1) + F(J) )                           01JL5
               YC = F(J-1) + 4.0 * F(J) + F(J+1) + HE3 * S(J) +         01JL5
     1              0.25 * H * ( ( R(J-1) + H * P(J-1) ) + ( R(J+1)     01JL5
     2              + H * P(J+1) ) )                                    01JL5
               YD = - 2.0 * ( F(J) + F(J+1) )                           01JL5
               YE = F(J+1) - 0.25 * H * ( R(J+1) + H * P(J+1) )         01JL5
          IF (KD-3) 7001, 7002, 7003                                    01JL5
 7001          AA = YA                                                  01JL5
               BB = YB                                                  01JL5
               CC = YC                                                  01JL5
               DD = YD                                                  01JL5
               EE = YE                                                  01JL5
               FF = HE3 * Q(J) - 0.5 * HE2 * ( T(J-1) - T(J+1) )        01JL5
          GO TO 7004                                                    01JL5
 7002          AA = XF * YA                                             01JL5
               BB = XF * YB                                             01JL5
               CC = XF * YC + 4.0 * H4T2 * RHO(J) + 2.0 * H41T * DF(J)  01JL5
               DD = XF * YD                                             01JL5
               EE = XF * YE                                             01JL5
               FF = HE3 * QTP - XB * (YA * W(J-2,K-1) + YB * W(J-1,K-1) 01JL5
     1              + YC * W(J,K-1) + YD * W(J+1,K-1) + YE * W(J+2,K-1))01JL5
     2              + 8.0 * H4T2 * RHO(J) * (W(J,K-1) + 0.5 * HT * WV(J)01JL5
     3              ) - 4.0 * H4T2 * RHO(J) * W(J,K-1) + 2.0 * H41T *   01JL5
     4              DF(J) * (W(J,K-1) + 0.5 * HT * WV(J))               01JL5
          GO TO 7004                                                    01JL5
 7003          AA = XF * YA                                             01JL5
               BB = XF * YB                                             01JL5
               CC = XF * YC + H4T2 * RHO(J) + H41T * DF(J)              01JL5
               DD = XF * YD                                             01JL5
               EE = XF * YE                                             01JL5
               FF = HE3 * QTP - XB * (YA * W(J-2,K-2) + YB * W(J-1,K-2) 01JL5
     1              + YC * W(J,K-2) + YD * W(J+1,K-2) + YE * W(J+2,K-2))01JL5
     2              + 2.0 * H4T2 * RHO(J) * W(J,K-1) - H4T2 * RHO(J)    01JL5
     3              * W(J,K-2) + H41T * DF(J) * W(J,K-1)                01JL5
```

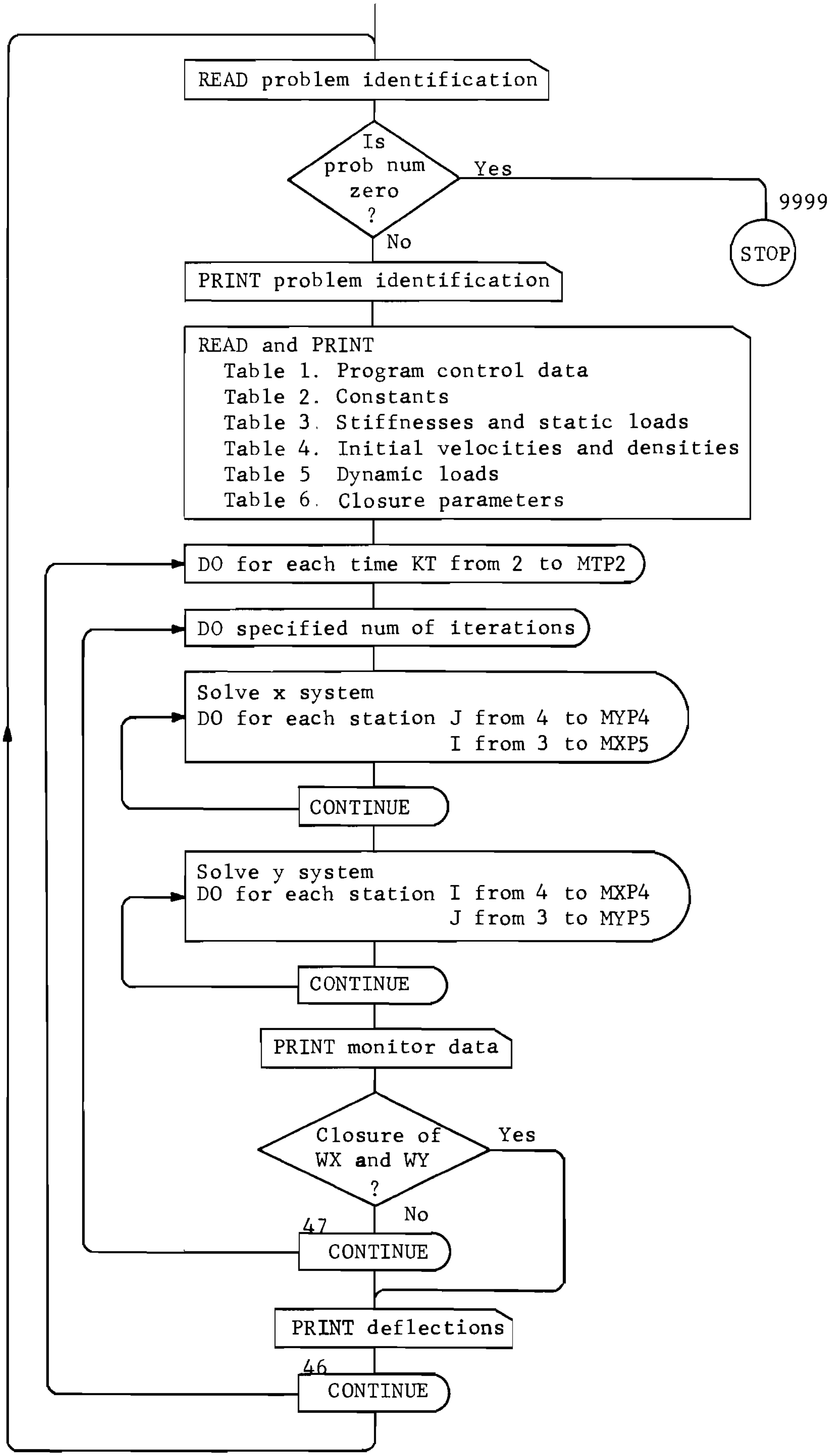
```
C-----COMPUTE RECURSION OR CONTINUITY COEFFS AT EACH STA                10JE3
 7004     CONTINUE                                                      01JL5
               E = AA * B(J-2) + BB                                     01JL5
               DENOM = E * B(J-1) + AA * C(J-2) + CC                    28MY3
          IF ( DENOM ) 6010, 6005, 6010                                 28MY3
C-----NOTE IF DENOM IS ZERO, BEAM DOES NOT EXIST, D = 0 SETS DEFL = 0.  10JE3
 6005          D = 0.0                                                  28MY3
          GO TO 6015                                                    28MY3
 6010          D = - 1.0 / DENOM                                        28MY3
 6015          C(J) = D * EE                                            28MY3
               B(J) = D * ( E * C(J-1) + DD )                           28MY3
               A(J) = D * ( E * A(J-1) + AA * A(J-2) - FF )             28MY3
C-----CONTROL RESET ROUTINES FOR SPECIFIED CONDITIONS                   10JE3
               KEYJ = KEY(J)                                            04JE3
          GO TO ( 6060, 6020, 6030, 6020, 6050 ), KEYJ                  20JA4
C-----RESET FOR SPECIFIED DEFLECTION                                    20JA4
 6020          C(J) = 0.0                                               05JE3
               B(J) = 0.0                                               28MY3
               A(J) = WS(NS)                                            05JE3
          IF ( KEYJ - 3 ) 6059, 6030, 6060                              20JA4
C-----RESET FOR SPECIFIED SLOPE AT NEXT STA                             17JA4
 6030          DTEMP = D                                                05JE3
               CTEMP = C(J)                                             28MY3
               BTEMP = B(J)                                             28MY3
               ATEMP = A(J)                                             28MY3
               C(J) = 1.0                                               28MY3
               B(J) = 0.0                                               28MY3
               A(J) = - HT2 * DWS(NS)                                   05JE3
          GO TO 6060                                                    04JE3
C-----RESET FOR SPECIFIED SLOPE AT PRECEDING STATION                    23MR4
 6050          DREV = 1.0 / ( 1.0 - ( BTEMP * B(J-1) + CTEMP - 1.0 ) *  05JE3
     1                D / DTEMP )                                       04JE3
               CREV = DREV * C(J)                                       28MY3
               BREV = DREV * ( B(J) + ( BTEMP * C(J-1) ) * D / DTEMP )  28MY3
               AREV = DREV * ( A(J) + ( HT2 * DWS(NS) + ATEMP + BTEMP   05JE3
     1                * A(J-1) ) * D / DTEMP )                          04JE3
               C(J) = CREV                                              28MY3
               B(J) = BREV                                              28MY3
               A(J) = AREV                                              28MY3
 6059          NS = NS + 1                                              20JA4
 6060     CONTINUE                                                      28MY3
C-----COMPUTE DEFLECTIONS                                               23MR4
          DO 6100  L = 3, MP5                                           23MR4
               J = M + 8 - L                                            30MY3
               W(J,K) = A(J) + B(J) * W(J+1,K) + C(J) * W(J+2,K)        01JL5
 6100     CONTINUE                                                      30MY3
          IF ( 8 - K ) 7007, 7007, 7000                                 01JL5
 7007          KSA = KD - 8                                             01JL5
               KSB = KD - 7                                             01JL5
               KSC = KD - 6                                             01JL5
               KSD = KD - 5                                             01JL5
               KSE = KD - 4                                             01JL5
      PRINT 617, KSA, KSB, KSC, KSD, KSE                                01JL5
          DO 7008 J = 3, MP5                                            01JL5
               JSTA = J - 4                                             01JL5
```

```
      PRINT 618, JSTA, W(J,2), W(J,3), W(J,4), W(J,5), W(J,6)                01JL5
 7008     CONTINUE                                                             01JL5
               K = 3                                                           01JL5
          DO 7010 J = 3, MP5                                                   01JL5
               W(J,2) = W(J,7)                                                 01JL5
               W(J,3) = W(J,8)                                                 01JL5
 7010     CONTINUE                                                             01JL5
 7000     CONTINUE                                                             01JL5
      CALL TIME                                                                18FE5 ID
          GO TO 1010                                                           26AG3 ID
 9990 CONTINUE                                                                 12MR5 ID
 9999 CONTINUE                                                                 04MY3 ID
      PRINT 11                                                                 08MY3 ID
      PRINT 1                                                                  18FE5 ID
      PRINT 13, ( AN1(N), N = 1, 32 )                                          18FE5 ID
      PRINT 19                                                                 26AG3 ID
      END                                                                      25JE4
      END                                                                      04MA3
         FINIS                                                                 01JL5
-EXECUTE.                                                                      01JL5
```

APPENDIX 5

SUMMARY FLOW DIAGRAM, GUIDE FOR DATA INPUT, AND LISTING FOR PROGRAM DPI1

SUMMARY FLOW DIAGRAM - DPI1



GUIDE FOR DATA INPUT FOR PROGRAM DPI1 (PLATE)

with Supplementary Notes

DPI1 GUIDE FOR DATA INPUT -- Card forms

IDENTIFICATION OF PROGRAM AND RUN (2 alphanumeric cards per run)

1	80
1	80

IDENTIFICATION OF PROBLEM (1 alphanumeric card each problem)

NPROB	DESCRIPTION OF PROBLEM (alphanumeric)
1 5	11 80

TABLE 1 CONTROL DATA (One card)

NUMBER CARDS IN TABLE					
3	4	5	6	MAX NUM ITER	CLOSURE TOLERANCE
6 10	16 20	26 30	36 40	46 50	56 65

MONITOR MESH POINTS (specify the I and J stations for three mesh points)

I ,	J	I ,	J	I ,	J
6 10	16 20	26 30	36 40	46 50	56 60

TABLE 2 CONSTANTS (One card)

NUM X INCRS	NUM Y INCRS	NUM TIME INCRS	X INCR LENGTH	Y INCR LENGTH	TIME INCR LENGTH	POISSON'S RATIO
6 10	16 20	26 30	40	50	60	70

TABLE 3. STIFFNESS AND STATIC LOADING (number of cards according to TABLE 1)

TABLE 4. INITIAL VELOCITIES AND DENSITIES (number of cards according to TABLE 1)

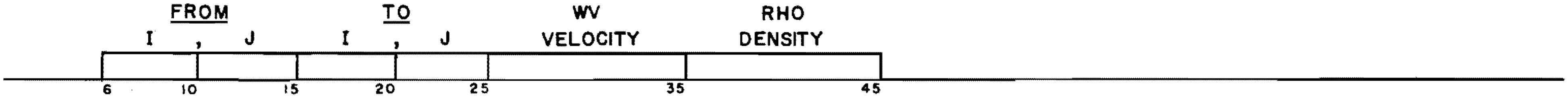


TABLE 5. DYNAMIC LOADING (Number of cards according to TABLE 1)

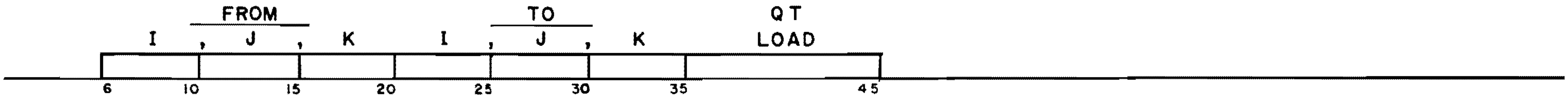


TABLE 6. CLOSURE PARAMETERS (Number of cards according to TABLE 1) Use one card for each parameter.

STOP CARD (One blank card at end of each run)

GENERAL PROGRAM NOTES

Two cards containing any desired alphanumeric information are required (for identification purposes only) at the beginning of the data for each new run.

The data cards must be stacked in proper order for the program to run.

A consistent system of units must be used for all input data; for example, pounds, inches, and seconds.

All integer data words must be right justified in the field provided.

All data words of 5 spaces or less are integers . . . . `-1234`

All data words of 10 spaces are to be entered as floating-point decimal numbers in an E format `-1.234E+03`

Blank data fields are interpreted as zeros in an integer or floating point mode.

One card with a problem number in columns 1-5 is required as the first card of each problem. This number may be alphanumeric. The remainder of the card may contain any information desired.

Any number of problems may be stacked in one run.

One card with problem number blank is required to stop the run.

The calculated deflections for the monitor mesh points are printed after each iteration.

When the closure tolerance is satisfied at all mesh points, or when the maximum number of iterations is reached, the calculated deflections for all mesh points are printed.

TABLE 1. CONTROL DATA

The maximum number of iterations is 999.

A closure tolerance of 1.0×10^{-6} in. is usually adequate.

TABLE 2. CONSTANTS

The maximum number of x and y plate increments is 15.

There is no maximum number of time increments.

TABLE 3. STIFFNESSES AND STATIC LOADING

Typical units:

variables:	D	T	S	Q
values per station:	lb-in	lb-in	lb/in	lb

In the foregoing, $D = \frac{Eh^3}{12\ (1 - \nu^2)}$, wherein h is the thickness of the plate, and $T = D\ (1 - \nu)$. The remaining symbols have been previously defined.

For a rectangular plate that is divided into an M × N grid, $i = 0, 1, \ldots, M$ and $j = 0, 1, \ldots, N$. The variables D, S, and Q are input at any grid or mesh point by specifying i and j in the FROM and TO columns. However, the variable T defines the torsional stiffness which is assumed to be concentrated at the center of each rectangular grid. In the program, T is numbered according to the mesh point that is located in the upper right corner of each grid, and it is assumed that the i station numbers increase from left to right and the j station numbers increase from bottom to top. Thus, for an M × N grid, T is specified from $i=1$, $j=1$ to $i=M$, $j=N$.

There are no restrictions on the sequential order of the cards. The input is cumulative with full values at each mesh point.

TABLE 4. INITIAL VELOCITIES AND DENSITIES

Typical units:

variables:	WV	RHO
values per station:	in/sec	lb sec^2/in^3

The variables WV and RHO are input at any mesh point by specifying i and j in the FROM and TO columns.

A zero initial velocity is automatically established in the program. Thus only non-zero velocities must be specified.

TABLE 4. Continued

There are no restrictions on the sequential order of the cards. The input is cumulative with full values at each mesh point.

TABLE 5. DYNAMIC LOADING

Typical units:
variable: QT
values per station: lb

The variable QT is input at any mesh point and time station by specifying i , j , and k in the FROM and TO columns.

There are no restrictions on the sequential order of the cards. The input is cumulative with full values at each station.

The loading may be specified for any mesh point and for a maximum of 28 time stations. Therefore, k maximum is 28.

TABLE 6. CLOSURE PARAMETERS

Typical units:
variable: RP
values per station: lb/in^3

The maximum number of parameters that may be input is nine.

The parameters are used in the cyclic order in which they are input.

The parameters are calculated on the basis of an average stiffness D and the increment length h in the x-direction from the equation.

$$(RP)_m = \frac{4D}{h_x^4}(1 - \cos\frac{m\pi}{M})(2 - \cos\frac{m\pi}{M}) \; ; \quad m = 1, 2, 3 \quad \ldots M - 1 .$$

The parameters for the y system are calculated internally in the program.

```
9COOP,CE051015, MATLOCK-SALANI, S/2S, 10, 6000.  DPI1
9FTN,E.
      PROGRAM DPI1
    1 FORMAT (5X,52HPROGRAM  DPI1 - MASTER DECK  -  HJ SALANI, H MATLOCK22JL5 ID
     1                                  28H     REVISION DATE 22 JUL 65)-----  *
C-----SOLVES FOR THE DYNAMIC RESPONSE OF A PLATE BY AN IMPLICIT METHOD  01JL5
C-----NOTATION                                                          01JL5
C         ANA(N),ANB(N)       ALPHA NUMERIC IDENTIFICATION              01JL5
C         A(N),B(N),C(N)      COEFFICIENTS                              01JL5
C         CTOL                CLOSURE TOLERANCE                         01JL5
C         D(I,J)              PLATE STIFFNESS PER UNIT AREA             01JL5
C         D(I,J)              (EHHH)/((12) (1-VV))                      01JL5
C         DN,RHON,TN,QN,QTN   TEMP VALUES OF D,RHO,T,Q,QT               01JL5
C         HX,HY,HT            INCREMENT LENGTHS IN X,Y AND Z DIRECTIONS 01JL5
C         ITEST               BLANK FIELD FOR ALPHANUMERIC ZERO         22JL5
C         ITMAX               MAX NUM ITERATIONS                        01JL5
C         I                   X PLATE AXIS                              01JL5
C         J                   Y PLATE AXIS                              01JL5
C         K                   TIME AXIS                                 01JL5
C         IM1,JM1 ETC         MONITOR STAS FOR DEFL                     01JL5
C         MX,MY,MT            NUMBER OF INCREMENTS IN X,Y AND Z         01JL5
C         MX,MY,MT            DIRECTIONS. MAX MX=MAX MY= 15,NO MAX MT   01JL5
C         NCT3,...NCT6        NUMBER CARDS IN TABLES 3 THRU 6           01JL5
C         NPROB               PROBLEM NUMBER,ZERO TO EXIT               01JL5
C         PR                  POISSON&S RATIO                           01JL5
C         Q(I,J)              TRANSVERSE STATIC LOAD PER MESH POINT     01JL5
C         QT(I,J,K)           TRANSVERSE DYNAMIC LOAD PER MESH POINT    01JL5
C         QT                  MAX NUM QT =28                            01JL5
C         RHO(I,J)            MASS DENSITY OF PLATE PER UNIT AREA       01JL5
C         RP(N)               CLOSURE PARMETER                          01JL5
C         S(I,J)              SPRING SUPPORT PER MESH POINT             01JL5
C         T(I,J)              STIFFNESS PER UNIT AREA, (1-V)(D)         01JL5
C         WV(I,J)             INITIAL VELOCITY                          01JL5
C         WY(I,J,K)           TRANSVERSE DEFLECTION FOR Y SYSTEM        01JL5
C         WX(I,J,K)           TRANSVERSE DEFLECTION FOR X SYSTEM        01JL5
      DIMENSION     AN1(32), AN2(14),                                   18FE5 ID
     1             Q(22,22), WV(22,22),                                 01JL5
     2             S(22,22),RHO(22,22),QT(22,22,30),A(22),B(22),C(22),  01JL5
     3             RP(9),WX(22,22,4),WY(22,22,4),JSTA(25)               01JL5
      COMMON/1/D(22,22),T(22,22)/2/X1,X2,X3,X4,X5,X6,X7,X8,X9,X10,      16MR5
     1         X11,Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9,Y10,Y11,XY1,XY2,XY3,      16MR5
     2         XY4,XY5,XY6,XY7,XY8,XY9,I,J,HA,HB,HC,HD,HXYA,            16MR5
     3         HXYB,HXYC,HXY1                                           16MR5
   10 FORMAT (    5H     , 80X, 10HI-----TRIM )                         27FE4 ID
   11 FORMAT (    5H1    , 80X, 10HI-----TRIM )                         27FE4 ID
   12 FORMAT (  16A5  )                                                 04MY3 ID
   13 FORMAT (        5X, 16A5 )                                        27FE4 ID
   14 FORMAT ( A5, 5X, 14A5 )                                           18FE5 ID
   15 FORMAT (///10H     PROB , /5X, A5, 5X, 14A5 )                     18FE5 ID
   16 FORMAT (///17H     PROB (CONTD), /5X, A5, 5X, 14A5 )              18FE5 ID
   19 FORMAT (///48H     RETURN THIS PAGE TO TIME RECORD FILE -- HM )   12MR5 ID
   20 FORMAT (5(2X,I3),5X,E10.3)                                        01JL5
   21 FORMAT (///30H     TABLE 1. CONTROL DATA    ,/                    01JL5
     1           30H          NUM CARDS TABLE 3   , 40X,I5, /           01JL5
     2           30H          NUM CARDS TABLE 4   , 40X,I5, /           01JL5
     3           30H          NUM CARDS TABLE 5   , 40X,I5, /           01JL5
     4           30H          NUM CARDS TABLE 6   , 40X,I5, /           01JL5
     5           30H          MAX NUM ITERATIONS  , 40X,I5, /           01JL5
     6           30H          CLOSURE TOLERANCE   , 35X,E10.3    )      01JL5
   22 FORMAT ( 8I5)                                                     01JL5
   23 FORMAT (   30H          MONITOR STAS I,J    , 20X,3(I2,2X,I2,4X)) 01JL5
   24 FORMAT (3I10,4E10.3)                                              01JL5
```

```
   25 FORMAT (///30H     TABLE 2. CONSTANTS        ,/                   01JL5
    1           30H          NUM INCREMENTS MX    , 40X,I5, /           01JL5
    2           30H          NUM INCREMENTS MY    , 40X,I5, /           01JL5
    3           30H          NUM INCREMENTS MT    , 40X,I5, /           01JL5
    4           30H          INCR LENGTH    HX    , 35X,E10.3,  /       01JL5
    5           30H          INCR LENGTH    HY    , 35X,E10.3,  /       01JL5
    6           30H          INCR LENGTH    HT    , 35X,E10.3,  /       01JL5
    7           30H          POISSON&S RATIO      , 35X,E10.3        )  01JL5
   26 FORMAT (///45H     TABLE 3. STIFFNESSES AND STATIC LOADING ,/     01JL5
    1           29H          FROM(I,J) THRU(I,J),6X,1HD,9X,1HT,10X ,    01JL5
    2            2HS  ,10X,3HQ      )                                   01JL5
   28 FORMAT (4I5,4E10.3)                                               01JL5
   29 FORMAT (   10X,I2,2X,I2,4X,I2,2X,I2,4X,4(E10.3,2X)             )  01JL5
   33 FORMAT (///50H     TABLE 4. INITIAL VELOCITIES AND DENSITIES    ,/01JL5
    1           34H          FROM(I,J)      THRU(I,J), 20X,2HWV  ,      01JL5
    2           12X, 3HRHO  )                                           01JL5
   35 FORMAT (     10X,I2,2X,I2,9X,I2,2X,I2,19X,E10.3 ,5X,E10.3      )  01JL5
   36 FORMAT (///30H     TABLE 5. DYNAMIC LOADING , /                   01JL5
    1           36H          FROM(I,J,K)    THRU(I,J,K), 18X,2HQT    )  01JL5
   38 FORMAT (6I5,E10.3)                                                01JL5
   39 FORMAT (   10X,I2,2X,I2,2X,I2,5X,I2,2X,I2,2X,I2,15X,E10.3      )  01JL5
   40 FORMAT (///35H     TABLE 6. CLOSURE PARAMETERS                 )  01JL5
   42 FORMAT (   E10.3)                                                 01JL5
   43 FORMAT (     10X,E10.3   )                                        01JL5
   45 FORMAT (///30H     *** MONITOR DEFLS ***     ,                    01JL5
    1        /  10X,  3HITR,7X,2HSF,8X,3HNOT,7X,4HTIME,16X,3HI,J ,      01JL5
    2        /  10X,  3HNUM,17X,6HCLOSED,10X,I2,1X,I2,7X,I2,1X,I2,      01JL5
    3           7X,I2,1X,I2 )                                           01JL5
   77 FORMAT (  5X,2HWX,3X,I4, 2X, E10.3, 4X, I5, 5X, I3, 2X, E10.3, 2X,01JL5
    1          E10.3, 2X, E10.3, /, 5X, 2HWY, 36X, 3(2X,E10.3)  )       01JL5
   85 FORMAT (///36H     *** D E F L E C T I O N S ***   ,/,5X,4HTIME,  01JL5
    1          1X,I4, 10X,  15HSTAS NOT CLOSED, I4    )                 01JL5
   87 FORMAT (/17X,5(2HJ=,I2,11X ) )                                    01JL5
   88 FORMAT (/  5X,2HI=,I2,2X,2HWX,2X, 5(E10.3,5X ))                   01JL5
   91 FORMAT (  11X,2HWY,2X, 5(E10.3,5X) )                              01JL5
   95 FORMAT  (5X, 11E10.3)                                             01JL5
  104 FORMAT (    )                                                     01JL5
               ITEST = 5H                                               19MRS ID
 1000 PRINT 10                                                          12JL3 ID
      CALL TIME                                                         18FE5 ID
C-----PROGRAM AND PROBLEM IDENTIFICATION                                04MY3 ID
      READ  12, ( AN1(N), N = 1, 32 )                                   18FE5 ID
 1010 READ  14,  NPROB, ( AN2(N), N = 1, 14 )                           28AG3 ID
          IF ( NPROB - ITEST ) 1020, 9990, 1020                         26FE5 ID
 1020 PRINT 11                                                          26AG3 ID
      PRINT 1                                                           18FE5 ID
      PRINT 13, ( AN1(N), N = 1, 32 )                                   18FE5 ID
      PRINT 15, NPROB, ( AN2(N), N = 1, 14 )                            26AG3 ID
C-----INPUT TABLE 1, CONTROL DATA                                       01JL5
      READ   20,NCT3,NCT4,NCT5,NCT6,ITMAX,CTOL                          01JL5
      PRINT  21,NCT3,NCT4,NCT5,NCT6,ITMAX,CTOL                          01JL5
      READ   22,IM1,JM1,IM2,JM2,IM3,JM3                                 01JL5
      PRINT  23,IM1,JM1,IM2,JM2,IM3,JM3                                 01JL5
C-----INPUT TABLE 2, CONSTANTS                                          01JL5
      READ   24,MX,MY,MT,HX,HY,HT,PR                                    01JL5
      PRINT  25,MX,MY,MT,HX,HY,HT,PR                                    01JL5
             MXP3=MX+3  $ MYP3=MY+3  $ MXP2= MX+2  $ MYP2=MY+2          01JL5
             MTP2=MT+2  $ MXP7=MX+7  $ MYP7=MY+7   $ MXP4=MX+4          01JL5
             MYP4=MY+4  $ MXP5=MX+5  $ MYP5=MY+5                        01JL5
             HXE4=HX**4     $ HYE4=HY**4    $ HTE2=HT*HT                16MR5
             HP=HX*HY       $ HXY=HP*HP     $ HT2=2.0*HT                16MR5
             HA=1.0/HXE4    $ HB=2.0*HA     $ HXY1=1.0/HXY              16MR5
```

```
               HXYA=(1.0/HXY)*PR         $ HXYB=2.0*HXYA                16MR5
               HXYC=4.0*HXYA       $ HC=1.0/HYE4        $ HD=2.0*HC     16MR5
C-----CLEAR STORAGE                                                     01JL5
          DO 30 I=1,MXP7                                                01JL5
               A(I)=B(I)=C(I) = 0.0                                     01JL5
          DO 30 J=1,MYP7                                                01JL5
               D(I,J)=T(I,J)=Q(I,J)=WV(I,J)=S(I,J)=RHO(I,J)= 0.0        01JL5
          DO 31 K=1,4                                                   01JL5
               WX(I,J,K)=WY(I,J,K)= 0.0                                 01JL5
   31     CONTINUE                                                      01JL5
          DO 32 KK=1,30                                                 01JL5
               QT(I,J,KK)=0.0                                           01JL5
   32     CONTINUE                                                      01JL5
   30     CONTINUE                                                      01JL5
C-----INPUT TABLE 3, STIFFNESSES AND STATIC LOADING                     01JL5
      PRINT  26                                                         01JL5
          DO 27 N=1,NCT3                                                01JL5
      READ   28,IN1,JN1,IN2,JN2,DN,TN,SN,QN                             01JL5
      PRINT  29,IN1,JN1,IN2,JN2,DN,TN,SN,QN                             01JL5
               I1=IN1+4 $ J1=JN1+4  $ I2=IN2+4  $ J2=JN2+4              01JL5
          DO 27 I=I1,I2                                                 01JL5
          DO 27 J=J1,J2                                                 01JL5
               D(I,J)=D(I,J)+DN   $  T(I,J)=T(I,J)+TN                   01JL5
               Q(I,J)=Q(I,J)+QN   $  S(I,J)=S(I,J)+SN                   01JL5
   27     CONTINUE                                                      01JL5
C-----INPUT TABLE 4. INITIAL VELOCITIES AND DENSITY                     01JL5
      PRINT  33                                                         01JL5
          DO 34 N=1,NCT4                                                01JL5
      READ   28,IN1,JN1,IN2,JN2, WVN , RHON                             01JL5
      PRINT  35,IN1,JN1,IN2,JN2, WVN , RHON                             01JL5
               I1=IN1+4  $  J1=JN1+4  $ I2=IN2+4   $ J2=JN2+4           01JL5
          DO 34 I=I1,I2                                                 01JL5
          DO 34 J=J1,J2                                                 01JL5
               WV(I,J)=WV(I,J)+WVN                                      01JL5
               RHO(I,J)=RHO(I,J)+RHON                                   01JL5
   34     CONTINUE                                                      01JL5
C-----INPUT TABLE 5. DYNAMIC LOADING                                    01JL5
      PRINT  36                                                         01JL5
          DO 37 N=1,NCT5                                                01JL5
      READ   38,IN1,JN1,KN1,IN2,JN2,KN2,QTN                             01JL5
      PRINT  39,IN1,JN1,KN1,IN2,JN2,KN2,QTN                             01JL5
               I1=IN1+4 $  J1=JN1+4   $ K1=KN1+2                        01JL5
               I2=IN2+4 $  J2=JN2+4   $ K2=KN2+2                        01JL5
          DO 37 I=II,I2                                                 01JL5
          DO 37 J=J1,J2                                                 01JL5
          DO 37 K=K1,K2                                                 01JL5
               QT(I,J,K) = QT(I,J,K) + QTN                              23FE5
   37     CONTINUE                                                      01JL5
C-----INPUT TABLE 6. CLOSURE PARAMETERS                                 01JL5
          PRINT 40                                                      01JL5
          DO 41 N = 1, NCT6                                             01JL5
      READ   42,RP(N)                                                   01JL5
      PRINT  43,RP(N)                                                   01JL5
   41     CONTINUE                                                      01JL5
C-----SET ERRONEOUSLY STORED DATA TO ZERO                               01JL5
          DO 44  I=3,MXP5                                               01JL5
          DO 44  J=3,MYP5                                               01JL5
               D(I,MYP5)=0.0  $  D(MXP5,J)=0.0                          01JL5
               T(I,MYP5)=0.0  $  T(MXP5,J)=0.0                          09JL5
   44     CONTINUE                                                      01JL5
C-----CALCULATE JSTA(N)                                                 01JL5
```

```
          DO 89 N=4,25   $ JSTA(3)= -1  $ JSTA(N)=JSTA(N-1)+1           11FE5
   89 CONTINUE                                                          01JL5
C-----SOLUTION OF PROBLEM-----------------------------------------------01JL5
               K=1                                                      01JL5
          DO 46 KT=2,MTP2                                               01JL5
               KSTA=KT-2     $ K=K+1                                    01JL5
          IF(4-K) 82,83,83                                              01JL5
   82          K=3                                                      01JL5
          DO 84 I=3,MXP5                                                01JL5
          DO 84 J=3,MYP5                                                01JL5
               WX(I,J,1)= WX(I,J,3)    $   WY(I,J,1)= WY(I,J,3)         01JL5
               WX(I,J,2)= WX(I,J,4)    $   WY(I,J,2)= WY(I,J,4)         01JL5
   84     CONTINUE                                                      01JL5
   83     CONTINUE                                                      01JL5
               ITER=0    $  N=0                                         01JL5
      PRINT  45, IM1,JM1,IM2,JM2,IM3,JM3                                01JL5
          DO 47 NIT=1,ITMAX                                             01JL5
               KCTOL =0                                                 01JL5
          ITER=ITER + 1     $ N=N+1                                     01JL5
          IF (NCT6-N) 78,79,79                                          01JL5
   78          N=1                                                      01JL5
   79     CONTINUE                                                      01JL5
C-----SOLVE X SYSTEM                                                    01JL5
  204     DO 48 J=4,MYP4                                                01JL5
          DO 49 I=3,MXP5                                                01JL5
          IF (D(I,J)) 74,74,96                                          01JL5
   74          SF=0.0                                                   01JL5
          GO TO 99                                                      01JL5
   96          SF= RP(N)  * (( HX / HY ) ** 4 )                         01JL5
   99     IF(28-KT)50,51,51                                             01JL5
   50          QTP=0.0                                                  01JL5
          GO TO 52                                                      01JL5
   51          QTP=QT(I,J,KT-1)                                         01JL5
   52 CALL COXY                                                         16MR5
          IF(KT-3)53,54,55                                              01JL5
   53          AA = X1                                                  01JL5
               BB = X2 + XY2                                            01JL5
               CC = X3 + XY5 + S (I,J) / HP + SF                        01JL5
               DD = X4 + XY8                                            01JL5
               EE = X5                                                  01JL5
               F1 = Q (I,J) / HP + SF * WY (I,J,K) - X6 * WX (I-1,J+1,K)01JL5
     1    - X7 * WX (I,J+1,K) - X8 * WX (I+1,J+1,K) - X9 * WX (I-1,J-1, 01JL5
     2    K) - X10 * WX (I,J-1,K) - X11 * WX (I+1,J-1,K) - Y1 * WY (I,J 01JL5
     3    -2,K) - Y2 * WY (I,J-1,K) - Y3 * WY (I,J,K) -Y4 * WY (I,J+1,K)01JL5
     4    - Y5 * WY (I,J+2,K) - Y6 * WY (I-1,J-1,K) - Y7 * WY (I-1,J,K) 01JL5
     5    - Y8 * WY (I-1,J+1,K) - Y9 * WY (I+1,J-1,K) - Y10 * WY (I+1,  01JL5
     6    J,K) - Y11 * WY (I+1,J+1,K)                                   01JL5
               F2 = - XY1 * ( WX (I-1,J-1,K) + WY (I-1,J-1,K) ) - XY2 * 01JL5
     1    WY (I-1,J,K) - XY3 * ( WX (I-1,J+1,K) + WY (I-1,J+1,K) ) -    01JL5
     2    XY4 * ( WX (I,J-1,K) + WY (I,J-1,K) ) - XY5 * WY (I,J,K) -    01JL5
     3    XY6 * ( WX (I,J+1,K) + WY (I,J+1,K) ) - XY7 * ( WX (I+1,J-1,K)01JL5
     4    + WY (I+1,J-1,K) ) - XY8 * WY (I+1,J,K) - XY9 * ( WX (I+1,J+1,01JL5
     5    K) + WY (I+1,J+1,K) )                                         01JL5
               FF = F1 + F2                                             01JL5
          GO TO 56                                                      01JL5
   54          AA = 0.5 * X1                                            01JL5
               BB = 0.5 * (X2 + XY2)                                    01JL5
               CC = 0.5 * (X3 + XY5 + S(I,J) / HP + SF) + 4.0 * (RHO(   01JL5
     1              I,J) / HTE2)                                        01JL5
               DD = 0.5 * (X4 + XY8)                                    01JL5
               EE = 0.5 * X5                                            01JL5
               F1 = QTP / HP + 0.5 * SF * WY(I,J,K) + 4.0 * (RHO(I,J)   01JL5
```

```
  1   / HT) * WV(I,J) + 4.0 * (RHO(I,J) / HTE2) * WX(I,J,K-1)       01JL5
          F2 = -0.5 * (X1 * WX(I-2,J,K-1) + (X2 + XY2) * WX(I-1,J,K01AP5
  1   -1) + (X3 + XY5 + S(I,J) / HP) * WX(I,J,K-1) + (X4 + XY8) *    01AP5
  2   WX(I+1,J,K-1) + X5 * WX(I+2,J,K-1) + (X6 + XY3) * (WX(I-1,J+1,01AP5
  3   K-1) + WX(I-1,J+1,K)) + (X7 + XY6) * (WX(I,J+1,K-1) + WX(I,J+101AP5
  4   ,K)) + (X8 + XY9) * (WX(I+1,J+1,K-1) + WX(I+1,J+1,K)) +        01AP5
  5   (X9 + XY1) * (WX(I-1,J-1,K-1) + WX(I-1,J-1,K)) + (X10 + XY4)   01AP5
  6   * (WX(I,J-1,K-1) + WX(I,J-1,K)) + (X11 +XY7) * (WX(I+1,J-1,    01AP5
  7   K-1) + WX(I+1,J-1,K)))                                         01JL5
         F3 = -0.5 * (Y1 *(WY(I,J-2,K-1) + WY(I,J-2,K)) + (Y2 +      01JL5
  1   XY4) * (WY(I,J-1,K-1) + WY(I,J-1,K)) + (Y3 + XY5) * (WY(I,J,   01AP5
  2   K-1) + WY(I,J,K)) + (Y4 + XY6) * (WY(I,J+1,K-1) + WY(I,J+1,    01AP5
  3   K)) + Y5 * (WY(I,J+2,K-1) + WY(I,J+2,K)) + (Y6 + XY1) * (WY(   01AP5
  4   I-1,J-1,K-1) + WY(I-1,J-1,K)) + (Y7 + XY2) * (WY(I-1,J,K-1) +  01AP5
  5   WY(I-1,J,K)) + (Y8 + XY3) * (WY(I-1,J+1,K-1) + WY(I-1,J+1,K))  01AP5
  6   + (Y9 + XY7) * (WY(I+1,J-1,K-1) + WY(I+1,J-1,K)) + (Y10 +      01AP5
  7   XY8) * (WY(I+1,J,K-1) + WY(I+1,J,K)) + (Y11+ XY9) * (WY(I+1,   01JL5
  8   J+1,K-1) + WY(I+1,J+1,K)))                                     01JL5
               FF = F1 + F2 + F3                                     01JL5
      GO TO 56                                                       01JL5
 55       AA = 0.5 * X1                                              01AP5
          BB = 0.5 * (X2 + XY2)                                      01AP5
          CC = 0.5 * (X3 + XY5 + S(I,J) / HP + SF) + RHO(I,J) /      01AP5
  1          HTE2                                                    01AP5
          DD = 0.5 * (X4 + XY8)                                      01AP5
          EE = 0.5 * X5                                              01AP5
          F1 = QTP / HP + 0.5 * SF * WY(I,J,K) - (RHO(I,J) / HTE2)  01AP5
  1   * WX(I,J,K-2) + 2.0 * (RHO(I,J) / HTE2) * WX(I,J,K-1)         01AP5
          F2 = -0.5 * (X1 * WX(I-2,J,K-2) + (X2 + XY2) * WX(I-1,J,K01AP5
  1   -2) + (X3 + XY5 + S(I,J) / HP) * WX(I,J,K-2) + (X4 + XY8) *    01AP5
  2   WX(I+1,J,K-2) + X5 * WX(I+2,J,K-2) + (X6 + XY3) * (WX(I-1,J+1,01AP5
  3   K-2) + WX(I-1,J+1,K)) + (X7 + XY6) * (WX(I,J+1,K-2) + WX(I,J+101AP5
  4   ,K)) + (X8 + XY9) * (WX(I+1,J+1,K-2) + WX(I+1,J+1,K)) +        01AP5
  5   (X9 + XY1) * (WX(I-1,J-1,K-2) + WX(I-1,J-1,K)) + (X10 + XY4)   01AP5
  6   * (WX(I,J-1,K-2) + WX(I,J-1,K)) + (X11 +XY7) * (WX(I+1,J-1,    01AP5
  7   K-2) + WX(I+1,J-1,K)))                                         01AP5
         F3 = -0.5 * (Y1 *(WY(I,J-2,K-2) + WY(I,J-2,K)) + (Y2 +      01JL5
  1   XY4) * (WY(I,J-1,K-2) + WY(I,J-1,K)) + (Y3 + XY5) * (WY(I,J,   01AP5
  2   K-2) + WY(I,J,K)) + (Y4 + XY6) * (WY(I,J+1,K-2) + WY(I,J+1,    01AP5
  3   K)) + Y5 * (WY(I,J+2,K-2) + WY(I,J+2,K)) + (Y6 + XY1) * (WY(   01AP5
  4   I-1,J-1,K-2) + WY(I-1,J-1,K)) + (Y7 + XY2) * (WY(I-1,J,K-2) +  01AP5
  5   WY(I-1,J,K)) + (Y8 + XY3) * (WY(I-1,J+1,K-2) + WY(I-1,J+1,K))  01AP5
  6   + (Y9 + XY7) * (WY(I+1,J-1,K-2) + WY(I+1,J-1,K)) + (Y10 +      01AP5
  7   XY8) * (WY(I+1,J,K-2) + WY(I+1,J,K)) + (Y11+ XY9) * (WY(I+1,   01JL5
  8   J+1,K-2) + WY(I+1,J+1,K)))                                     01AP5
               FF = F1 + F2 + F3                                     01AP5
 56   CONTINUE                                                       01JL5
202       E = AA *B(I-2) + BB                                        01JL5
          DENOM=E*B(I-1)+AA*C(I-2)+CC                                01JL5
      IF (DENOM) 57,58,57                                            01JL5
 58       D=0.0                                                      01JL5
      GO TO 59                                                       01JL5
 57       D= -1.0/DENOM                                              01JL5
 59       C(I)= D*EE                                                 01JL5
          B(I)= D*(E*C(I-1)+DD)                                      01JL5
          A(I)= D*(E*A(I-1)+AA*A(I-2)-FF)                            01JL5
 49   CONTINUE                                                       01JL5
      DO 60 L=3,MXP5                                                 01JL5
          I=MX+8-L                                                   01JL5
          WX(I,J,K)= A(I)+ B(I)* WX(I+1,J,K) + C(I)*WX(I+2,J,K)      01JL5
 60   CONTINUE                                                       01JL5
 48   CONTINUE                                                       01JL5
```

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C-----SOLVE Y SYSTEM                                                    01JL5
  206     DO 61 I=4,MXP4                                                01JL5
          DO 62 J=3,MYP5                                                01JL5
          IF (D(I,J)) 97,97,98                                          01JL5
   97          SF=0.0                                                   01JL5
          GO TO 100                                                     01JL5
   98          SF=RP(N)                                                 01JL5
  100     IF(28-KT)63,64,64                                             01JL5
   63          QTP=0.0                                                  01JL5
          GO TO 65                                                      01JL5
   64          QTP=QT(I,J,KT-1)                                         01JL5
   65 CALL COXY                                                         16MR5
          IF(KT-3)66,67,68                                              01JL5
   66          AA = Y1                                                  01JL5
               BB = Y2 + XY4                                            01JL5
               CC = Y3 + XY5 + S (I,J) / HP + SF                        01JL5
               DD = Y4 + XY6                                            01JL5
               EE = Y5                                                  01JL5
               F1 = Q (I,J) / HP + SF * WX (I,J,K) - Y6 * WY (I-1,J-1,K)01JL5
    1     - Y7 * WY (I-1,J,K) - Y8 * WY (I-1,J+1,K) - Y9 * WY (I+1,J-1, 01JL5
    2     K) - Y10 * WY (I+1,J,K) - Y11 * WY (I+1,J+1,K) - X1 * WX (I-2,01JL5
    3     J,K) - X2 * WX (I-1,J,K) - X3 * WX (I,J,K) -X4 *WX (I+1,J,K) -01JL5
    4     X5 * WX (I+2,J,K) - X6 * WX (I-1,J+1,K) - X7 * WX (I,J+1,K) - 01JL5
    5     X8 * WX (I+1,J+1,K) - X9 * WX (I-1,J-1,K) - X10 * WX (I,J-1,K)01JL5
    6     - X11 * WX (I+1,J-1,K)                                        01JL5
               F2 = - XY1 * ( WX (I-1,J-1,K) + WY (I-1,J-1,K) ) - XY2 * 01JL5
    1     ( WX (I-1,J,K) + WY (I-1,J,K) ) - XY3 * ( WX (I-1,J+1,K) + WY 01JL5
    2     (I-1,J+1,K) ) - XY4 * WX (I,J-1,K) - XY5 * WX (I,J,K) - XY6 * 01JL5
    3     WX (I,J+1,K) - XY7 * ( WX (I+1,J-1,K) + WY ( I+1,J-1,K) ) -   01JL5
    4     XY8 * ( WX (I+1,J,K) + WY (I+1,J,K) ) - XY9 * ( WX (I+1,J+1,K)01JL5
    5     + WY (I+1,J+1,K) )                                            01JL5
               FF = F1 + F2                                             01JL5
          GO TO 69                                                      01JL5
   67          AA = 0.5 * Y1                                            01JL5
               BB = 0.5 * (Y2 + XY4)                                    01JL5
               CC = 0.5 * (Y3 + XY5 + S(I,J) / HP + SF) + 4.0 * (RHO    01JL5
    1               (I,J) / HTE2 )                                      01JL5
               DD = 0.5 * (Y4 + XY6)                                    01JL5
               EE = 0.5 * Y5                                            01JL5
               F1 = QTP / HP + 0.5 * SF * WX(I,J,K) + 4.0 * (RHO(I,J) / 01JL5
    1     HT ) * WV(I,J) + 4.0 * (RHO(I,J) / HTE2) * WY(I,J,K-1)        01JL5
               F2 = -0.5 * (Y1 * WY(I,J-2,K-1) + (Y2 + XY4) * WY(I,J-1, 01AP5
    1     K-1) + (Y3 + XY5 + S(I,J) / HP) * WY(I,J,K-1) + (Y4 + XY6)    01AP5
    2     * WY(I,J+1,K-1) + Y5 * WY(I,J+2,K-1) + (Y6 + XY1) * (WY(I-1,  01JL5
    3     J-1,K-1) + WY(I-1,J-1,K)) + (Y7 + XY2) * (WY(I-1,J,K-1) +     01AP5
    4     WY(I-1,J,K)) + (Y8 +XY3) * (WY(I-1,J+1,K-1) + WY(I-1,J+1,K)) +09JL5
    5     (Y9 + XY7) * (WY(I+1,J-1,K-1) + WY(I+1,J-1,K)) + (Y10 + XY8)  01AP5
    6     * (WY(I+1,J,K-1) + WY(I+1,J,K)) + (Y11 + XY9) * (WY(I+1,J+1,  01AP5
    7     K-1) + WY(I+1,J+1,K)))                                        01JL5
               F3 = -0.5 * (X1 * (WX(I-2,J,K-1) + WX(I-2,J,K)) + (X2 +  01AP5
    1     XY2) * (WX(I-1,J,K-1) + WX(I-1,J,K)) + (X3 + XY5) * (WX(      01AP5
    2     I,J,K-1) + WX(I,J,K)) + (X4 + XY8) * (WX(I+1,J,K-1)           01JL5
    3            + WX(I+1,J,K)) + X5 * (WX(I+2,J,K-1) + WX(I+2,J,K))    01JL5
    4     + (X6 + XY3) * (WX(I-1,J+1,K-1) + WX(I-1,J+1,K)) + (X7 + XY6) 01JL5
    5     * (WX(I,J+1,K-1) + WX(I,J+1,K)) + (X8 + XY9) * (WX(I+1,J+1,K  01AP5
    6     -1) + WX(I+1,J+1,K)) + (X9 + XY1) * (WX(I-1,J-1,K-1) +        01JL5
    7     WX(I-1,J-1,K)) + (X10 + XY4) * (WX(I,J-1,K-1) + WX(I,J-1,K))  01AP5
    8     + (X11 + XY7) * (WX(I+1,J-1,K-1) + WX(I+1,J-1,K)))            01JL5
                    FF = F1 + F2 + F3                                   01JL5
          GO TO 69                                                      01JL5
   68          AA = 0.5 * Y1                                            01AP5
               BB = 0.5 * (Y2 + XY4)                                    01AP5
```

```
               CC = 0.5 * (Y3 + XY5 + S(I,J) / HP + SF) + RHO(I,J) /    01AP5
     1              HTE2                                                01AP5
               DD = 0.5 * (Y4 + XY6)                                    01AP5
               EE = 0.5 * Y5                                            01AP5
               F1 = QTP / HP + 0.5 * SF * WX(I,J,K) - (RHO(I,J) / HTE2) 01AP5
     1    * WY(I,J,K-2) + 2.0 * (RHO(I,J) / HTE2) * WY(I,J,K-1)         01AP5
               F2 = -0.5 * (Y1 * WY(I,J-2,K-2) + (Y2 + XY4) * WY(I,J-1, 01AP5
     1    K-2) + (Y3 + XY5 + S(I,J) / HP) * WY(I,J,K-2) + (Y4 + XY6)    01AP5
     2    * WY(I,J+1,K-2) + Y5 * WY(I,J+2,K-2) + (Y6 + XY1) * (WY(I-1,  01JL5
     3    J-1,K-2) + WY(I-1,J-1,K)) + (Y7 + XY2) * (WY(I-1,J,K-2) +     01AP5
     4    WY(I-1,J,K)) + (Y8 +XY3) * (WY(I-1,J+1,K-2) + WY(I-1,J+1,K)) +01JL5
     5    (Y9 + XY7) * (WY(I+1,J-1,K-2) + WY(I+1,J-1,K)) + (Y10 + XY8)  01AP5
     6    * (WY(I+1,J,K-2) + WY(I+1,J,K)) + (Y11 + XY9) * (WY(I+1,J+1,  01AP5
     7    K-2) + WY(I+1,J+1,K)))                                        01AP5
               F3 = -0.5 * (X1 * (WX(I-2,J,K-2) + WX(I-2,J,K)) + (X2 +  01AP5
     1    XY2) * (WX(I-1,J,K-2) + WX(I-1,J,K)) + (X3 + XY5) * (WX(      01AP5
     2    I,J,K-2) + WX(I,J,K)) + (X4 + XY8) * (WX(I+1,J,K-2)           01JL5
     3           + WX(I+1,J,K)) + X5 * (WX(I+2,J,K-2) + WX(I+2,J,K))    01JL5
     4    + (X6 + XY3) * (WX(I-1,J+1,K-2) + WX(I-1,J+1,K)) + (X7 + XY6) 01JL5
     5    * (WX(I,J+1,K-2) + WX(I,J+1,K)) + (X8 + XY9) * (WX(I+1,J+1,K  01AP5
     6    -2) + WX(I+1,J+1,K)) + (X9 + XY1) * (WX(I-1,J-1,K-2) +        01AP5
     7    WX(I-1,J-1,K)) + (X10 + XY4) * (WX(I,J-1,K-2) + WX(I,J-1,K))  01AP5
     8    + (X11 + XY7) * (WX(I+1,J-1,K-2) + WX(I+1,J-1,K)))            01AP5
                    FF = F1 + F2 + F3                                   01AP5
   69     CONTINUE                                                      01JL5
  201          E = AA * B(J-2) + BB                                     01JL5
               DENOM= E*B(J-1)+AA*C(J-2)+CC                             01JL5
          IF (DENOM) 70,71,70                                           01JL5
   71          D=0.0                                                    01JL5
          GO TO 72                                                      01JL5
   70          D= -1.0 /DENOM                                           01JL5
   72          C(J)= D*EE                                               01JL5
               B(J)= D*(E*C(J-1)+DD)                                    01JL5
               A(J)= D*(E*A(J-1)+AA*A(J-2)-FF)                          01JL5
   62     CONTINUE                                                      01JL5
          DO 73 L=3,MYP5                                                01JL5
               J=MY+8-L                                                 01JL5
               WY(I,J,K)= A(J)+ B(J)*WY(I,J+1,K)+ C(J)* WY(I,J+2,K)     01JL5
   73     CONTINUE                                                      01JL5
   61     CONTINUE                                                      01JL5
C-----COUNT STAS WHERE  WX AND WY NOT CLOSED                            01JL5
          DO 113 I=4,MXP4                                               01JL5
          DO 113 J=4,MYP4                                               01JL5
          IF(ABSF(WX(I,J,K)-WY(I,J,K))-CTOL) 94,94,76                   01JL5
   76     KCTOL =KCTOL + 1                                              01JL5
   94     CONTINUE                                                      01JL5
  113     CONTINUE                                                      01JL5
C-----PRINT MONITOR DATA                                                01JL5
      PRINT 77,ITER,RP(N) ,KCTOL, KSTA,WX(IM1+4,JM1+4,K),               01JL5
     1         WX(IM2+4,JM2+4,K),WX(IM3+4,JM3+4,K),WY(IM1+4,JM1+4,K),   01JL5
     2         WY(IM2+4,JM2+4,K), WY(IM3+4,JM3+4,K)                     01JL5
          IF (KCTOL) 75,75,81                                           01JL5
   81     CONTINUE                                                      01JL5
   47     CONTINUE                                                      01JL5
   75     CONTINUE                                                      01JL5
C-----PRINT DEFLS                                                       01JL5
      PRINT 11                                                          08MY3 ID
      PRINT 1                                                           18FE5 ID
      PRINT 13, ( AN1(N), N = 1, 32 )                                   18FE5 ID
      PRINT 16, NPROB, ( AN2(N), N = 1, 14 )                            28AG3 ID
  109 PRINT  85, KSTA , KCTOL                                           01JL5
               JI=3  $ JF=7   $ JTEST= MYP5/5                           01JL5
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```
  107     DO   92 JKE=1,JTEST                                           01JL5
      PRINT  87,(JSTA(N),N=JI,JF )                                      01JL5
          DO 86 I=3,MXP5                                                01JL5
               ISTA= I-4                                                01JL5
      PRINT  88, ISTA , (WX(I,J,K), J=JI,JF)                            01JL5
      PRINT  91,        (WY(I,J,K), J=JI,JF)                            01JL5
   86     CONTINUE                                                      01JL5
               JI=JI+5       $  JF=JF+5                                 01JL5
   92     CONTINUE                                                      01JL5
          IF ( MYP5 - JI ) 93, 108, 108                                 01JL5
  108          JF = MYP5                                                01JL5
               JTEST = 1                                                01JL5
          GO TO 107                                                     01JL5
   93     CONTINUE                                                      01JL5
          DO 101 I= 4,MXP4                                              01JL5
          DO 101 J= 4,MYP4                                              01JL5
               WY(I,J,K) =             0.5* ( WX(I,J,K) + WY(I,J,K) )   01JL5
               WX(I,J,K) = WY(I,J,K)                                    01JL5
  101     CONTINUE                                                      01JL5
      CALL TIME                                                         01JL5
   46     CONTINUE                                                      01JL5
      CALL TIME                                                         18FE5 ID
          GO TO 1010                                                    26AG3 ID
 9990 CONTINUE                                                          12MR5 ID
 9999 CONTINUE                                                          04MY3 ID
      PRINT 11                                                          08MY3 ID
      PRINT 1                                                           18FE5 ID
      PRINT 13, ( AN1(N), N = 1, 32 )                                   18FE5 ID
      PRINT 19                                                          26AG3 ID
      END                                                               04MY3 ID
C-----SUBROUTINE                                                        01JL5
      SUBROUTINE COXY                                                   16MR5
      COMMON/1/D(22,22),T(22,22)/2/X1,X2,X3,X4,X5,X6,X7,X8,X9,X10,      16MR5
     1         X11,Y1,Y2,Y3,Y4,Y5,Y6,Y7,Y8,Y9,Y10,Y11,XY1,XY2,XY3,      16MR5
     2         XY4,XY5,XY6,XY7,XY8,XY9,I,J,HA,HB,HC,HD,HXYA,            16MR5
     3         HXYB,HXYC,HXY1                                           16MR5
               X1 = HA * D (I-1,J)                                      01JL5
               X2 = - HB * ( D (I-1,J) + D (I,J) ) - HXYB * D (I-1,J)   01JL5
               X3 = HA * ( D (I-1,J) + 4.0 * D (I,J) + D (I+1,J) )      01JL5
     1                   + HXYC * D (I,J)                               01JL5
               X4 = - HB * ( D (I,J) + D (I+1,J) ) - HXYB * D (I+1,J)   01JL5
               X5 = HA * D (I+1,J)                                      01JL5
               X6 = HXYA * D (I-1,J)                                    01JL5
               X7 = - HXYB * D (I,J)                                    01JL5
               X8 = HXYA * D (I+1,J)                                    01JL5
               X9 = X6                                                  01JL5
               X10 = X7                                                 01JL5
               X11 = X8                                                 01JL5
               Y1 = HC * D (I,J-1)                                      01JL5
               Y2 = - HD * ( D (I,J-1) + D (I,J) ) - HXYB * D (I,J-1)   01JL5
               Y3 = HC * ( D (I,J-1) + 4.0 * D (I,J) + D (I,J+1) )      01JL5
     1                   + HXYC * D (I,J)                               01JL5
               Y4 = - HD * ( D (I,J) + D (I,J+1) ) - HXYB * D (I,J+1)   01JL5
               Y5 = HC * D (I,J+1)                                      01JL5
               Y6 = HXYA * D (I,J-1)                                    01JL5
               Y7 = X7                                                  01JL5
               Y8 = HXYA * D (I,J+1)                                    01JL5
               Y9 = Y6                                                  01JL5
               Y10 = Y7                                                 01JL5
               Y11 = Y8                                                 01JL5
               XY1 = HXY1 * T(I,J)                                      09JL5
               XY2 = - HXY1 * ( T (I,J) + T(I,J+1) )                    09JL5
```

```
                  XY3 = HXY1 * T (I,J+1)                                   09JL5
                  XY4 = - HXY1 * ( T(I,J) + T(I+1,J) )                     09JL5
                  XY5 = HXY1 * ( T(I,J) + T(I,J+1) + T(I+1,J) + T(I+1,J+1))09JL5
                  XY6 = - HXY1 * ( T(I,J+1) + T(I+1,J+1) )                 09JL5
                  XY7 = HXY1 * T(I+1,J)                                    09JL5
                  XY8 = -HXY1 * ( T(I+1,J) + T(I+1,J+1) )                  09JL5
                  XY9 = HXY1 * T(I+1,J+1)                                  09JL5
       END
       END
          FINIS
9EXECUTE.
```